A New Algorithm for Attitude Determination Using Global Positioning System Signals

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Abstract

In this paper, a novel technique for finding a point-by-point (deterministic) attitude solution of a vehicle using Global Positioning System phase difference measurements is presented. This technique transforms a general cost function into a more numerically efficient form by determining three-dimensional vectors in either the body or reference coordinate system. Covariance relationships for the new algorithm, as well as methods which minimize the general cost function, are also derived. The equivalence of the general cost function and transformed cost function is shown for the case of orthogonal baselines or sightlines. Simulation results are shown which demonstrate the usefulness of the new algorithm and covariance expressions.

Introduction

The utilization of phase difference measurements from Global Positioning System (GPS) receivers provides a novel approach for three-axis attitude determination and/or estimation. These measurements have been successfully used to determine the attitude of both aircraft[1] and spacecraft.[2,3] Recently, much attention has been placed on spacecraft-based applications. One of the first space-based GPS experiments for attitude determination was flown on the RADCAL (RADar CALibration) spacecraft.[4] To obtain maximum GPS visibility, and to reduce signal interference due to multipath reflection, GPS patch antennas were placed on the top surface of the spacecraft bus. Although the antenna baselines were short for attitude determination, accuracies between 0.5 to 1.0 degrees (root-mean-square) were achieved.

In this paper, the problem of finding the attitude from GPS phase difference measurements using deterministic approaches is addressed. Error sources, such as integer sign ambiguity,[5] are

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not investigated. These errors are assumed to be accounted for before the attitude determination problem is solved. The most common GPS attitude determination scheme minimizes a cost function constituting the sum weighted two-norm residuals between the estimated and determined phase difference quantities. However, as of this writing, the optimal attitude solution which minimizes this general cost function can only be found using iterative techniques, such as gradient search methods. A suboptimal solution involves transforming the general cost function into a form which can be minimized without iterative intense methods. One such technique, developed by Cohen,[1] transforms the general cost function into a form identical to Wahba's problem.[6] Therefore, fast algorithms such as QUEST[7] and FOAM[8] can then be used to determine the attitude. Cohen showed that the solution based on Wahba's problem is almost an order of magnitude faster than a conventional nonlinear least-squares algorithm.

Cohen's approach involves a two step process. The first step involves finding a weighting matrix, using a Singular Value Decomposition (SVD), which transforms the baseline configuration to an equivalent orthonormal basis. At least three non-collinear baselines must exist to perform this transformation. If this is not the case, the transformation can still be accomplished as long as three non-collinear sightlines exist. However, a SVD must be performed for each new sightline, which can be computationally expensive, whereas the baseline transformation has to be done only once. The second step involves finding the attitude using the fast algorithms such as QUEST or FOAM. Since the weighting matrix transforms the baseline configuration to an equivalent orthonormal basis, suboptimal attitude solutions may arise if the baseline configuration does not already form an orthonormal basis.[1] An example of this scenario is when three baselines are coplanar. In order to determine the optimal attitude, iterative techniques which minimize the general cost function must be used. The method presented in this paper also is suboptimal for the case where the baseline (or sightline) configuration does not form an orthonormal basis. However, it does not require a SVD of a 3×3 matrix to perform the orthonormal transformation.

Bar-Itzhack et. al.[9] show another analytical conversion of the basic GPS scalar difference measurements into unit vectors to be used in Wahba's problem. This is accomplished by expressing the angle determined by one of the baselines, which describes a cone around the baseline vector, and likewise for the second baseline, into a three-dimensional vector resolved in a reference coordinate system. Attitude solutions are provided for baselines which constitute Cartesian and non-Cartesian coordinate systems; however, these solutions shown in Ref. [9] involve only two baseline vectors. This paper generalizes these results to multiple baseline vectors. Also, covariance relations are shown for the new approach, as well as for techniques which minimize the general cost function directly. This allows users to quantify any additional errors produced by transforming the general cost function into Wahba's form.

The organization of this paper proceeds as follows. First, the concept of the GPS phase difference measurement is introduced. Then, the general cost function used for GPS-based attitude determination is reviewed. Next, Cohen's method for transforming the general cost function into Wahba's problem is shown. Also, system observability using two baselines is discussed. Then, a general technique for transforming the general cost function is developed. Also, the equivalence of the general and transformed (Wahba) cost functions for orthogonal baselines and/or sightlines is discussed. Next, a covariance analysis is performed on the new algorithm, and on algorithms which minimize the general cost function directly. Finally, results are shown for a simulated vehicle with near-orthogonal baselines, non-orthogonal baselines, and baselines which are nearly collinear.

Background

In this section, a brief background of the GPS phase difference measurement is shown. The GPS constellation of spacecraft was developed for accurate navigation information of land-based, air, and spacecraft user systems. Spacecraft applications initially involved obtaining accurate orbit information and accurate time-tagging of spacecraft operations. However, attitude determination of vehicles, such as spacecraft or aircraft, has gained much attention. The main measurement used for attitude determination is the phase difference of the GPS signal received from two antennas separated by a baseline. The principle of the wavefront angle and wavelength, which are used to develop a phase difference, is illustrated in Figure 1.

Figure 1 GPS Wavelength and Wavefront Angle

The phase difference measurement is obtained by

$$b_l \cos\theta = \lambda\left(n + \Delta\phi^0 / 2\pi\right) \tag{1}$$

where b_l is the baseline length, θ is the angle between the baseline and the line of sight to the GPS spacecraft, n is the number of integer wavelengths between two receivers, $\Delta\phi^0$ is the actual phase difference measurement, and λ is the wavelength of the GPS signal. The two GPS frequency carriers are L1 at 1575.42 MHz and L2 at 1227.6 MHz. Then, assuming no integer offset, we define a normalized phase difference measurement $\Delta\phi$ by

$$\Delta\phi \equiv \frac{\lambda\,\Delta\phi^0}{2\pi\, b_l} = \underline{b}^T A\, \underline{s} \tag{2}$$

where $\underline{s} \in R^3$ is the normalized line of sight vector to the GPS spacecraft in an inertial frame, $\underline{b} \in R^3$ is the normalized baseline vector, which is the relative position vector from one antenna to another, and $A \in SO(3)$ is the attitude matrix, which is a Lie group of orthogonal matrices with determinant 1 (i.e., $A^T A = I$ and $\det A = 1$).

