Optimal Attitude and Position Determination from Line-of-Sight Measurements

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Abstract

In this paper an optimal solution to the problem of determining both vehicle attitude and position using line-of-sight measurements is presented. The new algorithm is derived from a generalized predictive filter for nonlinear systems. This uses a one time-step ahead approach to propagate a simple kinematics model for attitude and position determination. The new algorithm is noniterative and is computationally efficient, which has significant advantages over traditional nonlinear least squares approaches. The estimates from the new algorithm are optimal in a probabilistic sense since the attitude/position covariance matrix is shown to be equivalent to the Cramér-Rao lower bound. Also, a covariance analysis proves that attitude and position determination is unobservable when only two line-of-sight observations are available. The performance of the new algorithm is investigated using light-of-sight measurements from a simulated sensor incorporating Position Sensing Diodes in the focal plane of a camera. Results indicate that the new algorithm provides optimal attitude and position estimates, and is robust to initial condition errors.

Introduction

Recent developments in Position Sensing Diodes (PSDs) in the focal plane of a camera allow the inherent centroiding of a beacon's incident light, from which a line-of-sight (LOS) vector can be determined. The inverse (navigation) problem is essentially the "image resection" problem of close-range photogrammetry [1]. To date, the basic aspects of a first generation vision navigation (VISNAV) system based on PSDs have been demonstrated

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through proof of concept experimental studies [2]. Results show that a beacon's LOS vector can be determined with an accuracy of one part in 5,000 (of the sensor field-of-view angle) and at a distance of $30m$ with an update rate of $100Hz$, with essentially zero image processing overhead.

The fundamental mechanism used to determine the attitude and position from LOS observations involves an object to image projective transformation, achieved through the *colinearity* equations [3]. These equations involve the angle of the body from the sensor boresight in two mutually orthogonal planes, which can be reconstructed into unit vector form. While these equations are theoretically valid only for the pin-hole camera model, we have established a calibration procedure [2] which absorbs all non-ideal effects into calibration functions. Shuster [4] has shown an analysis of the probability density function for the measurement error involving LOS observations in unit vector form. A significant conclusion is that from a practical standpoint, the probability density on the sphere is indistinguishable from the corresponding density on the tangent plane, so that the reconstructed unit vector can in fact be used in standard forms, e.g. in Wahba's problem [5] for the attitude-only determination case. However, unlike Wahba's problem where the attitude is the only unknown, in the present work we treat both attitude and position as the unknowns (i.e. the full six degree-of-freedom problem).

Determining attitude from LOS observations commonly involves finding a proper orthogonal matrix that minimizes the scalar weighted norm-error between sets of 3×1 body vector observations and 3×1 known reference vectors mapped (via the attitude matrix) into the body frame. If the reference vectors are known, then at least two non-colinear unit vector observations are required to determine the attitude. Many methods have been developed that solve this problem efficiently and accurately [6, 7]. Determining the position from LOS observations involves triangulation from known reference base points. If the attitude is known, then at least two non-colinear unit vector observations are required to establish a three-dimensional position. Determining both attitude and position from LOS observations is more complex since more than two non-colinear unit vector observations are required (as will be demonstrated in this paper), and, unlike Wahba's problem, the unknown attitude and position are interlaced in a highly nonlinear fashion.

The most common approach to determine attitude and position uses the colinearity equations and involves a Gaussian Least Squares Differential Correction (GLSDC) process [2, 3]. However, this has several disadvantages. The GLSDC process is computationally inefficient and iterative, and may take several iterations to converge. Also, it is highly sensitive to initial guess errors. In this paper a new approach is presented, based on a predictive filter for nonlinear systems [8]. This approach uses a recursive (one time-step ahead) method to "predict" the required model error so that the propagated model produces optimal estimates. The filter developed in this paper is essentially reduced to a deterministic approach since the corrections required to update the model are not weighted in the loss function. The main advantages of this approach over the GLSDC process are: 1) the algorithm is not iterative at each time instant (convergence is given as a differential correction in time), 2) it is robust with respect to initial guess errors, 3) it determines angular and linear velocity as part of the solution, and 4) the algorithm is easy to implement. A covariance analysis will be used to show that the new algorithm produces estimates that have the same error covariance as the ideal one derived from maximum likelihood. Therefore, the new algorithm is optimal from a probabilistic viewpoint.

