APPLICATION OF VECTORIZED ATTITUDE DETERMINATION USING GLOBAL POSITIONING SYSTEM SIGNALS

John L. Crassidis
Senior Member AIAA
Assistant Professor
Department of Aerospace Engineering
Texas A&M University
College Station, TX 77843

F. Landis Markley
Fellow AIAA
Aerospace Engineer
GNC Center, Code 570
NASA-Goddard Space Flight Center
Greenbelt, MD 20771

E. Glenn Lightsey
Member AIAA
Aerospace Engineer
GNC Center, Code 570
NASA-Goddard Space Flight Center
Greenbelt, MD 20771

Abstract

In this paper, an analysis of two techniques for finding a point-by-point (deterministic) attitude solution of a vehicle using Global Positioning System phase difference measurements is preformed. These techniques transform a general loss function into a more numerically efficient form. One technique determines three-dimensional vectors in the body coordinate system, and the other in the reference coordinate system. Covariance relationships for both vectorized approaches show that they produce suboptimal estimates of the attitude unless the baseline or sightlines are proportional to an orthonormal set, in which case they produce optimal estimates. Both vectorized techniques are tested on a hardware dynamic simulator. Results from this study are useful to determine the circumstances for which vector transformation yields the more accurate results.

Introduction

The utilization of phase difference measurements from Global Positioning System (GPS) receivers provides a novel approach for three-axis attitude determination and/or estimation. These measurements have been successfully used to determine the attitude of both aircraft[1] and spacecraft.[2,3] Recently, much attention has been placed on spacecraft-based applications. One of the first space-based GPS experiments for attitude determination was flown on the RADCAL (RADar CALibration) spacecraft.[4] To obtain maximum GPS visibility, and to reduce signal interference due to multipath reflection, GPS patch antennas were placed on the top surface of the spacecraft bus. Although the antenna baselines were short for attitude determination, accuracies between 0.5 to 1.0 degrees (root-mean-square) were achieved.

In this paper, the problem of finding the attitude from GPS phase difference measurements using deterministic approaches is addressed. Error sources, such as integer ambiguity,[5] are not investigated. These errors are assumed to be accounted for before the attitude determination problem is solved. The most common GPS attitude determination scheme minimizes a loss function constituting the sum weighted two-norm residuals between the measured and determined phase difference quantities. A suboptimal solution involves transforming the general loss function into a form that can be minimized without iterative intense methods. One such technique, shown in Ref. [6], transforms the general loss function into a form identical to Wahba's problem.[7] Therefore, fast algorithms such as QUEST[8] and FOAM[9] can then be used to determine the attitude. Cohen [1] showed that the solution based on Wahba's problem is almost an order of magnitude faster than a conventional nonlinear least-squares algorithm.

The vectorized approach in Ref. [6] involves a two step process. The first step involves finding the sightline vectors in the body coordinate system. At least three non-coplanar baselines must exist to perform this transformation. If this is not the case, the transformation can still be accomplished as long as three non-coplanar sightlines exist. However, a 3×3 matrix inverse must be performed for each new sightline, which can be computationally expensive, whereas the baseline transformation has to be done only once. Also, Ref. [6] shows that body and reference transformations produce suboptimal estimates of the attitude unless the baseline or sightline vectors are proportional to an orthonormal set. The second step involves finding the attitude using the fast algorithms such as QUEST or FOAM. In order to determine the optimal attitude for the case that the baselines or sightlines are not proportional to an orthonormal set, other approaches, such as iterative techniques, which minimize the general loss function must be used.

Bar-Itzhack et. al.[10] show another analytical conversion of the basic GPS scalar difference measurements into unit vectors to be used in Wahba's problem. This is accomplished by expressing the angle determined by one of the baselines, which describes a cone around the baseline vector, and likewise for the second baseline, into a three-dimensional vector resolved in a reference coordinate system. Attitude solutions are provided for baselines which constitute Cartesian and non-Cartesian coordinate systems; however, these solutions shown in Ref. [10] involve only two baseline vectors. This paper generalizes these results to multiple baseline vectors. Also, covariance relations are shown for the new approach, as well as for techniques which minimize the general loss function directly. This allows users to quantify any additional errors produced by transforming the general loss function into Wahba's form.

Another technique in Ref. [11] uses a predictive (one time-step ahead) approach to solve for the body angular velocity components. These are then use to propagate a simple kinematics model to determine the attitude. Advantages of this approach include: (i) the algorithm is non-iterative, (ii) the algorithm works even for the case of coplanar baseline or coplanar sightline vectors, and (iii) the algorithm produces optimal estimates provided that the observation sampling is fairly frequent. Even though the algorithm requires an initial attitude estimate, it has been shown that it always converges to the correct solution. However, a point-by-point solution can be useful for initialization purposes and as an integrity check.

The organization of this paper proceeds as follows. First, the concept of the GPS phase difference measurement is introduced. Next, the quaternion attitude representation is reviewed. Then, the general loss function used for GPS-based attitude determination is shown, and an analysis is performed for the case that the baselines are proportional to an orthonormal set. Next, the vectorized transformations of the general loss function are reviewed, as well as the corresponding covariance expressions. Finally, these transformations are tested using a GPS hardware simulator.