Cohen's Method

In this section, Cohen's method[1] for determining the attitude of a vehicle using Equation (2) is reviewed. The general cost function to be minimized is given by

$$J(A) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}\left(\Delta\phi_{ij} - \underline{b}_i^T A\, \underline{s}_j\right)^2 \tag{3}$$

where m represents the number of baselines, and n represents the number of observed GPS spacecraft. The parameters, w_{ij}, serve to weight each individual phase measurement. The phase measurement can contain noise, which is modeled by

$$\Delta\phi_{ij} = \Delta\phi_{ij}^{\text{true}} + v_{ij} \tag{4}$$

where v_{ij} is a zero-mean stationary Gaussian process with covariance given by σ_{ij}^2. The maximum likelihood estimate for w_{ij} is given by $1/\sigma_{ij}^2$. If the weights w_{ij} factor into a satellite-dependent and a baseline-dependent factor, i.e. $w_{ij} = w_{bi} w_{sj}$, then the cost function in Equation (3) may be re-written as[1]

$$J(A) = \left\| W_B^{1/2} \left(\Delta\Phi - B^T A S \right) W_S^{1/2} \right\|_F^2 \tag{5}$$

where $\| \ \|_F^2$ denotes the Frobenius norm, and

$$\Delta\Phi = \begin{bmatrix} \Delta\phi_{11} & \Delta\phi_{12} & \cdots & \Delta\phi_{1n} \\ \Delta\phi_{21} & & & \\ & & \ddots & \vdots \\ \Delta\phi_{m1} & & \cdots & \Delta\phi_{mn} \end{bmatrix} \tag{6a}$$

$$B = \left[\underline{b}_1 \mid \underline{b}_2 \mid \cdots \mid \underline{b}_m \right] \tag{6b}$$

$$S = \left[\underline{s}_1 \mid \underline{s}_2 \mid \cdots \mid \underline{s}_n \right] \tag{6c}$$

The weighting matrices W_B and W_S are applicable to the rows (baselines) and columns (spacecraft) of $\Delta\Phi$, respectively.

If the quaternion[10] representation is used for the attitude matrix, then Equation (5) leads to a quartic dependence in the quaternions. In Wahba's problem, this dependence cancels out of the cost function. In order to cancel this dependence in Equation (5), Cohen chooses the following weighting matrix for W_B

$$W_B = V_B \Sigma_B^{-2} V_B^T \tag{7}$$

where V_B and Σ_B are given from a SVD of B, i.e. $B = U_B \Sigma_B V_B^T$. From Equation (7), the matrix B must be full rank, which means that at least three non-collinear baselines must be used. However, if this is not true a solution can still be found as long as three non-collinear sightlines exist. This can be accomplished by performing a SVD of S, and choosing W_S as in the same form in Equation (7). However, a SVD must be performed for each sightline. This is more computationally expensive than using Equation (7), which may be done once for constant baselines. It is also not obvious that Equation (7) is consistent with Equation (3). Substituting

Equation (7) into the general cost function in Equation (5) leads to Wahba's problem, which maximizes

$$J^{'}(A) = \text{trace}\left(A\,S\,W_S \Delta\Phi^T W_B B^T\right) \equiv \text{trace}\left(A\,G^T\right) \tag{8}$$

However, by constraining W_B or W_S, the solution using the transformed cost function in Equation (8) is suboptimal for non-orthogonal baselines or sightlines. The concept of the suboptimal solution will be discussed in detail later.

Attitude Determination from Vectorized Measurements

In this section, a new method for attitude determination from GPS phase measurements is developed. This new method extends the method shown in Ref. [9], which converts the phase measurements into vector measurements. The general method for the vectorized measurements is based on an algorithm given by Shuster.[11] Also, a covariance analysis is performed for the new method, and for methods which minimize the general cost function in Equation (3) directly.

The vectorized measurement problem involves determining the sightline vector in the body frame, denoted by $\underline{\tilde{s}}_j \equiv A\underline{s}_j$, or the baseline in an inertial frame, denoted by $\underline{\bar{b}}_i \equiv A^T\underline{b}_i$. For the sightline case, the following cost function is minimized

$$J_j\left(\underline{\tilde{s}}_j\right) = \frac{1}{2}\sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij} - \underline{b}_i^T\,\underline{\tilde{s}}_j\right)^2 \quad \text{for } j = 1,2,\ldots,n \tag{9}$$

The minimization of Equation (9) is straightforward and leads to[11]

$$\underline{\tilde{s}}_j = M_j^{-1}\,\underline{y}_j \tag{10}$$

where

$$M_j = \sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\underline{b}_i\underline{b}_i^T \quad \text{for } j = 1,2,\ldots,n \tag{11a}$$

$$\underline{y}_j = \sum_{i=1}^{m} \frac{1}{\sigma_{ij}^2} \Delta\phi_{ij} \underline{b}_i \quad \text{for } j = 1, 2, \ldots, n \tag{11b}$$

The error covariance of $\underline{\tilde{s}}_j$ is given by

$$P_j = M_j^{-1} \tag{12}$$

If the sightline in the body is required to be normalized, then the cost function in Equation (9) must be minimized subject to the constraint $\underline{\tilde{s}}^T \underline{\tilde{s}} = 1$. However, Shuster[11] showed that the error introduced by ignoring this constraint is on the order of $m^{-1}\text{trace}\left[\left|\underline{\tilde{s}}_j\right|^{-2}\left(I - \underline{\tilde{s}}_j \underline{\tilde{s}}_j^T\right) P_j \left(I - \underline{\tilde{s}}_j \underline{\tilde{s}}_j^T\right)\right]$, which is usually negligible. The solution of Wahba's problem as shown below will determine the optimal attitude (in the least-squares sense) which results in a normalized body vector. Therefore, the normalization constraint may be ignored. Also, the normalized error covariance is singular, as shown in Ref. [11]. This singularity is avoided by using the covariance given by Equation (12). For a discussion of singularity issues for measurement covariances see Ref. [12].