The organization of this paper proceeds as follows. First, a review of the colinearity equations is shown. Then, a generalized loss function derived from maximum likelihood for attitude and position determination is given. Next, the optimal estimate covariance is derived, followed by an analysis using only two unit vector observations. Then, a review of the predictive filter is given, followed by the application of this approach for attitude and position determination from LOS observations. Also, an estimate error covariance expression is derived for the new algorithm. Finally, the algorithm is tested using a simulated vehicle maneuver.

The Colinearity Equations and Covariance

In this section an analysis of the colinearity equations for attitude and position determination is shown. First, the observation model is reviewed. Then, the estimate (attitude and position) covariance matrix is derived using maximum likelihood. Finally, an analysis using two unit vector observations is demonstrated.

Colinearity Equations

If we choose the z-axis of the sensor coordinate system to be directed outward along the boresight, then given object space (X, Y, Z) and image space (x, y, z) coordinate frames (see Fig. 1), the ideal object to image space projective transformation (noiseless) can be written as follows:

$$x_i = -f \frac{A_{11}(X_i - X_c) + A_{12}(Y_i - Y_c) + A_{13}(Z_i - Z_c)}{A_{31}(X_i - X_c) + A_{32}(Y_i - Y_c) + A_{33}(Z_i - Z_c)}, \quad i = 1, 2, \ldots, N \tag{1a}$$

$$y_i = -f \frac{A_{21}(X_i - X_c) + A_{22}(Y_i - Y_c) + A_{23}(Z_i - Z_c)}{A_{31}(X_i - X_c) + A_{32}(Y_i - Y_c) + A_{33}(Z_i - Z_c)}, \quad i = 1, 2, \ldots, N \tag{1b}$$

where N is the total number of observations, (x_i, y_i) are the image space observations for the i^{th} light-of-sight, (X_i, Y_i, Z_i) are the known object space locations of the i^{th} beacon,

(X_c, Y_c, Z_c) is the unknown object space location of the sensor, f is the known focal length, and A_{jk} are the unknown coefficients of the attitude matrix (A) associated to the orientation from the object plane to the image plane.

The observation can be reconstructed in unit vector form as

$$\mathbf{b}_i = A\mathbf{r}_i, \quad i = 1, 2, \dots, N \tag{2}$$

where

$$\mathbf{b}_i \equiv \frac{1}{\sqrt{f^2 + x_i^2 + y_i^2}} \begin{bmatrix} -x_i \\ -y_i \\ f \end{bmatrix} \tag{3a}$$

$$\mathbf{r}_i \equiv \frac{1}{\sqrt{(X_i - X_c)^2 + (Y_i - Y_c)^2 + (Z_i - Z_c)^2}} \begin{bmatrix} X_i - X_c \\ Y_i - Y_c \\ Z_i - Z_c \end{bmatrix} \tag{3b}$$

When measurement noise is present, Shuster [6] has shown that nearly all the probability of the errors is concentrated on a very small area about the direction of $A\mathbf{r}_i$, so the sphere containing that point can be approximated by a tangent plane, characterized by

$$\tilde{\mathbf{b}}_i = A\mathbf{r}_i + \boldsymbol{\upsilon}_i, \quad \boldsymbol{\upsilon}_i^T A\mathbf{r}_i = 0 \tag{4}$$

where $\tilde{\mathbf{b}}_i$ denotes the i^{th} measurement, and the sensor error $\boldsymbol{\upsilon}_i$ is approximately Gaussian which satisfies

$$E\{\boldsymbol{\upsilon}_i\} = \mathbf{0} \tag{5a}$$

$$E\left\{\boldsymbol{\upsilon}_i \boldsymbol{\upsilon}_i^T\right\} = \sigma_i^2 \left[I - (A\mathbf{r}_i)(A\mathbf{r}_i)^T\right] \tag{5b}$$

and $E\{\,\}$ denotes expectation. Equation (5b) makes the small field-of-view assumption of Ref. [6]; however, for a large field-of-view lens with significant radial distortion, this covariance model should be modified appropriately.

