Sliding Mode Control Using Modified Rodrigues Parameters

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Introduction

Spacecraft pointing poses a complex problem involving nonlinear dynamics with either linear and/or nonlinear control laws. Primary control actuators usually include thrusters for rapid and coarse attitude maneuvers, and reaction wheels for slow and precise attitude maneuvers. Other types of control mechanisms include gravity-gradient stabilization and magnetic torquer assemblies. Control algorithms can be divided into open-loop systems and closed-loop (feedback) systems. Open-loop systems usually require a pre-determined pointing maneuver, and are typically determined using optimal control techniques which involve the solution of a two-point-boundary-problem. An example of open-loop control is the time-optimal attitude maneuver (e.g., see the excellent survey paper by Scrivener and Thompson [1]). Closed-loop systems can provide robustness with respect to spacecraft modeling uncertainties and unexpected disturbances.

The control technique used in this note is based upon sliding mode (variable structure) control (see [2]). This type of control has been successfully applied for spacecraft pointing and regulation using both a Rodrigues (Gibbs vector) representation [3] and a quaternion representation [4]. An advantage of the quaternion representation is that singularities in the kinematic equations can be avoided. However, the use of quaternions requires an extra parameter which leads to a non-minimal parameterization. The Rodrigues parameters provide a minimal (i.e., three dimensional) parameterization. However, a singularity exists for 180° rotations, which hinders this parameterization for extremely large angle rotations. This difficulty may be overcome by applying successive rotations, each less than 180°. However, the overall maneuver may require extra control authority and power requirements which may not be necessary. In this note, a sliding mode controller is developed based upon the modified

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Rodrigues parameters [5-6]. Advantages of using this attitude representation include: (1) rotations of up to 360° are possible, and (2) the parameters form a minimal parameterization.

Attitude Kinematics and Dynamics

In this section, a brief review of the kinematic equations of motion using the modified Rodrigues parameters is shown. This parameterization is derived by applying a stereographic projection of the quaternions. The quaternion representation is given by

$$\underline{q} \equiv \begin{bmatrix} \underline{q}_{13} \\ q_4 \end{bmatrix} \tag{1}$$

with

$$\underline{q}_{13} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underline{\hat{n}} \sin\left(\frac{\theta}{2}\right) \tag{2a}$$

$$q_4 = \cos\left(\frac{\theta}{2}\right) \tag{2b}$$

where $\underline{\hat{n}}$ is a unit vector corresponding to the axis of rotation and θ is the angle of rotation. The modified Rodrigues parameters are defined by [5]

$$\underline{p} = \frac{\underline{q}_{13}}{1+q_4} = \underline{\hat{n}} \tan\left(\frac{\theta}{4}\right) \tag{3}$$

where $\underline{p}$ is a 3×1 vector. The kinematic equations of motion are derived by using the spacecraft's angular velocity ($\underline{\omega}$), given by [6]

$$\underline{\dot{p}} = \frac{1}{4}\left\{\left(1-\underline{p}^T\,\underline{p}\right) I_{3\times3} + 2\left[\underline{p}\times\right] + 2\,\underline{p}\,\underline{p}^T\right\}\underline{\omega} \tag{4}$$

where $I_{3\times3}$ is a 3×3 identity matrix, and $\left[\underline{p}\times\right]$ is a 3×3 "cross product" matrix defined by

$$\left[\underline{p}\times\right] \equiv \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \tag{5}$$

The dynamic equation of motion are given by Euler's equation, defined by

$$\underline{\dot{\omega}} = J^{-1}\left[J\underline{\omega}\times\right]\underline{\omega} + J^{-1}\underline{u} \tag{6}$$

where J is the spacecraft's inertia (3×3) matrix, and $\underline{u}$ is a torque input.

Sliding Mode Controller

In this section a sliding mode controller is developed using the modified Rodrigues parameters. It is assumed that measurements of both the spacecraft attitude and angular rate are available, which may be provided by a Kalman filter. The nonlinear model for spacecraft motion is summarized by

$$\underline{\dot{p}} = F\left(\underline{p}\right)\underline{\omega} \tag{7a}$$

$$\underline{\dot{\omega}} = \underline{f}\left(\underline{\omega}\right) + J^{-1}\underline{u} \tag{7b}$$

where

$$F\left(\underline{p}\right) \equiv \frac{1}{4}\left\{\left(1-\underline{p}^T\underline{p}\right)I_{3\times 3} + 2\left[\underline{p}\times\right] + 2\underline{p}\,\underline{p}^T\right\} \tag{8a}$$

$$\underline{f}\left(\underline{\omega}\right) \equiv J^{-1}\left[J\underline{\omega}\times\right]\underline{\omega} \tag{8b}$$