From Equation (10) it is seen that at least three non-collinear baselines are required to determine the sightlines in the body frame. This is analogous to the problem posed by Cohen.[1] However, if only two non-collinear baselines exist, a solution is again possible as long as three non-collinear sightlines exist. This approach determines the baselines in the inertial frame, using the following cost function

$$J_i(\underline{\bar{b}}_i) = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{\sigma_{ij}^2} \left(\Delta\phi_{ij} - \underline{\bar{b}}_i^T \underline{s}_j\right)^2 \quad \text{for } i = 1, 2, \ldots, m \tag{13}$$

The minimization of Equation (13) is again straightforward and leads to

$$\underline{\bar{b}}_i = N_i^{-1} \underline{z}_i \tag{14}$$

where

$$N_i = \sum_{j=1}^{n} \frac{1}{\sigma_{ij}^2} \underline{s}_j \underline{s}_j^T \quad \text{for } i = 1, 2, \ldots, m \tag{15a}$$

$$\underline{z}_i = \sum_{j=1}^{n} \frac{1}{\sigma_{ij}^2} \Delta\phi_{ij} \underline{s}_j \quad \text{for } i = 1, 2, \ldots, m \tag{15b}$$

The error covariance of $\underline{\bar{b}}_i$ is given by

$$Q_i = N_i^{-1} \tag{16}$$

The case with two non-collinear baselines and two non-collinear sightlines can also be solved for either the baseline inertial case or sightline body case. Solving for the latter case yields

$$\underline{\tilde{s}}_j = a_{1j} \underline{b}_1 + a_{2j} \underline{b}_2 + a_{3j} \left(\underline{b}_1 \times \underline{b}_2 \right) \quad \text{for } j = 1, 2 \tag{17}$$

where

$$a_{1j} = \left| \underline{b}_1 \times \underline{b}_2 \right|^{-2} \left[\Delta\phi_{1j} - \Delta\phi_{2j} \left(\underline{b}_1 \cdot \underline{b}_2 \right) \right] \quad \text{for } j = 1, 2 \tag{18a}$$

$$a_{2j} = \left| \underline{b}_1 \times \underline{b}_2 \right|^{-2} \left[\Delta\phi_{2j} - \Delta\phi_{1j} \left(\underline{b}_1 \cdot \underline{b}_2 \right) \right] \quad \text{for } j = 1, 2 \tag{18b}$$

$$a_{3j} = \pm \left| \underline{b}_1 \times \underline{b}_2 \right|^{-2} \left\{ f_j \left| \underline{b}_1 \times \underline{b}_2 \right|^2 - \Delta\phi_{1j}^2 + 2\, \Delta\phi_{1j} \Delta\phi_{2j} \left(\underline{b}_1 \cdot \underline{b}_2 \right) + \Delta\phi_{2j}^2 \right\}^{1/2} \quad \text{for } j = 1, 2 \tag{18c}$$

where $f_j = \left| \underline{\tilde{s}}_j \right|^2$. Equation (18c) involves knowledge of $\left| \underline{\tilde{s}}_j \right|^2$. However, this quantity can be assumed to be 1 with reasonable accuracy. Also, from Equation (18c), there are two possible solutions for the body sightlines. However, this sign ambiguity can usually be resolved from the geometry of vehicle to the GPS spacecraft. The error covariance is given by[11]

$$P_j = T\, L_j\, T^T \tag{19}$$

where

$$T = \left[\underline{b}_1 \;\vdots\; \underline{b}_2 \;\vdots\; \underline{b}_1 \times \underline{b}_2 \right] \tag{20a}$$

$$L_j = \left[\begin{array}{c|c} D_j & \underline{l}_j \\ \hline \underline{l}_j^T & d_j \end{array}\right] \tag{20b}$$

and

$$D_j = U\, P_{\phi_j}\, U \tag{21a}$$

$$U = \left|\underline{b}_1 \times \underline{b}_2\right|^{-2} \begin{bmatrix} 1 & -\underline{b}_1 \cdot \underline{b}_2 \\ -\underline{b}_1 \cdot \underline{b}_2 & 1 \end{bmatrix} \tag{21b}$$

$$P_{\phi_j} = \begin{bmatrix} \sigma_{1j}^2 & 0 \\ 0 & \sigma_{2j}^2 \end{bmatrix} \tag{21c}$$

$$\underline{l}_j = \mp\left|\underline{b}_1 \times \underline{b}_2\right|^{-1} \left[1 - \underline{\psi}_j^T D_j \underline{\psi}_j\right]^{-1/2} U\, P_{\phi_j} U\, \underline{\psi}_j \tag{21d}$$

$$d_j = \left|\underline{b}_1 \times \underline{b}_2\right|^{-2} \left[1 - \underline{\psi}_j^T D_j \underline{\psi}_j\right]^{-1} \underline{\psi}_j^T U\, P_{\phi_j} U\, \underline{\psi}_j \tag{21e}$$

$$\underline{\psi}_j \equiv \begin{bmatrix} \Delta\phi_{1j} \\ \Delta\phi_{2j} \end{bmatrix} \tag{21f}$$

The covariance in Equation (19) is singular. However, this does not affect the determination of the attitude error covariance, as will be shown. Also, the method can be trivially modified to determine the baselines in inertial space.