Maximum Likelihood Estimation and Covariance

Attitude and position determination using LOS measurements involves finding estimates of the proper orthogonal matrix A and position vector $\mathbf{p} \equiv [X_c \; Y_c \; Z_c]^T$ that minimize the following loss function:

$$J(\hat{A},\, \hat{\mathbf{p}}) = \frac{1}{2}\sum_{i=1}^{N}\sigma_i^{-2}\|\tilde{\mathbf{b}}_i - \hat{A}\hat{\mathbf{r}}_i\|^2 \tag{6}$$

where $\wedge$ denotes estimate. An estimate error covariance can be derived from the loss function in equation (6). This is accomplished by using results from maximum likelihood estimation [6,9]. The Fisher information matrix for a parameter vector $\mathbf{x}$ is given by

$$F_{xx} = E\left\{\frac{\partial}{\partial\mathbf{x}\,\partial\mathbf{x}^T}J(\mathbf{x})\right\}_{\mathbf{x}_{\text{true}}} \tag{7}$$

where $J(\mathbf{x})$ is the negative log-likelihood function, which is the loss function in this case (neglecting terms independent of A and $\mathbf{p}$). Asymptotically, the Fisher information matrix tends to the inverse of the estimate error covariance so that $\lim\limits_{N\to\infty} F_{xx} = P^{-1}$. The true attitude matrix is approximated by

$$A = e^{-[\boldsymbol{\delta\alpha}\times]}\hat{A} \approx (I_{3\times 3} - [\boldsymbol{\delta\alpha}\times])\,\hat{A} \tag{8}$$

where $\boldsymbol{\delta\alpha}$ represents a small angle error and $I_{3\times3}$ is a 3×3 identity matrix. The 3×3 matrix $[\boldsymbol{\delta\alpha}\times]$ is referred to as a cross-product matrix because $\mathbf{a}\times\mathbf{b} = [\mathbf{a}\times]\,\mathbf{b}$, with

$$[\mathbf{a}\times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{9}$$

The parameter vector is now given by $\mathbf{x} = [\boldsymbol{\delta\alpha}^T\ \hat{\mathbf{p}}^T]^T$, and the covariance is defined by $P = E\left\{\mathbf{x}\,\mathbf{x}^T\right\} - E\left\{\mathbf{x}\right\}E^T\left\{\mathbf{x}\right\}$. Substituting equation (8) into equation (6), and after taking the appropriate partials the following optimal error covariance can be derived:

$$P = \begin{bmatrix} -\sum\limits_{i=1}^{N}\sigma_i^{-2}[A\,\mathbf{r}_i\times]^2 & \sum\limits_{i=1}^{N}\sigma_i^{-2}\zeta_i^{-1/2}A\,[\mathbf{r}_i\times] \\ \sum\limits_{i=1}^{N}\sigma_i^{-2}\zeta_i^{-1/2}[\mathbf{r}_i\times]^T A^T & -\sum\limits_{i=1}^{N}\sigma_i^{-2}\zeta_i^{-1}[\mathbf{r}_i\times]^2 \end{bmatrix}^{-1} \tag{10}$$

where $\zeta_i \equiv (X_i - X_c)^2 + (Y_i - Y_c)^2 + (Z_i - Z_c)^2$. The terms A and $\mathbf{r}_i$ are evaluated at their respective *true* values (although in practice the estimates are used). It should be noted that equation (10) gives the Cramér-Rao lower bound [9] (any estimator whose error covariance is equivalent to equation (10) is an *efficient*, i.e. optimal estimator).

Two Vector Observation Case

In this section an analysis of the covariance matrix using two vector observations ($N = 2$) is shown. Although using one vector observation provides some information, the physical interpretation of this case is difficult to visualize and demonstrate analytically. Note, as the range to the beacons becomes large, the angular separation decreases and the beacons ultimately approach co-location. The result is a geometric dilution of precision, and ultimately, a loss of observability analogous to the one beacon case. The two vector case does provide some physical insight, which is also worthy of study. From equation (10) the $1-1$ partition of the inverse covariance (P^{-1}) is equivalent to the inverse of the QUEST covariance matrix [4, 6]. This 3×3 matrix is nonsingular if at least two non-colinear vector observations exist and the reference vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are known. The $2-2$ partition is nonsingular if at least two non-colinear vectors exist, which is independent of attitude knowledge. The $1-2$ partition has at most rank 2 for any number of observations since it is given as a sum of cross product matrices. We now will prove that the two vector observation case for the coupled problem involving both attitude and position determination is unobservable. We first partition the inverse covariance matrix into 3×3 sub-matrices as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}^{-1}, \quad P = \begin{bmatrix} \mathcal{P}_{11}^{-1} & \mathcal{P}_{12}^{-1} \\ \mathcal{P}_{12}^{-T} & \mathcal{P}_{22}^{-1} \end{bmatrix} \tag{11}$$