Background

In this section, a brief background of the GPS phase difference measurement is shown. The GPS constellation of spacecraft was developed for accurate navigation information of land-based, air, and spacecraft user systems. Spacecraft applications initially involved obtaining accurate orbit information and accurate time-tagging of spacecraft operations. However, attitude determination of vehicles, such as spacecraft or aircraft, has gained much attention. The main measurement used for attitude determination is the phase difference of the GPS signal received from two antennas separated by a baseline. The principle of the wavefront angle and wavelength, which are used to develop a phase difference, is illustrated in Figure 1.

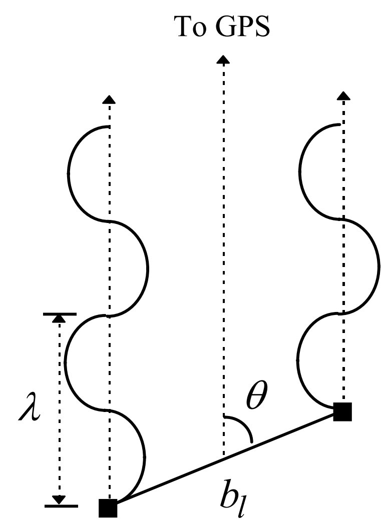

Fig. 1 GPS Wavelength and Wavefront Angle

The phase difference measurement is obtained by

$$b_l \cos\theta = \lambda(\Delta\phi - n) \tag{1}$$

where b_l is the baseline length (in cm), θ is the angle between the baseline and the line of sight to the GPS spacecraft, n is the number of integer wavelengths between two antennas, $\Delta\phi$ is the phase difference (in cycles), and λ is the wavelength (in cm) of the GPS signal. The two GPS frequency carriers are L1 at 1575.42 MHz and L2 at 1227.6 MHz. As of this writing, non-military applications generally use the L1 frequency. Then, the phase difference $\Delta\phi$ can be expressed by

$$\Delta\phi = \underline{b}^T A\,\underline{s} + n \tag{2}$$

where $\underline{s} \in R^3$ is the normalized line of sight vector to the GPS spacecraft in a reference frame, $\underline{b} \in R^3$ is the baseline vector (in wavelengths), which is the relative position vector from one receiver to another, and $A \in R^{3\times3}$ is the attitude matrix, which is an orthogonal matrix with determinant 1 (i.e., $A^T A = I_{3\times3}$). The measurement model is given by

$$\Delta\tilde{\phi}_{ij} = \underline{b}_i^T A\,\underline{s}_j + n_{ij} + v_{ij} \tag{3}$$

where $\Delta\tilde{\phi}_{ij}$ denotes the phase difference measurement for the i^{th} baseline and j^{th} sightline, and v_{ij} represents a zero-mean Gaussian measurement error with standard deviation σ_{ij} given by 0.5cm / λ = 0.026 wavelengths for typical phase noise.

Quaternion Representation

In this paper the attitude matrix is assumed to be represented by the quaternion, defined as

$$\underline{q} = \begin{bmatrix} \underline{q}_{13} \\ q_4 \end{bmatrix} \tag{4}$$

with

$$\underline{q}_{13} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underline{\hat{n}} \sin\left(\frac{\vartheta}{2}\right) \tag{4a}$$

$$q_4 = \cos\left(\frac{\vartheta}{2}\right) \tag{4b}$$

where $\underline{\hat{n}}$ is a unit vector corresponding to the axis of rotation and ϑ is the angle of rotation. The attitude matrix using the quaternion representation is given by

$$A(\underline{q}) = -\Xi^T(\underline{q})\Psi(\underline{q}) \tag{5}$$

with

$$\Xi(\underline{q}) \equiv \begin{bmatrix} q_4\, I_{3\times3} + [\underline{q}_{13}\times] \\ \hdashline -\underline{q}_{13}^T \end{bmatrix} \tag{6a}$$

$$\Psi(\underline{q}) \equiv \begin{bmatrix} -q_4\, I_{3\times3} + [\underline{q}_{13}\times] \\ \hdashline \underline{q}_{13}^T \end{bmatrix} \tag{6b}$$

where $I_{3\times3}$ is a 3×3 identity matrix., and $[\underline{q}_{13}\times]$ is a cross-product matrix because $\underline{a}\times\underline{b} = [\underline{a}\times]\underline{b}$, with

$$[\underline{a}\times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{7}$$

Because a three-degree-of-freedom system is represented by a four-dimensional vector, the quaternion components cannot by independent of each other, which is shown by the following normality constraint

$$\underline{q}^T\underline{q} = \underline{q}_{13}^T\underline{q}_{13} + q_4^2 = 1 \tag{8}$$