Attitude Determination

The attitude determination problem using body sightlines is very similar to that using inertial baselines, so we may consider only the former case. The attitude is determined by using the following cost function

$$J(A) = \frac{1}{2}\sum_{j=1}^{n}\left(\tilde{\underline{s}}_j - A\,\underline{s}_j\right)^T M_j\left(\tilde{\underline{s}}_j - A\,\underline{s}_j\right) \tag{22}$$

This cost function is not identical to Wahba's problem since the quartic dependence in the quaternion does not cancel, unless the baselines form an orthonormal basis so that M_j is given by a scalar times the identity matrix. The cost function in Equation (22) is in fact equivalent to the general cost function in Equation (3). This is shown by substituting Equation (10) and (11) into (22) and expanding terms, giving

$$J(A) = \frac{1}{2}\sum_{j=1}^{n}\left(\underline{y}_j^T M_j^{-1}\underline{y}_j - 2\,\underline{y}_j^T A\underline{s}_j + \underline{s}_j^T A^T M_j A\underline{s}_j\right) \tag{23}$$

Expanding Equation (23) now yields

$$J(A) = \frac{1}{2}\sum_{j=1}^{n}\left(\underline{y}_j^T M_j^{-1}\underline{y}_j - \sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}^2\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij} - \underline{b}_i^T A\underline{s}_j\right)^2 \tag{24}$$

Since the first term in Equation (24) is independent of attitude, it is clear that this cost function is equivalent to the general cost function in Equation (3). In order to reduce the cost function in Equation (22) into a form corresponding to Wahba's problem the condition that M_j is given by a scalar times the identity matrix must be valid. Therefore, if the baselines do not form an orthonormal basis, then the attitude solution is suboptimal.

Attitude Covariance

Wahba posed the three-axis determination problem in terms of finding the proper orthogonal attitude matrix that minimizes the least-squares cost function, given by

$$J(A) = \frac{1}{2}\sum_{j=1}^{n} a_j\left|\underline{\tilde{s}}_j - A\underline{s}_j\right|^2 \tag{25}$$

Several efficient algorithms have been developed to solve this problem (e.g., QUEST[7] and FOAM[8]). Another solution for the attitude matrix is given by performing a SVD of the following matrix

$$F = \sum_{j=1}^{n} a_j\,\underline{\tilde{s}}_j\,\underline{s}_j^T = U\,\Sigma V^T \tag{26}$$

The optimal solution for the attitude matrix is given by[13]

$$A_{\text{opt}} = U_+ V_+^T \tag{27}$$

where

$$U_+ = U\left[\text{diag}(1, 1, \det U)\right] \tag{28a}$$

$$V_+ = V\left[\text{diag}(1, 1, \det V)\right] \tag{28b}$$

The covariance of the estimation error angle vector in the body frame is given by[13]

$$E\left\{\delta\underline{\alpha}\,\delta\underline{\alpha}^T\right\} = P_{\text{body}} = \left(I - F\,A_{opt}^T\right)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} a_i a_j \left[\underline{\tilde{s}}_j \times\right] E\left\{\underline{e}_i\,\underline{e}_j^T\right\} \left[\underline{\tilde{s}}_j \times\right]^T \left(I - F\,A_{opt}^T\right)^{-1} \tag{29}$$

where $\left[\underline{\tilde{s}}_j \times\right]$ represents the cross product matrix (see Ref. [13]), $\delta\underline{\alpha}$ is the small error angle, and

$$\underline{e}_k \equiv A_{opt}\delta\underline{s}_k - \delta\underline{\tilde{s}}_k, \quad \text{for any } k \tag{30}$$

The terms $\delta\underline{s}$ and $\delta\underline{\tilde{s}}$ represent variations in the inertial and body sightlines, respectively. The expectation in Equation (29) can be written as

$$E\left\{\underline{e}_i\,\underline{e}_j^T\right\} = A_{opt} E\left\{\delta\underline{s}_i \delta\underline{s}_j^T\right\} A_{opt}^T + E\left\{\delta\underline{\tilde{s}}_i \delta\underline{\tilde{s}}_j^T\right\} \tag{31}$$

Assuming that the only errors are in the effective phase measurements reduces Equation (29) to

$$P_{\text{body}} = \left(I - F\,A_{opt}^T\right)^{-1} \sum_{j=1}^{n} a_j^2 \left[\underline{\tilde{s}}_j \times\right] P_j \left[\underline{\tilde{s}}_j \times\right]^T \left(I - F\,A_{opt}^T\right)^{-1} \tag{32}$$

Now using the approximation of

$$\underline{\tilde{s}}_j \approx A_{\text{opt}}\,\underline{s}_j \tag{33}$$

yields

$$I - F\,A_{opt}^T \approx \sum_{j=1}^{n} a_j \left(I - \underline{\tilde{s}}_j \underline{\tilde{s}}_j^T\right) = \sum_{j=1}^{n} a_j \left[\underline{\tilde{s}}_j \times\right]\left[\underline{\tilde{s}}_j \times\right]^T \equiv X \tag{34}$$

and thus the error angle covariance is given by

$$P_{\text{body}} \approx X^{-1}\left\{\sum_{j=1}^{n} a_j^2 \left[\underline{\tilde{s}}_j \times\right] P_j \left[\underline{\tilde{s}}_j \times\right]^T\right\} X^{-1} \tag{35}$$

Note that if the covariances P_j are multiples of the identity, $P_j = \sigma_j^2 I$, and then setting $a_j = \sigma_j^{-2}$ would yield

$$P_{\text{body}} \approx \left[\sum_{j=1}^{n} \sigma_j^{-2} \left[\underline{\tilde{s}}_j \times\right]\left[\underline{\tilde{s}}_j \times\right]^T\right]^{-1} = X^{-1} \tag{36}$$