with obvious definitions for P_{11}, P_{12} and P_{22} from equation (10). The relationships between $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{22}$ and P_{11}, P_{12}, P_{22} are given by [10]

$$\mathcal{P}_{11} = (P_{11} - P_{12} P_{22}^{-1} P_{12}^T) \tag{12a}$$

$$\mathcal{P}_{12} = P_{11}^{-1} P_{12} (P_{12}^T P_{11}^{-1} P_{12} - P_{22}) \tag{12b}$$

$$\mathcal{P}_{22} = (P_{22} - P_{12}^T P_{11}^{-1} P_{12}) \tag{12c}$$

We shall first concentrate on the attitude part of the covariance, given by equation (12a). With two unit vector observations, P_{22}^{-1} is given by [6]

$$\begin{aligned} P_{22}^{-1} = \tilde{\sigma}_{\text{tot}}^2 I_{3\times 3} + \|\mathbf{r}_1 \times \mathbf{r}_2\|^{-2} [(\tilde{\sigma}_2^2 - \tilde{\sigma}_{\text{tot}}^2) \mathbf{r}_1 \mathbf{r}_1^T + (\tilde{\sigma}_1^2 - \tilde{\sigma}_{\text{tot}}^2) \mathbf{r}_2 \mathbf{r}_2^T \\ + \tilde{\sigma}_{\text{tot}}^2 (\mathbf{r}_1^T \mathbf{r}_2)(\mathbf{r}_1 \mathbf{r}_2^T + \mathbf{r}_2 \mathbf{r}_1^T)] \end{aligned} \tag{13}$$

where

$$\tilde{\sigma}_i^2 \equiv \left[(X_i - X_c)^2 + (Y_i - Y_c)^2 + (Z_i - Z_c)^2\right] \sigma_i^2, \quad i = 1,\, 2 \tag{14a}$$

$$\tilde{\sigma}_{\text{tot}}^2 \equiv \frac{\tilde{\sigma}_1^2 \tilde{\sigma}_2^2}{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2} \tag{14b}$$

Next, we make use of the following identity:

$$[A\mathbf{r}_i \times] = A\,[\mathbf{r}_i \times]\,A^T \tag{15}$$

which allows us to factor out the attitude matrix from equation (12a), so that

$$\mathcal{P}_{11} = A\mathcal{G}A^T \tag{16}$$

where

$$\mathcal{G} \equiv -\sum_{i=1}^{2} \sigma_i^{-2}[\mathbf{r}_i \times]^2 + \left\{\sum_{i=1}^{2} \sigma_i^{-2}\zeta_i^{-1/2}[\mathbf{r}_i \times]\right\} P_{22}^{-1} \left\{\sum_{i=1}^{2} \sigma_i^{-2}\zeta_i^{-1/2}[\mathbf{r}_i \times]\right\} \tag{17}$$

Substituting equation (13) into equation (17), after considerable algebra, leads to

$$\mathcal{G} = \frac{1}{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}\left\{-\left([\boldsymbol{\beta}_1 \times] - [\boldsymbol{\beta}_2 \times]\right)^2 + \eta\,[\boldsymbol{\beta}_1 \times]\boldsymbol{\beta}_2\boldsymbol{\beta}_2^T[\boldsymbol{\beta}_1 \times]\right\} \tag{18}$$

where

$$\boldsymbol{\beta}_i = \begin{bmatrix} X_i - X_c \\ Y_i - Y_c \\ Z_i - Z_c \end{bmatrix}, \quad i = 1,\, 2 \tag{19a}$$

$$\eta = \|\boldsymbol{\rho}\|^2 / \|\boldsymbol{\gamma}\| \tag{19b}$$

and

$$\boldsymbol{\rho} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2 \tag{20a}$$

$$\boldsymbol{\gamma} = \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \tag{20b}$$