The matrix $\Xi(\underline{q})$ obeys the following helpful relations

$$\Xi^T(\underline{q})\Xi(\underline{q}) = (\underline{q}^T\underline{q})I_{3\times3} \tag{9a}$$

$$\Xi(\underline{q})\Xi^T(\underline{q}) = (\underline{q}^T\underline{q})I_{4\times4} - \underline{q}\,\underline{q}^T \tag{9b}$$

$$\Xi^T(\underline{q})\underline{q} = \underline{0}_{3\times1} \tag{9c}$$

$$\Xi^T(\underline{q})\underline{\rho} = -\Xi^T(\underline{\rho})\underline{q} \quad \text{for any } \underline{\rho}_{4\times1} \tag{9d}$$

The matrix $\Psi(\underline{q})$ also obeys the equivalent relations in Equation (9). Also, other useful identities are given by

$$\Xi(\underline{q})\underline{\omega} = \Omega(\underline{\omega})\underline{q} \quad \text{for any } \underline{\omega}_{3\times1} \tag{10a}$$

$$\Psi(\underline{q})\underline{\omega} = \Gamma(\underline{\omega})\underline{q} \quad \text{for any } \underline{\omega}_{3\times1} \tag{10b}$$

where

$$\Omega(\underline{\omega}) \equiv \left[\begin{array}{c:c} -[\underline{\omega}\times] & \underline{\omega} \\ \hdashline -\underline{\omega}^T & 0 \end{array}\right] \tag{11a}$$

$$\Gamma(\underline{\omega}) \equiv \left[\begin{array}{c:c} -[\underline{\omega}\times] & -\underline{\omega} \\ \hdashline \underline{\omega}^T & 0 \end{array}\right] \tag{11b}$$

Some useful relations for $\Omega(\underline{\omega})$ and $\Gamma(\underline{\omega})$ also include

$$\Omega(\underline{\omega})\Omega(\underline{\omega}) = -(\underline{\omega}^T\underline{\omega})I_{4\times4} \tag{12a}$$

$$\Gamma(\underline{\omega})\Gamma(\underline{\omega}) = -(\underline{\omega}^T\underline{\omega})I_{4\times4} \tag{12b}$$

GPS Loss Function

In this section an analysis of the GPS standard loss function is shown. The general loss function to be minimized is given by

$$J(\underline{q}) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left\{\Delta\phi_{ij} - \underline{b}_i^T A(\underline{q})\underline{s}_j\right\}^2 \tag{13}$$

$$\text{subject to } \underline{q}^T\underline{q} = 1$$

where m represents the number of baselines, and n represents the number of observed GPS spacecraft. The extrema of $J(\underline{q})$, subject to the normalization constraint, can be found by the method of Lagrange multipliers, which maximizes

$$g(\underline{q}) = g_1(\underline{q}) - g_2(\underline{q}) - \gamma\,\underline{q}^T\underline{q} + \Upsilon \tag{14}$$

with

$$g_1(\underline{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}\,\underline{b}_i^T A(\underline{q})\underline{s}_j \tag{15a}$$

$$g_2(\underline{q}) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left\{\underline{b}_i^T A(\underline{q})\underline{s}_j\right\}^2 \tag{15b}$$

where γ is the Lagrange multiplier and Υ is independent of $\underline{q}$. Substituting Equation (5) into $\underline{b}_i^T A(\underline{q})\underline{s}_j$ and using Equation (10) leads to the following identity

$$\underline{b}_i^T A(\underline{q})\underline{s}_j = \underline{q}^T\Omega(\underline{b}_i)\Gamma(\underline{s}_j)\underline{q} \tag{16}$$

This identity can also be used to form the K matrix in Wahba's problem (see Ref. [12]). Differentiating Equation (14) with respect to $\underline{q}$ and setting the result to zero leads to

$$(K_1 - K_2 - \gamma\,I_{4\times4})\underline{q} = \underline{0}_{4\times1} \tag{17}$$

where

$$K_1 = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}\,\Gamma(\underline{s}_j)\Omega(\underline{b}_i) \tag{18a}$$

$$K_2 = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\,\Gamma(\underline{s}_j)\Omega(\underline{b}_i)\underline{q}\,\underline{q}^T\Omega(\underline{b}_i)\Gamma(\underline{s}_j) \tag{18b}$$

Note that $\Gamma(\underline{s}_j)\Omega(\underline{b}_i) = \Omega(\underline{b}_i)\Gamma(\underline{s}_j)$ (i.e., the two matrices commute). Therefore, minimizing Equation (13) is equivalent to finding an orthonormal basis $(\underline{q})$ for the null space of $K_1 - K_2$. This is not straightforward since K_2 explicitly depends on $\underline{q}$.