Under ideal conditions, the state trajectories move onto a sliding manifold $\left(\underline{s} = \underline{0}\right)$, where $\underline{s}$ is given by

$$\underline{s} \equiv \underline{\omega} - \underline{m}\left(\underline{p}\right) \tag{9}$$

The quantity $\underline{m}\left(\underline{p}\right)$ is obtained using a desired vector field from the kinematic relations (see [3]), given by

$$\underline{m}\left(\underline{p}\right) = F^{-1}\left(\underline{p}\right)\underline{d}\left(\underline{p}\right) \tag{10}$$

where

$$\underline{F}^{-1}\left(\underline{p}\right) = 4\left(1+\underline{p}^T\underline{p}\right)^{-2}\left\{\left(1-\underline{p}^T\underline{p}\right)I_{3\times 3} - 2\left[\underline{p}\times\right] + 2\underline{p}\,\underline{p}^T\right\} \tag{11}$$

The quantity $\underline{d}\left(\underline{p}\right)$ is formed by allowing a linear behavior in the sliding motion, given by

$$\underline{d}\left(\underline{p}\right)=\Lambda\left(\underline{p}-\underline{p}_d\right) \tag{12}$$

where $\underline{p}_d$ is the desired reference trajectory, and Λ is a diagonal matrix with negative elements. This allows for decoupled sliding motions and exponential convergence towards the final desired orientation. The sliding mode controller which produces a negative definite derivative of the Lyapunov function $\underline{s}^T\underline{s}$ is given by ([3])

$$\underline{u}=-J\left\{\underline{f}\left(\underline{\omega}\right)-\frac{\partial\underline{m}}{\partial\underline{p}}\left[F\left(p\right)\underline{m}\left(p\right)+F\left(p\right)\underline{s}\right]+K\,\mathrm{sat}\left(\underline{s},\varepsilon\right)\right\} \tag{13}$$

where K is a 3×3 positive definite, diagonal matrix. The saturation function is used to minimize chattering in the control torques. This function is defined by

$$\mathrm{sat}\left(s_i,\varepsilon\right)\equiv\begin{cases}1 & \text{for} \quad s_i>\varepsilon\\ \dfrac{s_i}{\varepsilon} & \text{for} \quad \left|s_i\right|\le\varepsilon\\ -1 & \text{for} \quad s_i<-\varepsilon\end{cases} \qquad i=1,2,3 \tag{14}$$

where ε is a small positive quantity.

Regulation

The regulation problem requires that the final spacecraft position be zero (i.e., unity quaternion). This corresponds to desired modified Rodrigues parameters given by $\underline{p}_d=0$. Also the Λ matrix in Equation (12) is assumed to be given by a scalar $\left(\lambda\right)$ times the identity matrix, which leads to

$$\underline{m}\left(\underline{p}\right)=4\lambda\left(1+\underline{p}^T\underline{p}\right)^{-2}\left\{\left(1-\underline{p}^T\underline{p}\right)I_{3\times 3}-2\left[\underline{p}\times\right]+2\underline{p}\,\underline{p}^T\right\}\underline{p} \tag{15}$$

This relation can be simplified significantly by applying some cross product relations, which leads to

$$\underline{m}\left(\underline{p}\right)=4\lambda\left(1+\underline{p}^T\underline{p}\right)^{-1}\underline{p} \tag{16}$$

The partial derivative of Equation (16) with respect to $\underline{p}$ is given by

$$\frac{\partial \underline{m}}{\partial \underline{p}} = 4\lambda\left(1+\underline{p}^T\,\underline{p}\right)^{-1}\left\{I_{3\times 3} - 2\left(1+\underline{p}^T\,\underline{p}\right)^{-1}\underline{p}\,\underline{p}^T\right\} \tag{17}$$

Tracking

The tracking problem requires the system attitude to follow a desired reference trajectory. Equation (10) for the tracking problem is derived to be

$$\underline{m}(\underline{p}) = 4\lambda\left(1+\underline{p}^T\,\underline{p}\right)^{-1}\underline{p} - 4\lambda\left(1+\underline{p}^T\,\underline{p}\right)^{-2}\left\{\left(1-\underline{p}^T\,\underline{p}\right)I_{3\times 3} - 2\left[\underline{p}\times\right] + 2\,\underline{p}\,\underline{p}^T\right\}\underline{p}_d \tag{18}$$