Equation (18) can also be given by

$$\mathcal{G} = \frac{1}{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}\,\mathbf{g}\,\mathbf{g}^T \tag{21}$$

with

$$\mathbf{g} = \begin{bmatrix} \pm\{(\rho_2^2+\rho_3^2) - \|\boldsymbol{\rho}\|^2\gamma_1^2/\|\boldsymbol{\gamma}\|^2\}^{1/2} \\ \pm\{(\rho_1^2+\rho_3^2) - \|\boldsymbol{\rho}\|^2\gamma_2^2/\|\boldsymbol{\gamma}\|^2\}^{1/2} \\ \pm\{(\rho_1^2+\rho_2^2) - \|\boldsymbol{\rho}\|^2\gamma_3^2/\|\boldsymbol{\gamma}\|^2\}^{1/2} \end{bmatrix} \tag{22}$$

The $\pm$ terms define the sign of the off-diagonal elements of equation (18). Equation (21) clearly has rank 1, which shows that only one angle is observable. An eigenvalue/eigenvector decomposition of equation (16) can be used to assess the observability. The eigenvalues of equation (16) are given by $(0,\, 0,\, [\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2]^{-1}\|\boldsymbol{\rho}\|^2)$, and the eigenvector associated with the non-zero eigenvalue is given by $\mathbf{v} = A\mathbf{g}/\|\mathbf{g}\|$, which defines the axis of rotation for the observable attitude angle. The eigenvector can easily be shown to lie in the plane of the two body vector observations since $\mathbf{v}^T A(\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2) = 0$. This vector is in essence a weighted average of the body observations with

$$\|A\boldsymbol{\beta}_1\| \cos a_1 = \|A\boldsymbol{\beta}_2\| \cos a_2 \tag{23}$$

where a_1 is the angle between $A\boldsymbol{\beta}_1$ and $\mathbf{v}$, and a_2 is the angle between $A\boldsymbol{\beta}_2$ and $\mathbf{v}$, as shown in Fig. 2 ($a_1 + a_2$ is the angle between $A\boldsymbol{\beta}_1$ and $A\boldsymbol{\beta}_2$). Equation (23) indicates that the observable axis of rotation is closer to the vector with less length. For example, if the magnitude of $A\boldsymbol{\beta}_1$ is much smaller than the magnitude of $A\boldsymbol{\beta}_2$ then the eigenvector will be closer to $A\boldsymbol{\beta}_1$. This is because a slight change in the smallest vector produces more change in the attitude than the same change in the largest vector. Also, if $\|\boldsymbol{\beta}_1\| = \|\boldsymbol{\beta}_2\|$ or if $\boldsymbol{\beta}_1^T\boldsymbol{\beta}_2 = 0$, then the eigenvector reduces to $\mathbf{v} = \pm A(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)/\|A\boldsymbol{\beta}_1 + A\boldsymbol{\beta}_2\|$, which is the *bisector* of the body observations.

At first glance, the observable angle with two vector observations is counterintuitive to standard attitude determination results because the matrix $-[A\mathbf{r}_i\times]^2 = I_{3\times3} - A\mathbf{r}_i\,\mathbf{r}_i^T A^T$ in P is the projection operator onto the space *perpendicular* to the body observation. Also, the observable axis is independent of the measurement weights, which again is counterintuitive to standard attitude determination results. This indicates that if one sensor is less accurate than the other sensor, then the entire attitude solution will degrade due to the measurement variance scaling in equation (18), but the observable axis of rotation remains the same. Note, the limiting case of using only one sensor (i.e. where the measurement variance of one sensor approaches infinity) cannot be analyzed using the analysis shown in this section. In the standard attitude determination problem, the position is assumed to be known (or immaterial for the case of line-of-sight vectors to stars since the measured vectors do not depend on position). However, the coupling effects of determining both the attitude and position from line-of-sight observations to nearby objects alters our intuitive perception. The remaining angles are unobservable since we cannot discern an attitude error about an unobservable axes from a position error along the respective perpendicular axes. For example, rotations about the vector perpendicular to the plane formed by the two body observations cannot be distinguished from translations within the plane.