A straightforward solution to Equation (17) can be found if either the baselines or the sightlines are proportional to an orthonormal set. First, we note that σ_{ij} is the same for all baselines and sightlines (which will be denoted by σ). Next, the term $K_2\,\underline{q}$ in Equation (17) can be re-written using the relations in Equation (10), so that

$$K_2\,\underline{q} = -\frac{1}{\sigma^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Gamma(\underline{s}_j)\Xi(\underline{q})\underline{b}_i\,\underline{b}_i^T\Xi^T(\underline{q})\Psi(\underline{q})\,\underline{s}_j \tag{19}$$

If the baselines are proportional to an orthonormal set, then

$$\sum_{i=1}^{m}\underline{b}_i\,\underline{b}_i^T = b\,I_{3\times3} \tag{20}$$

where b is a scalar. Equation (19) now reduces to

$$K_2\,\underline{q} = -\frac{b}{\sigma^2}\sum_{j=1}^{n}\Gamma(\underline{s}_j)\Xi(\underline{q})\Xi^T(\underline{q})\Psi(\underline{q})\,\underline{s}_j \tag{21}$$

Using the relations in Equations (9) and (12) leads to the following simplification

$$K_2\,\underline{q} = \frac{n\,b}{\sigma^2}\underline{q} \tag{22}$$

Therefore, Equation (17) simplifies to the following form

$$\left\{\frac{1}{\sigma^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta\phi_{ij}\,\Gamma(\underline{s}_j)\Omega(\underline{b}_i)\right\}\underline{q} = \left(\frac{n\,b}{\sigma^2} + \gamma\right)\underline{q} \tag{23}$$

Equation (23) is equivalent to the q-method solution of Wahba's problem.[12] Many efficient and optimal algorithm exist for finding the solution to this problem (e.g., see Refs. [8-9]). A similar form to Equation (23) can be easily developed for the case that the sightlines are proportional to an orthonormal set

If neither the baselines nor the sightlines are proportional to an orthonormal set, then Equation (17) must be solved directly. As of this writing the optimal (point-by-point) solution to this equation can only be found by numerically intense optimization techniques. Cohen[1] proposed a linearized least-squares technique that is numerically efficient, but is sensitive to initial guesses. Another technique by Crassidis et. al.[11] provides an optimal solution that is both numerically efficient and insensitive to initial guesses, but relies on sequential operations.

Attitude Determination from Vectorized Measurements

In this section, a previously found method for attitude determination from GPS phase measurements is summarized (see Ref. [6] for more details). The general method for the vectorized measurements is based on an algorithm given by Shuster.[13] Also, a covariance analysis is performed for the new method, and for methods which minimize the general loss function in Equation (13) directly.

The vectorized measurement problem involves determining the sightline vector in the body frame, denoted by $\underline{\tilde{s}}_j \equiv A\underline{s}_j$, or the baseline in the reference frame, denoted by $\underline{\bar{b}}_i \equiv A^T\underline{b}_i$. For the sightline case, the following loss function is minimized

$$J_j\left(\underline{\tilde{s}}_j\right)=\frac{1}{2}\sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij}-\underline{b}_i^T\underline{\tilde{s}}_j\right)^2 \quad \text{for } j=1,2,\ldots,n \tag{24}$$

The minimization of Equation (24) leads to[11]

$$\underline{\tilde{s}}_j = M_j^{-1}\underline{y}_j \tag{25}$$

where

$$M_j=\sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\underline{b}_i\underline{b}_i^T \quad \text{for } j=1,2,\ldots,n \tag{26a}$$

$$\underline{y}_j=\sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}\underline{b}_i \quad \text{for } j=1,2,\ldots,n \tag{26b}$$

The error covariance of $\underline{\tilde{s}}_j$ is given by

$$P_j = M_j^{-1} \tag{27}$$

If the sightline in the body is required to be normalized, then the loss function in Equation (24) must be minimized subject to the constraint $\underline{\tilde{s}}^T\underline{\tilde{s}}=1$. However, Shuster[13] showed that the error introduced by ignoring this constraint is usually negligible. The solution to Wahba's problem determines the optimal attitude (in the least-squares sense) which results in a normalized body vector. Therefore, the normalization constraint may be ignored. Also, the normalized error covariance is singular, as shown in Ref. [13]. This singularity is avoided by using the covariance given by Equation (27). For a discussion of singularity issues for measurement covariances see Ref. [14].

From Equation (25) it is seen that at least three non-coplanar baselines are required to determine the sightlines in the body frame. This is analogous to the problem posed by Cohen.[1] However, if only two non-collinear baselines exist, a solution is again possible as long as three non-coplanar sightlines exist. This approach determines the baselines in the reference frame, by minimizing the following loss function

$$J_i\left(\underline{\bar{b}}_i\right)=\frac{1}{2}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij}-\underline{\bar{b}}_i^T\underline{s}_j\right)^2 \quad \text{for } i=1,2,\ldots,m \tag{28}$$

The minimization of Equation (28) is again straightforward and leads to

$$\underline{\bar{b}}_i = N_i^{-1}\underline{z}_i \tag{29}$$

where

$$N_i=\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\underline{s}_j\underline{s}_j^T \quad \text{for } i=1,2,\ldots,m \tag{30a}$$

$$\underline{z}_i=\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}\underline{s}_j \quad \text{for } i=1,2,\ldots,m \tag{30b}$$

The error covariance of $\underline{\bar{b}}_i$ is given by

$$Q_i = N_i^{-1} \tag{31}$$

The case with two non-collinear baselines and two non-collinear sightlines can also be solved for either the baseline reference case or sightline body case (see Ref. [6] or [10] for details).