Therefore, in this case the covariance in Equation (36) would be identical to the covariance given by QUEST.[7] The best suboptimal weighting factor a_j in Equation (35) can be found by minimizing the trace of P_{body}. However, this is extremely complex. If Equation (36) is still a good approximation, then a_j can be chosen to minimize some matrix norm of the following

$$J(a_j) = \left\| a_j P_j - I \right\| \tag{37}$$

An alternative to Equation (37) is to minimize the following cost function for some matrix norm

$$J(a_j) = \left\| a_j I - P_j^{-1} \right\| \tag{38}$$

For example, minimizing Equation (38) with a Frobenius norm results in

$$a_j = \frac{1}{3}\text{trace}\left(P_j^{-1}\right) = \frac{1}{3}\text{trace}\left(M_j\right) \tag{39}$$

Once a proper weight is determined, then Wahba's problem in Equation (25) can be solved. The covariance of the attitude errors is given by Equation (35).

Transforming the general cost function in Equation (3) results in a suboptimal solution. In order to quantify the errors introduced by the suboptimal solution, the error attitude covariance for the general cost function is derived. This is accomplished by using results from maximum likelihood estimation.[14] The Fisher information matrix for a parameter vector $\underline{x}$ is given by

$$F_{xx} = E\left\{\frac{\partial}{\partial \underline{x} \partial \underline{x}^T} J(\underline{x})\right\}_{\underline{x}_{\text{true}}} \tag{40}$$

where $J(\underline{x})$ is the negative log likelihood function, which is the loss function in this case. If the measurements are Gaussian and linear in the parameter vector, then the error covariance is given by

$$P_{xx} = F_{xx}^{-1} \tag{41}$$

Since the cost function in Equation (22) is equivalent to the full cost function in Equation (3), Equation (22) can be used to determine the covariance of the optimal solution. First, the attitude matrix is approximated by

$$A = e^{-[\delta\underline{\alpha}\times]} A_{\text{true}} \approx \left(I - [\delta\underline{\alpha}\times] + \frac{1}{2}[\delta\underline{\alpha}\times]^2\right) A_{\text{true}} \tag{42}$$

Equations (42) and (33) are next substituted into Equations (22) and (40) to determine the Fisher information matrix. First-order terms vanish in the partials, and third-order terms become zero since $E\{\delta\underline{\alpha}\} = \underline{0}$. Also, assuming that the quartic terms are negligible leads to the following simple form for the optimal covariance

$$P_{\text{opt}} \approx \left[\sum_{j=1}^{n} [\tilde{\underline{s}}_j \times] P_j^{-1} [\tilde{\underline{s}}_j \times]^T\right]^{-1} \tag{43}$$

Note that the optimal covariance in Equation (43) reduces to the covariance in Equation (36) if the condition $P_j = \sigma_j^2 I$ is true. The errors introduced when using a suboptimal solution can now be compared to the expected performance of minimizing the general cost function in Equation (3). Also, for the case of two baselines and two sightlines, the optimal covariance can be derived by using the cost function in Equation (3) in the Fisher information matrix, which leads to

$$P_{\text{opt}} \approx \left[\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{\sigma_{ij}^2} \left(\tilde{\underline{s}}_j \times \underline{b}_i\right)\left(\tilde{\underline{s}}_j \times \underline{b}_i\right)^T\right]^{-1} \tag{44}$$

The covariance analysis can be easily extended to the case where the baselines in inertial space are determined. The body covariance for the transformed cost function in this case becomes

$$P_{\text{body}} \approx \left[\sum_{i=1}^{m} a_i \left[\underline{b}_i \times \right]^2 \right]^{-1} \sum_{i=1}^{m} a_i^2 \left[\underline{b}_i \times \right] A Q_i \, A^T \left[\underline{b}_i \times \right]^T \left[\sum_{i=1}^{m} a_i \left[\underline{b}_i \times \right]^2 \right]^{-1} \tag{45}$$

The error covariance for the optimal solution is given by

$$P_{\text{opt}} \approx \left[\sum_{i=1}^{m} \left[\underline{b}_i \times \right] A Q_i^{-1} A^T \left[\underline{b}_i \times \right]^T \right]^{-1} \tag{46}$$

Simulation Results

In this section, simulation results are shown using the new algorithm and covariance expressions. Three case are presented. The first case involves three baselines which are nearly orthogonal. The second involves three baselines which do not constitute an orthogonal set. The third case involves three baselines, where the first two baselines are far from constituting an orthogonal set (i.e., nearly collinear). Although the third case would most likely never be used in a practical application, it provides a radical test comparison between the optimal and suboptimal solutions. It is assumed that the vehicle is always in the view of two GPS spacecraft with constant and normalized sightlines given by

$$\underline{s}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \qquad \underline{s}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 1\end{bmatrix}^T \tag{47}$$

The three normalized baseline cases are given by

Case 1

$$\underline{b}_1 = \frac{1}{\sqrt{1.09}}\begin{bmatrix}1 & 0.3 & 0\end{bmatrix}^T \qquad \underline{b}_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \qquad \underline{b}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \tag{48a}$$

Case 2

$$\underline{b}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T \qquad \underline{b}_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \qquad \underline{b}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \tag{48b}$$

Case 3

$$\underline{b}_1 = \frac{1}{\sqrt{1.02}}\begin{bmatrix}0.1 & 1 & 0.1\end{bmatrix}^T \qquad \underline{b}_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \qquad \underline{b}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \tag{48c}$$

The noise for each phase difference measurement is assumed to have a normalized standard deviation of $\sigma = 0.001$ (corresponding to an attitude error of about 0.5 degrees). Also, the attitude of the vehicle is assumed to be Earth-pointing with a rotation of 236 deg/hr about the vehicle's y-axis (negative orbit-normal), while holding the remaining axis rotations to zero. The spacecraft z-axis is defined to be pointed nadir, and the x-axis completes the triad.