In a similar fashion, the position information matrix can be shown to be given by

$$\mathcal{P}_{22} = \frac{1}{\sigma_1^2 + \sigma_2^2} \left\{ -\left([\boldsymbol{\delta}_1 \times] - [\boldsymbol{\delta}_2 \times] \right)^2 + \lambda \, [\boldsymbol{\delta}_1 \times] \boldsymbol{\delta}_2 \boldsymbol{\delta}_2^T [\boldsymbol{\delta}_1 \times] \right\} \tag{24}$$

where

$$\boldsymbol{\delta}_i = \boldsymbol{\beta}_i / \|\boldsymbol{\beta}_i\|^2, \quad i = 1, \, 2 \tag{25a}$$

$$\lambda = \|\boldsymbol{\varrho}\|^2 / \|\boldsymbol{\vartheta}\| \tag{25b}$$

and

$$\boldsymbol{\varrho} = \boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 \tag{26a}$$

$$\boldsymbol{\vartheta} = \boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2 \tag{26b}$$

Equation (24) can also be given by

$$\mathcal{P}_{22} = \frac{1}{\sigma_1^2 + \sigma_2^2} \, \mathbf{h} \, \mathbf{h}^T \tag{27}$$

with

$$\mathbf{h} = \begin{bmatrix} \pm \left\{ (\varrho_2^2 + \varrho_3^2) - \|\boldsymbol{\varrho}\|^2 \vartheta_1^2 / \|\boldsymbol{\vartheta}\|^2 \right\}^{1/2} \\ \pm \left\{ (\varrho_1^2 + \varrho_3^2) - \|\boldsymbol{\varrho}\|^2 \vartheta_2^2 / \|\boldsymbol{\vartheta}\|^2 \right\}^{1/2} \\ \pm \left\{ (\varrho_1^2 + \varrho_2^2) - \|\boldsymbol{\varrho}\|^2 \vartheta_3^2 / \|\boldsymbol{\vartheta}\|^2 \right\}^{1/2} \end{bmatrix} \tag{28}$$

The eigenvalues of equation (24) are given by $(0, \, 0, \, [\sigma_1^2 + \sigma_2^2]^{-1} \|\boldsymbol{\varrho}\|^2)$, and the eigenvector associated with the non-zero eigenvalue is given by $\mathbf{w} = \mathbf{h}/\|\mathbf{h}\|$, which defines the observable position axis. The eigenvector can be shown to lie in the plane of the two reference vectors since $\mathbf{w}^T(\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2) = 0$. The weighted average relationship for the observable position axis is given by

$$\|\boldsymbol{\beta}_1\| / \cos\alpha_1 = \|\boldsymbol{\beta}_2\| / \cos\alpha_2 \tag{29}$$

where α_1 is the angle between $\boldsymbol{\beta}_1$ and $\mathbf{w}$, and α_2 is the angle between $\boldsymbol{\beta}_2$ and $\mathbf{w}$ ($\alpha_1 + \alpha_2$ is the angle between $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$). Equation (29) indicates that the observable position axis is closer to the vector with greater length, which intuitively makes sense because the position solution is more sensitive to the magnitude of the vectors. A slight change in the largest vector produces more change in the position than the same change in the smallest vector. Also, if $\|\boldsymbol{\beta}_1\| = \|\boldsymbol{\beta}_2\|$ or if $\boldsymbol{\beta}_1^T\boldsymbol{\beta}_2 = 0$, then the eigenvector reduces to $\mathbf{w} = \pm(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)/\|\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2\|$, which is the bisector of the reference vectors. As before, the information given by the two observation vectors is used to calculate the part of the attitude needed to compute the observable position.

The above analysis indicates that the beacon that is closest to the target provides the most attitude information, but has the least position information. The converse is true as well (i.e. the beacon that is farthest from the target provides the most position information, but has the least attitude information). The covariance analysis can be useful to trade off the relative importance between attitude and position requirements with beacon locations. Also, when three vector observations are available, the covariance is nonsingular; however, two geometric solutions for the attitude and position are possible in practice [2], although a rigorous theoretical proof of this is difficult. Therefore, in practice, four unit vector observations are required to unambiguously determine both attitude and position.