Attitude Determination

The attitude determination problem using body sightlines is very similar to that using reference baselines, so we may consider only the former case. The attitude is determined by minimizing the following loss function

$$J(A)=\frac{1}{2}\sum_{j=1}^{n}\left(\underline{\tilde{s}}_j-A\underline{s}_j\right)^T M_j\left(\underline{\tilde{s}}_j-A\underline{s}_j\right) \tag{32}$$

This loss function is not identical to Wahba's problem since the quartic dependence in the quaternion does not cancel, unless the baselines are proportional to an orthonormal set so that M_j is given by a scalar times the identity matrix. The loss function in Equation (32) is in fact equivalent to the general loss function in Equation (13). This is shown by substituting Equations (25) and (26) into (32) and expanding terms, giving

$$J(A)=\frac{1}{2}\sum_{j=1}^{n}\Big(\underline{y}_j^T M_j^{-1}\underline{y}_j - 2\,\underline{y}_j^T A\underline{s}_j + \underline{s}_j^T A^T M_j A\underline{s}_j\Big) \tag{33}$$

Expanding Equation (33) now yields

$$J(A)=\frac{1}{2}\sum_{j=1}^{n}\left(\underline{y}_j^T M_j^{-1}\underline{y}_j - \sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}^2\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij}-\underline{b}_i^T A\underline{s}_j\right)^2 \tag{34}$$

Since the first term in Equation (34) is independent of attitude, it is clear that this loss function is equivalent to the general loss function in Equation (13). In order to reduce the loss function in Equation (32) into a form corresponding to Wahba's problem the condition that M_j is given by a scalar times the identity matrix must be valid. Therefore, if the baselines are not proportional to an orthonormal set, then the attitude solution is suboptimal.

Attitude Covariance

An attitude error covariance can also derived from the GPS loss function in Equation (13). This is accomplished by using results from maximum likelihood estimation.[6,15] The Fisher information matrix for a parameter vector $\underline{x}$ is given by

$$F_{xx}=E\left\{\frac{\partial}{\partial\underline{x}\,\partial\underline{x}^T}J(\underline{x})\right\}_{\underline{x}_{\text{true}}} \tag{35}$$

where $E\{\ \}$ denotes expectation, and $J(\underline{x})$ is the negative log likelihood function, which is the loss function in this case. If the measurements are Gaussian and linear in the parameter vector, then the error covariance is given by

$$P_{xx}=F_{xx}^{-1} \tag{36}$$

Now, the attitude matrix is approximated by

$$\begin{aligned} A &= e^{-[\underline{\delta\alpha}\times]}A_{\text{true}} \\ &\approx \left(I_{3\times3}-[\underline{\delta\alpha}\times]+\frac{1}{2}[\underline{\delta\alpha}\times]^2\right)A_{\text{true}} \end{aligned} \tag{37}$$

where $\underline{\delta\alpha}$ represents a small angle error (for the quaternion $2\,\underline{\delta q}_{13}\approx\underline{\delta\alpha}$). Equation (37) is next substituted into Equation (13) to determine the Fisher information matrix. First-order terms vanish in the partials, and third-order terms are small because we assume the probability distribution to be approximately symmetric about the mean. Also, assuming that the quartic terms are negligible (see [16] for a Gaussian approximation to fourth-order terms) leads to the following form for the optimal covariance

$$P_{\text{body}}^{o}\approx\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^{-2}\left[A\underline{s}_j\times\right]\underline{b}_i\underline{b}_i^T\left[A\underline{s}_j\times\right]^T\right]^{-1} \tag{38}$$

Note that the optimal covariance requires knowledge of the attitude matrix. However, if the baselines are non-coplanar then the optimal covariance can be determined without the attitude knowledge by replacing $A\underline{s}_j$ with $\underline{\tilde{s}}_j$.

For the sightline transformation case the following loss function, which is equivalent to Wahba's form, is minimized

$$J(A)=\frac{1}{2}\sum_{j=1}^{n}a_j\left|\underline{\tilde{s}}_j - A\underline{s}_j\right|^2 \tag{39}$$

Several efficient algorithms have been developed to solve this problem (e.g., QUEST[8] and FOAM[9]). Also, various methods for determining optimal values for a_j are shown in Ref. [6]. Minimizing the loss function in Equation (39) produces suboptimal attitude estimates unless the baselines are proportional to an orthonormal set. The attitude error covariance for the sightline transformation case can be shown to be given by

$$P_{\text{body}}^{s}\approx X^{-1}\left\{\sum_{j=1}^{n}a_j^2\left[\underline{\tilde{s}}_j\times\right]P_j\left[\underline{\tilde{s}}_j\times\right]^T\right\}X^{-1} \tag{40}$$

where

$$X\equiv\sum_{j=1}^{n}a_j\left[\underline{\tilde{s}}_j\times\right]\left[\underline{\tilde{s}}_j\times\right]^T \tag{41}$$

Note that if the covariances P_j are multiples of the identity, $P_j = \sigma_j^2 I$, and then setting $a_j = \sigma_j^{-2}$ yields

$$P_{\text{body}}^{s} \approx \left[\sum_{j=1}^{n} \sigma_j^{-2} \left[\underline{\tilde{s}}_j \times \right] \left[\underline{\tilde{s}}_j \times \right]^T \right]^{-1} = X^{-1} \quad (42)$$

Therefore, in this case the covariance in Equation (42) would be identical to the covariance given by QUEST.[8] This analysis is equivalent to the previous analysis on the GPS loss function itself for baselines that are proportional to an orthonormal set.