The partial derivative of Equation (18) with respect to $\underline{p}$ is given by

$$\begin{aligned}\frac{\partial \underline{m}}{\partial \underline{p}} &= 4\lambda\left(1+\underline{p}^T\,\underline{p}\right)^{-1}\left\{I_{3\times 3} - 2\left(1+\underline{p}^T\,\underline{p}\right)^{-1}\underline{p}\,\underline{p}^T\right\}\\ &\quad -8\lambda\left(1+\underline{p}^T\,\underline{p}\right)^{-2}\left\{\underline{p}\,\underline{p}_d^T - \underline{p}_d\,\underline{p}^T + \left[\underline{p}_d\times\right] + \left(\underline{p}_d^T\,\underline{p}\right)I_{3\times 3}\right\}\\ &\quad +16\lambda\left(1+\underline{p}^T\,\underline{p}\right)^{-3}\left\{\left(1-\underline{p}^T\,\underline{p}\right)I_{3\times 3} - 2\left[\underline{p}\times\right] + 2\,\underline{p}\,\underline{p}^T\right\}\underline{p}_d\,\underline{p}^T\end{aligned} \tag{19}$$

Spacecraft Simulation

The spacecraft simulation involves a multi-axis rest-to-rest maneuver. The inertia matrix for the simulated spacecraft is given by [3]

$$J = \text{diag}\left[114,\quad 86,\quad 87\right]\quad \text{kg}-\text{m}^2 \tag{20}$$

The initial conditions for the angular velocity are set to zero, and the initial conditions for the modified Rodrigues parameters are given by

$$\underline{p}(t_0) = \left[-0.1\quad 0.5\quad 1.0\right]^T \tag{21}$$

The desired attitude parameters are set to zero (i.e., the regulation case). The diagonal elements of K in Equation (13) are all set to 0.0015, and the constant λ is set to -0.015 sec^{-1}. The parameter ε in the saturation controller is set to 0.01. Also, the control torques are limited to 1.0 N-m. A plot of the closed-loop modified Rodrigues parameters is shown in Figure 1. Also, plots of the angular velocity trajectories and applied control torques are shown in Figures 2 and 3, respectively. Using Equation (3), the rotation for the initial conditions in Equation (21) is approximately 193°. Therefore, converting the modified Rodrigues parameters shown in Figure 1 to quaternions (see [6]) reveals that the scalar (fourth) quaternion crosses the zero. Therefore, the Gibbs vector control formulation in [3] becomes singular, but is easily handled by the modified Rodrigues parameter control formulation.

Figure 1 Plot of Closed-Loop Modified Rodrigues Trajectories

Figure 2 Plot of Closed-Loop Angular Velocity Trajectories

Figure 3 Plot of Applied Control Torques

Conclusions

In this note, a sliding mode controller was developed for attitude pointing using the modified Rodrigues parameters. The modified Rodriques parameters represent a minimal parameterization with a singularity at 360°. These parameters avoid the normalization constraint associated with the quaternion parameterization, and allow for rotations of greater than 180° for which the Gibbs vector parameterization becomes singular. Simulation results indicate that the new algorithm was able to accurately control the attitude of a spacecraft for large angle maneuvers.