If the baselines do not constitute an orthogonal set, the solution of the transformed cost function to Wahba's form is suboptimal. However, the covariance analysis shown in this paper can be used to assess the errors introduced from the transformation. All simulation results presented in the figures use a_j given by Equation (39). Figure 2 shows the attitude errors and three-sigma bounds by solving Wahba's form for Case 1. This shows the excellent agreement between theory and simulated measurement processes. In order to quantify the error introduced by using the suboptimal solution, the following error factor is used

$$f = \frac{1}{m_{\text{tot}}} \sum_{k=1}^{m_{\text{tot}}} \frac{\text{trace}\left\{\text{diag}\left[P_{\text{body}}^{1/2}\right]\right\}}{\text{trace}\left\{\text{diag}\left[P_{\text{opt}}^{1/2}\right]\right\}} \tag{49}$$

where m_{tot} represents the total number of measurements used in the simulation. A plot of the error factor at each time is shown in Figure 3. Equation (49) represents the average of the curve shown in Figure 3. Clearly, the suboptimal solution is adequate, with a maximum error of about 3%. A plot of the standard deviation errors for the suboptimal and optimal solutions for Case 2 is shown in Figure 4. The optimal standard deviation error is always lower than the suboptimal solution. A plot of the error factor at each time is shown in Figure 5. For this case, the suboptimal solution can produce large errors, with a maximum error of about 35%. This is due to the non-orthogonal baselines, and due to the attitude of the vehicle. Results using various values of a_j for all three case are shown in Table 1 (the subscripts max and min denote eigenvalues). Clearly, various choices for the weighting factors a_j do not affect system performance. Also, Case 3 where the baselines are nearly collinear results in a substantial degradation in performance when using the suboptimal solution as compared to the optimal solution. Therefore, the covariance analysis is extremely helpful for determining whether or not the suboptimal and/or the optimal solution meets required performance specifications.

Figure 2 Attitude Errors and Bounds for Case 1

Figure 3 Error Factor for Case 1

Figure 4 Standard Deviation Comparison for Case 2

Figure 5 Error Factor for Case 2

Table 1 Weighting Factor Performance Comparisons

	Case 1, f	Case 2, f	Case 3, f
$a_j = \dfrac{1}{(P_j)_{\max}}$	1.014	1.151	58.13
$a_j = \dfrac{2}{(P_j)_{\max} + (P_j)_{\min}}$	1.014	1.151	58.16
$a_j = \dfrac{(P_j^{-1})_{\max} + (P_j^{-1})_{\min}}{2}$	1.014	1.151	58.29
$a_j = \dfrac{\text{trace}(P_j)}{\text{trace}(P_j^{-1})}$	1.014	1.151	58.43
$a_j = \dfrac{1}{2}\text{trace}(P_j^{-1})$	1.014	1.150	58.13

Conclusions

The problem of determining the attitude of a vehicle using GPS phase measurements was addressed in this paper. A general method which transforms the general GPS cost function into a Wahba cost function was presented. Covariance equations for both the new method, and methods which solve the general cost function were developed. It was shown that the transformation produces suboptimal attitude solutions for non-orthogonal baselines and sightlines. The equivalence of both covariance equations for orthogonal baselines and/or sightlines was also shown. Simulation results indicate that the new method is adequate for nearly orthogonal baselines or sightlines, but can produce large errors for nearly collinear baselines or sightlines, as compared to methods which minimize the general cost function directly. This paper provides a means of accessing various performance criteria, such as computational efficiency versus attitude accuracy, for the particular application.

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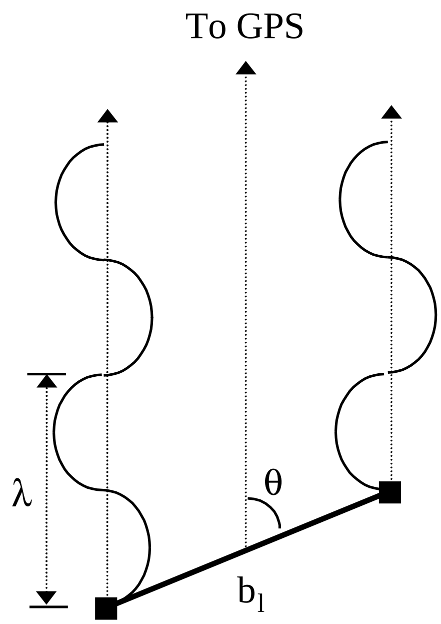
Figure 1 GPS Wavelength and Wavefront Angle