The covariance analysis can be easily extended to the case where the baselines in the reference frame are determined. The body covariance for the transformed loss function in this case becomes

$$P_{\text{body}}^{b} \approx A\, B_a \sum_{i=1}^{m} a_i^2 \left[\underline{\bar{b}}_i \times \right] Q_i \left[\underline{\bar{b}}_i \times \right]^T B_a\, A^T \quad (43a)$$

$$B_a = \left[\sum_{i=1}^{m} a_i \left[\underline{\bar{b}}_i \times \right]^2 \right]^{-1} \quad (43b)$$

It can easily be shown that $P_{\text{body}}^{s} \geq P_{\text{body}}^{o}$ and $P_{\text{body}}^{b} \geq P_{\text{body}}^{o}$. Equations (43) and (40) can be compared to Equation (38) to determine which transformation produces the best estimate. In general this will be a function of the baseline and sightline geometry.

Dilution of Precision

In order to access the relative performance of the two transformations, a number of dilution of precision (DOP) factors are used. The first, called the attitude-DOP (ADOP), uses the optimal covariance expression, with

$$\text{ADOP} = \sqrt{\text{trace}\left(P_{\text{body}}^{o} \right)} \quad (44)$$

The next two use the covariance for the sightline transformation case, called the sightline-attitude-DOP (SADOP), and the covariance for the baseline transformation case, called baseline-attitude-DOP (BADOP), with

$$\text{SADOP} = \sqrt{\text{trace}\left(P_{\text{body}}^{s} \right)} \quad (45a)$$

$$\text{BADOP} = \sqrt{\text{trace}\left(P_{\text{body}}^{b} \right)} \quad (45b)$$

Hardware Simulation Results

A hardware simulation of a typical spacecraft attitude determination application was undertaken to demonstrate the performance of the new algorithm. For this simulation, a Northern Telecom 40 channel, 4 RF output STR 2760 unit was used to generate the GPS signals that would be received at a user specified location and velocity. The signals are then provided directly (i.e., they are not actually radiated) to a GPS receiver that has been equipped with software tracking algorithms that allow it operate in space.

The receiver that was used was a Trimble TANS Vector; which is a 6 channel, 4 RF input multiplexing receiver that performs 3-axis attitude determination using GPS carrier phase and line of sight measurements. This receiver software was modified at Stanford University and NASA-Goddard to allow it to operate in space. This receiver model has been flown and operated successfully on several spacecraft: REX-II, OAST-Flyer, GANE, Orbcomm, Microlab, and others.

The simulated motion profile was for an actual spacecraft, the Small Satellite Technology Initiative (SSTI) Lewis satellite, which carried an experiment to assess the performance of GPS attitude determination on-orbit. Although the spacecraft was lost due to a malfunction not related to the GPS experiment shortly after launch, this motion profile is nonetheless very representative of the types of attitude determination applications. The orbit parameters and pointing profile used for the simulation are given in Table 1.

Table 1 SSTI Lewis Orbit parameters

Semimajor axis (a)	6901.137 km
Inclination (i)	97.45 deg
Right Ascension of Ascending Node	-157.1 deg
Eccentricity (e)	0.0001
Pointing profile	Earth pointed
Launch date	Aug. 22, 1997

Quantities such as line biases and integer ambiguities are first determined before the attitude determination algorithms are tested. The GPS raw measurements are processed at 1 Hz over a 40 minute simulation. A plot of the number of available GPS spacecraft sightlines for the simulated run is shown in Figure 2. During the beginning of the run there are 5 to 6 available sightlines, which drops down to about 4 near the end of the simulation.