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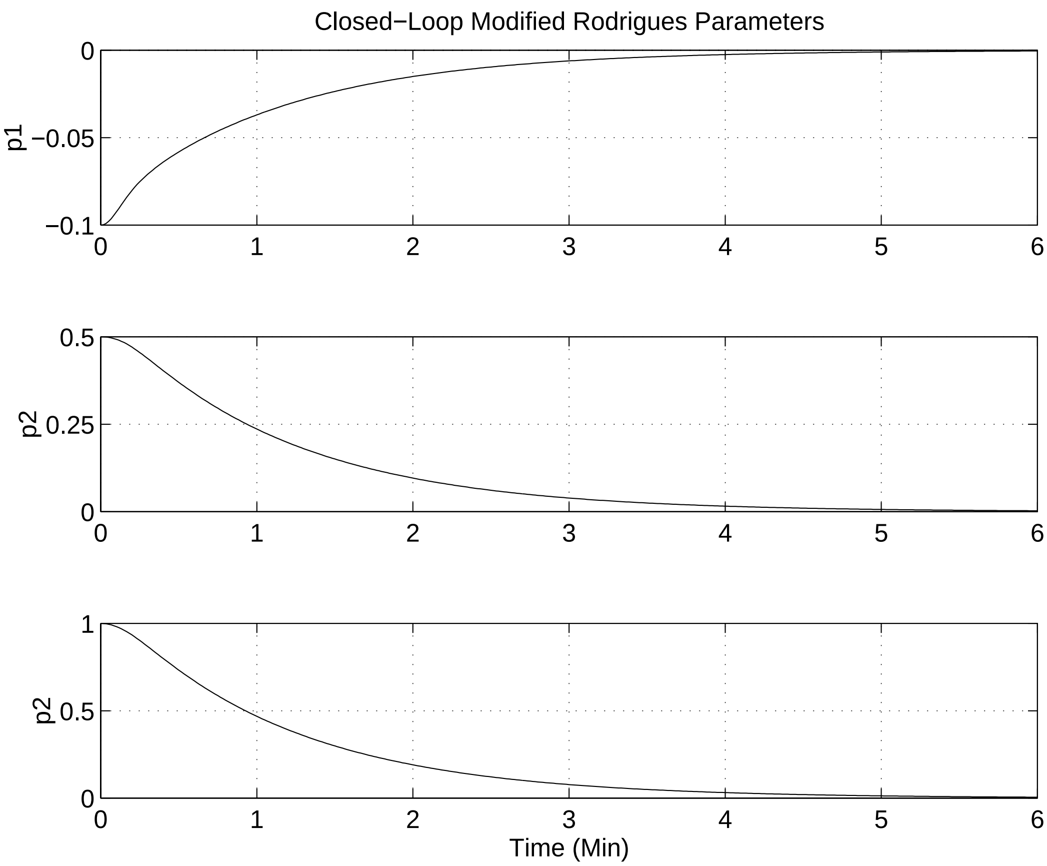
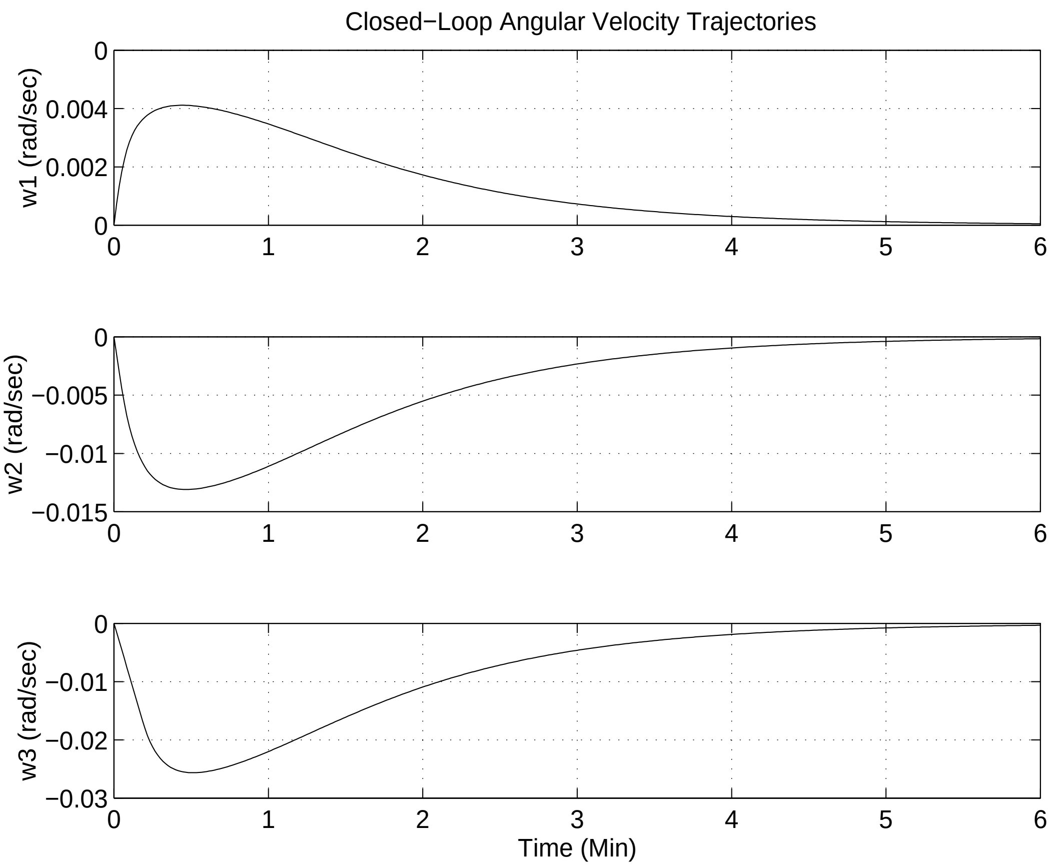