Figure 2 Attitude Errors and Bounds for Case 1

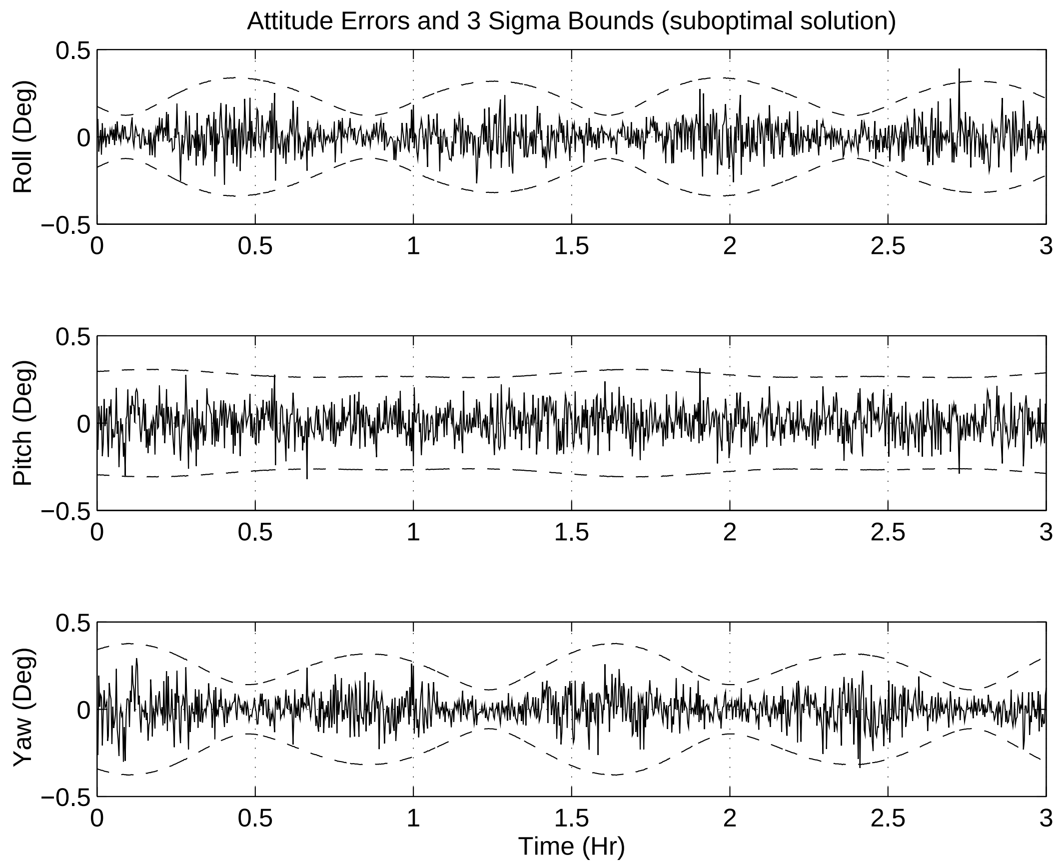
Figure 3 Error Factor for Case 1

Figure 4 Standard Deviation Comparison for Case 2

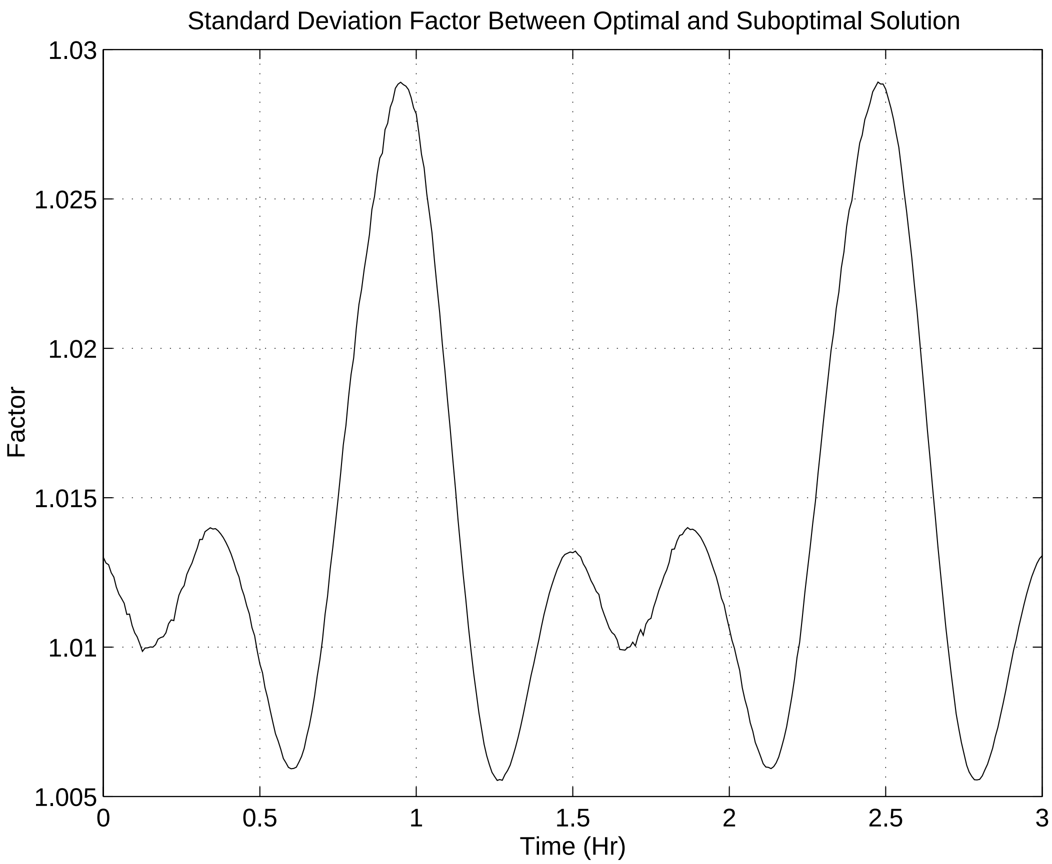
Figure 5 Error Factor for Case 2

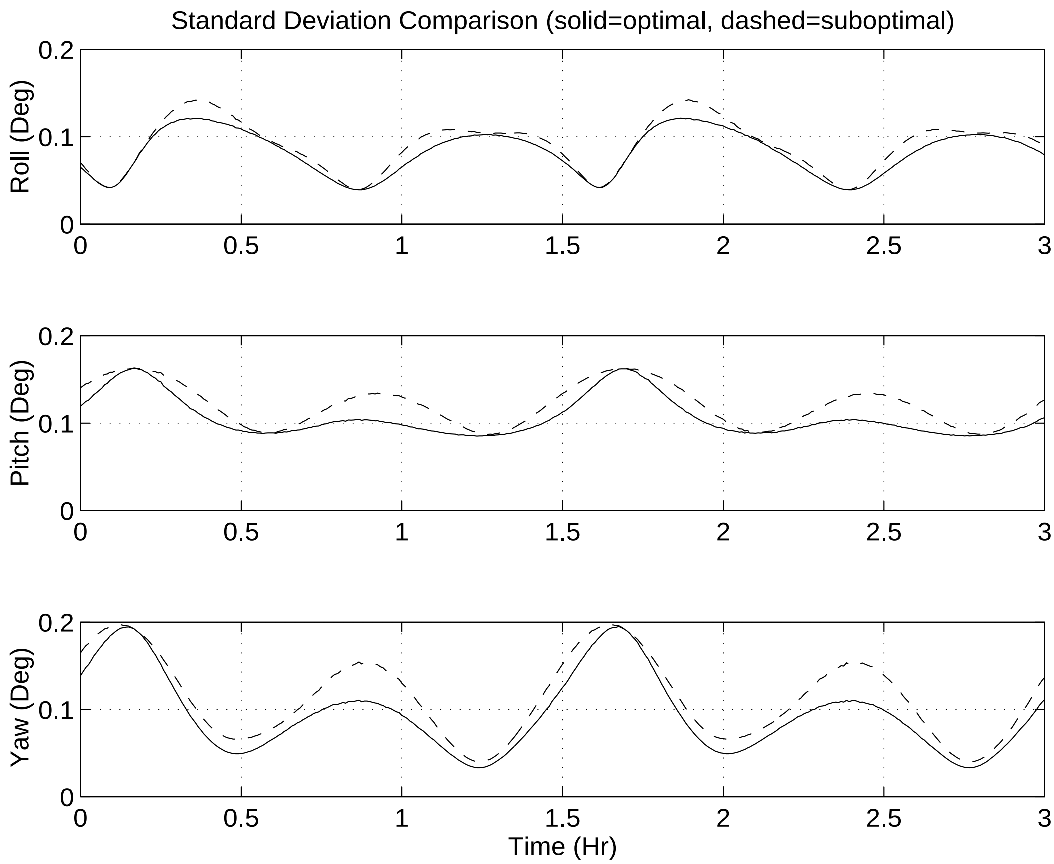


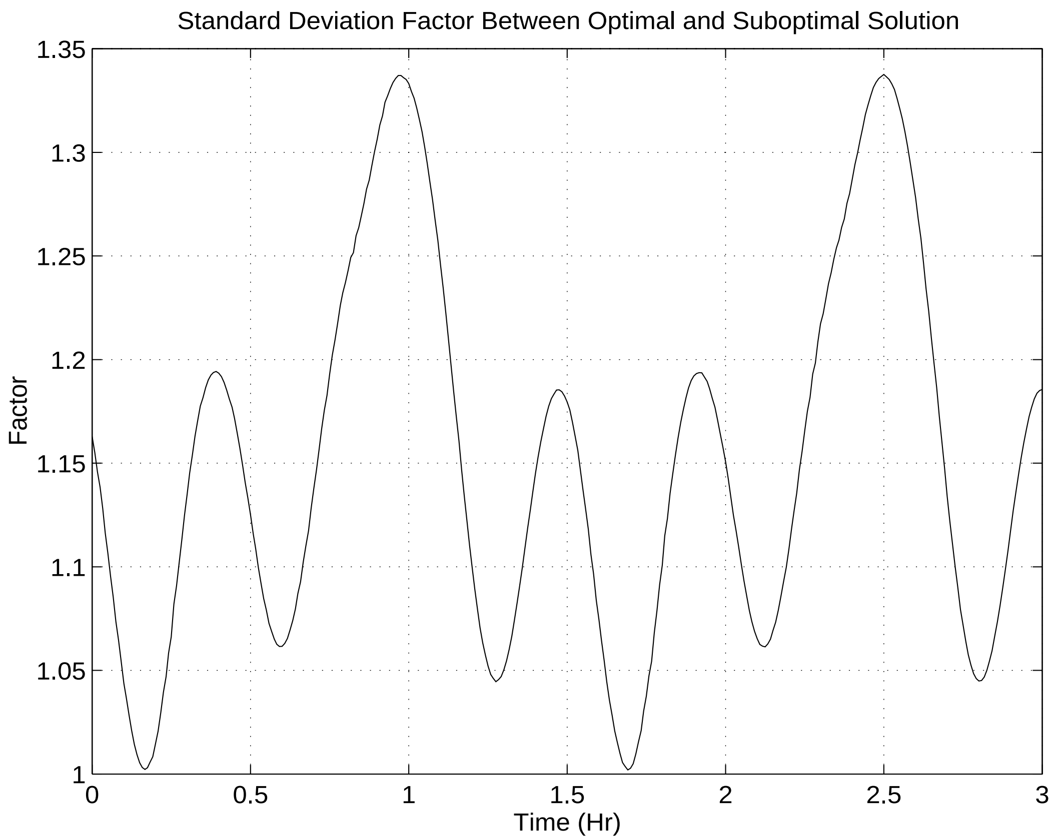
To GPS
λ
θ
b_l



Attitude Errors and 3 Sigma Bounds (suboptimal solution)
Roll (Deg)
Pitch (Deg)
Yaw (Deg)
0.5
0
−0.5
0
0.5
1
1.5
2
2.5
3
Time (Hr)


Standard Deviation Factor Between Optimal and Suboptimal Solution
1.03
1.025
1.02
1.015
1.01
1.005
Factor
0
0.5
1
1.5
2
2.5
3
Time (Hr)


Standard Deviation Comparison (solid=optimal, dashed=suboptimal)
Roll (Deg)
Pitch (Deg)
Yaw (Deg)
Time (Hr)
0
0.1
0.2
0
0.5
1
1.5
2
2.5
3


Standard Deviation Factor Between Optimal and Suboptimal Solution
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
Factor
0
0.5
1
1.5
2
2.5
3
Time (Hr)