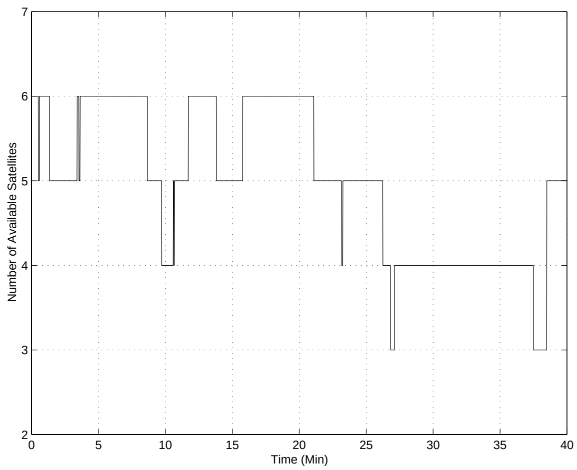

Fig. 2 Number of Available GPS Sightlines

For the first set of simulations the performances of the sightline and baseline transformations were investigated using the available sightlines in Figure 2 and three normalized baseline vectors. Two of these baselines are orthogonal, given by

$$\underline{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{46}$$

The third baseline is given by using an azimuth angle ψ and an elevation angle θ (see Figure 3), so that

$$\underline{b}_3 = \begin{bmatrix} \cos(\psi)\cos(\theta) \\ \sin(\psi)\cos(\theta) \\ \sin(\theta) \end{bmatrix} \tag{47}$$

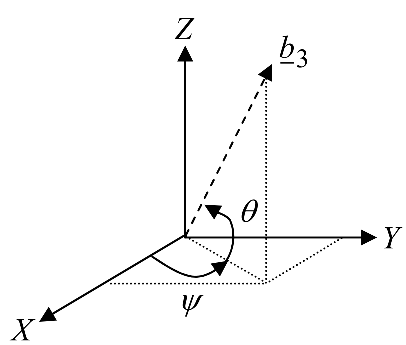

Fig. 3 Orientation of the Third Baseline

The assumed length for each baseline is 5 cycles (i.e., about 1 meter). Also, it is assumed that the third baseline sees the same sightlines for all azimuth and elevation angles. A 40 minute simulation was run each time for a given set of azimuth and elevation angles. A plot of the time-averaged ADOP for each angle is shown in Figure 4. Clearly, an orthogonal set of baselines does not yield the best results for this case, since the minimum ADOP occurs at $\psi = 1^\circ$ and $\theta = 45^\circ$. This is due to the geometry of the sightlines. A plot of the time-averaged SADOP divided by the time-averaged ADOP is shown in Figure 5. This case uses the baseline vectors to transform the sightlines into the body frame. Clearly, the performance is degraded as the elevation angle decreases and the baselines become coplanar. The minimum value (1.00) occurs at elevation and azimuth angles of 90°, as expected. A plot of the time-averaged BADOP divided by the time-averaged ADOP is shown in Figure 6. This case uses the sightline vectors to transform the baselines into the reference frame. Clearly, this case is less sensitive to changes in elevation and azimuth angles in the third baselines than the SADOP case. This is again due to the geometry of the sightline vectors. A plot of the time-averaged ADOP, SADOP, and BADOP values versus the elevation angle with azimuth angle set to 45° is shown in Figure 7. For this case SADOP becomes lower than BADOP when the elevation angle is greater than about 45°. This analysis shows that the DOP quantities in Equations (44) and (45) can be extremely useful to show which transformation (baseline or sightline) yields the more accurate results.

The next simulation uses the baseline vectors for the actual Lewis spacecraft. The antenna separation distances are 0.61 m, 1.12 m, and 1.07 m, respectively. One antenna (in baseline 3) is located 0.23 m out of plane (below) the other three antennas. On the spacecraft, the antennas are mounted on pedestals with ground planes to minimize signal reflections and multipath. For the simulation, the signal was provided to the GPS receiver without multipath noise. The simulated SSTI Lewis spacecraft has four GPS antennas that form three baselines. The baseline vector components in wavelengths are given by the following

$$\underline{b}_1 = \begin{bmatrix} 2.75 \\ 1.64 \\ -0.12 \end{bmatrix}, \quad \underline{b}_2 = \begin{bmatrix} 0.00 \\ 6.28 \\ -0.17 \end{bmatrix}, \quad \underline{b}_3 = \begin{bmatrix} -3.93 \\ 3.93 \\ -1.23 \end{bmatrix} \tag{48}$$

A plot of the ADOP, SADOP, and BADOP values during the 40 minute run is shown in Figure 8. The increase in ADOP and BADOP just after 25 minutes and just before 40 minutes is due to that fact that only three sightlines were available (see Figure 2). This run clearly shows that the case that involves transforming the baseline vectors into the reference frame yields the more accurate results, since the geometry of the sightline vectors is better than the geometry of the baseline vectors.