Closed−Loop Modified Rodrigues Parameters
p1
0
−0.05
−0.1
p2
0.5
0.25
0
p2
1
0.5
0
0 1 2 3 4 5 6
Time (Min)


Closed−Loop Angular Velocity Trajectories
w1 (rad/sec)
0
0.004
0.002
0
w2 (rad/sec)
0
−0.005
−0.01
−0.015
w3 (rad/sec)
0
−0.01
−0.02
−0.03
0 1 2 3 4 5 6
Time (Min)

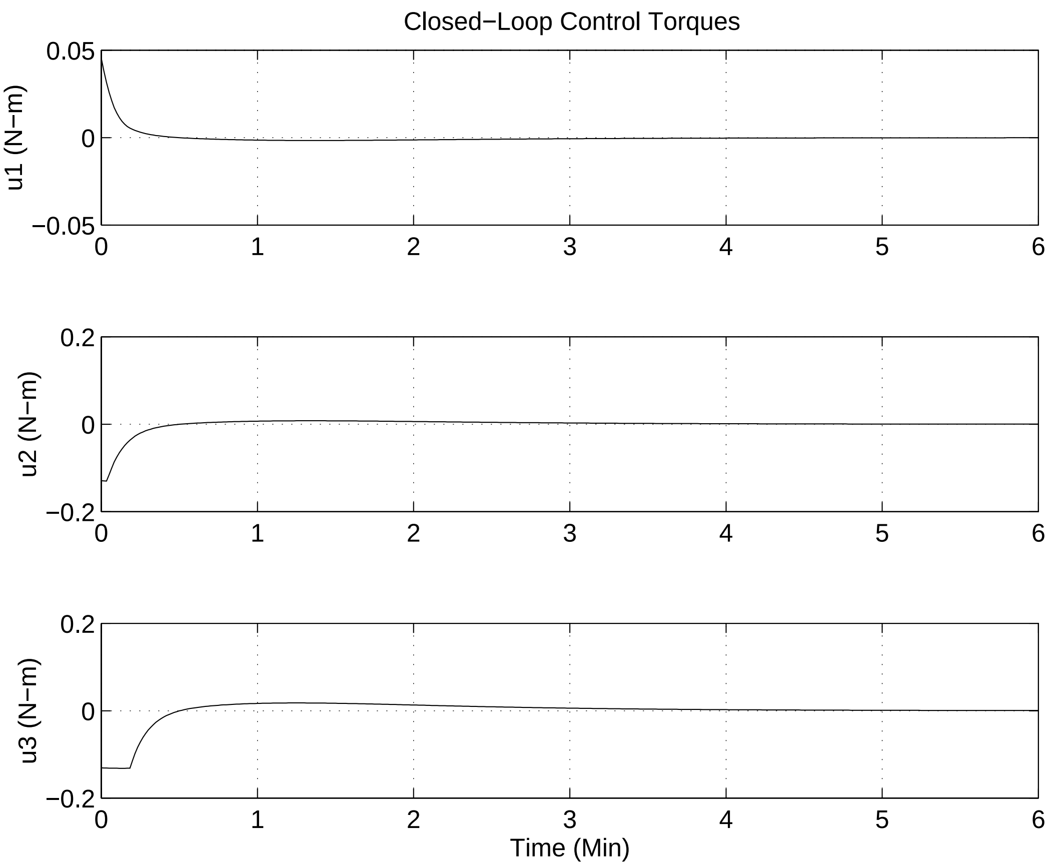

Closed−Loop Control Torques
u1 (N−m)
0.05
0
−0.05
u2 (N−m)
0.2
0
−0.2
u3 (N−m)
0.2
0
−0.2
0
1
2
3
4
5
6
Time (Min)

Figure 1 Plot of Closed-Loop Modified Rodrigues Trajectories

Figure 2 Plot of Closed-Loop Angular Velocity Trajectories

Figure 3 Plot of Applied Control Torques