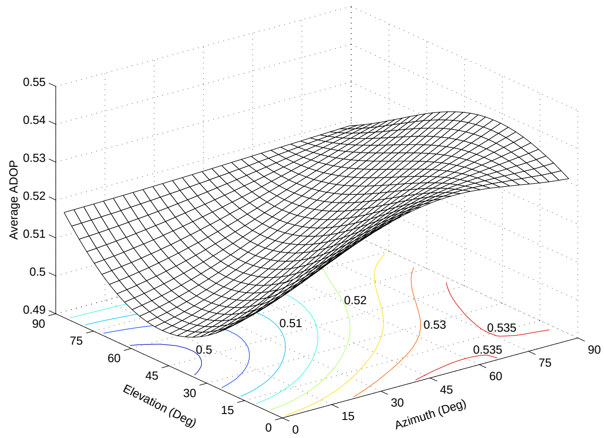

Fig. 4 Plot of Time-Averaged ADOP Values

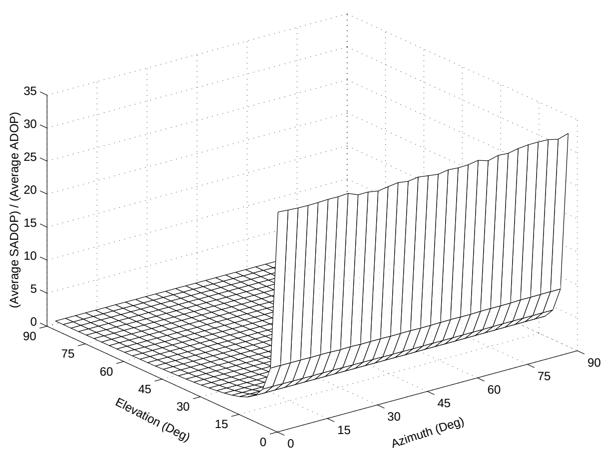

Fig. 5 Plot of Time-Averaged SADOP Values Divided by Time-Averaged-ADOP Values

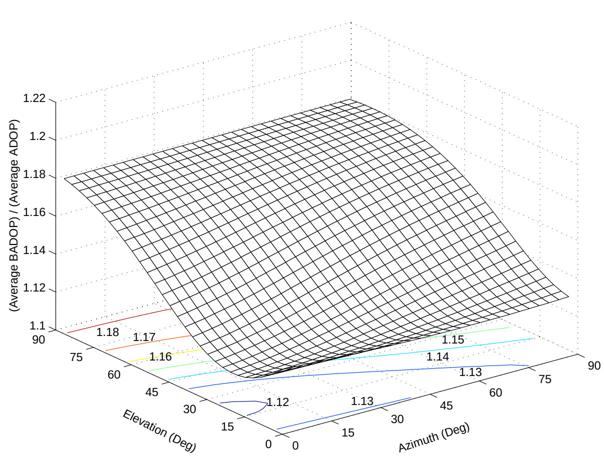

Fig. 6 Plot of Time-Averaged BADOP Values Divided by Time-Averaged ADOP Values

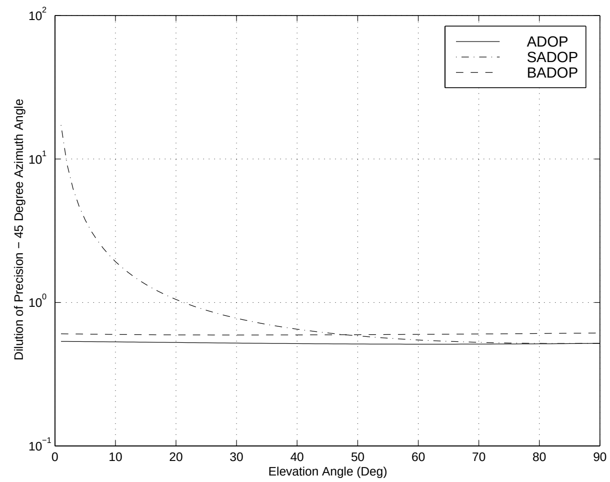

Fig. 7 Plot of DOP Values for 45° Azimuth Angle

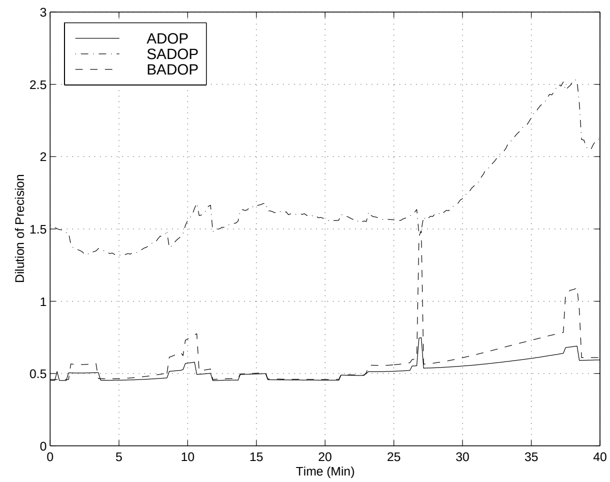

Fig. 8 Plot of DOP Values for Lewis Spacecraft

Conclusions

In this paper an investigation of two transformations of the GPS loss function into Wahba's form was performed using simulated and actual receiver data. One transformation involves determining the sightline vectors into the body frame; the other involves determining the baseline vectors in the reference frame. An analysis was performed to show that the GPS loss function reduces to Wahba's form when either the baselines or sightlines are proportional to an orthonormal set. Covariance expressions were developed and used to create various dilution-of-precision quantities. Since these transformations are simple, one can use both for on-board applications and then choose the best transformation based on the dilution-of-precision values. Also, since the algorithms in this paper provide point-by-point solutions, they can be used as integrity checks or for initializing iterative/sequential routines.

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