Predictive Attitude and Position Determination

In this section a new algorithm for attitude and position determination is derived using a nonlinear predictive approach. First, a brief review of the nonlinear predictive filter is shown (see Ref. [8] for more details). Then, the filter is reduced to a deterministic-type approach for attitude and position determination. Finally, a covariance expression for the attitude and position errors using the new algorithm is derived.

Predictive Filtering

In the nonlinear predictive filter it is assumed that the state and output estimates are given by a preliminary model and a to-be-determined model error vector, given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}[\hat{\mathbf{x}}(t), t] + G(t)\mathbf{d}(t) \tag{30a}$$

$$\hat{\mathbf{y}} = \mathbf{c}[\hat{\mathbf{x}}(t), t] \tag{30b}$$

where $\mathbf{f} \in \Re^n$ is the dynamics model vector, $\hat{\mathbf{x}}(t) \in \Re^n$ is the state estimate vector, $\mathbf{d}(t) \in \Re^q$ is the model error vector, $G(t) \in \Re^{n\times q}$ is the model-error distribution matrix, $\mathbf{c} \in \Re^m$ is the output model vector, and $\hat{\mathbf{y}}(t) \in \Re^m$ is the estimated output vector. State-observable discrete measurements are assumed for equation (30a) in the following form:

$$\tilde{\mathbf{y}}(t_k) = \mathbf{c}[\mathbf{x}(t_k), t_k] + \boldsymbol{\nu}(t_k) \tag{31}$$

where $\tilde{\mathbf{y}}(t_k) \in \Re^m$ is the measurement vector at time t_k, $\mathbf{x}(t_k) \in \Re^n$ is the true state vector, and $\boldsymbol{\nu}(t_k) \in \Re^m$ is the measurement noise vector, which is assumed to be a zero-mean, stationary, Gaussian noise distributed process with

$$E\left\{\boldsymbol{\nu}(t_k)\right\} = \mathbf{0} \tag{32a}$$

$$E\left\{\boldsymbol{\nu}(t_k)\boldsymbol{\nu}^T(t_{k'})\right\} = R\,\delta_{kk'} \tag{32b}$$

where $R \in \Re^{m\times m}$ is a positive-definite covariance matrix.

A Taylor series expansion using small Δt of the output estimate in equation (30b) is given by

$$\hat{\mathbf{y}}(t_{k+1}) = \hat{\mathbf{y}}(t_k) + \mathbf{z}(\hat{\mathbf{x}}_k, \Delta t) + \Lambda(\Delta t)S(\hat{\mathbf{x}}_k)\mathbf{d}(t_k) \tag{33}$$

where $\hat{\mathbf{x}}_k \equiv \hat{\mathbf{x}}(t_k)$, Δt is the measurement sampling interval, $S(\hat{\mathbf{x}}_k) \in \Re^{m\times q}$ is a generalized sensitivity matrix, and $\Lambda(\Delta t) \in \Re^{m\times m}$ is a diagonal matrix with elements given by

$$\lambda_{ii} = \frac{\Delta t}{p_i!}, \quad i = 1,\, 2,\, \ldots\,,\, m \tag{34}$$

where p_i, $i = 1,\, 2,\, \ldots\,,\, m$ is the lowest order of the time derivative of $c_i[\hat{\mathbf{x}}(t), t]$ in which any component of $\mathbf{d}(t)$ first appears due to successive differentiation and substitution for $\dot{\hat{x}}_i(t)$ on the right side. The i^{th} component of $\mathbf{z}(\hat{\mathbf{x}}, \Delta t)$ is given by

$$z_i(\hat{\mathbf{x}}, \Delta t) = \sum_{k=1}^{p_i} \frac{\Delta t^k}{k!} L_f^k(c_i) \tag{35}$$

where $L_f^k(c_i)$ is the k^{th} Lie derivative, defined by

$$\begin{aligned} L_f^0(c_i) &= c_i \\ \mathrm{L}_f^k(c_i) &= \frac{\partial L_f^{k-1}(c_i)}{\partial \hat{\mathbf{x}}}\mathbf{f}, \quad \text{for } k \geq 1 \end{aligned} \tag{36}$$

The i^{th} row of $S(\hat{\mathbf{x}})$ is given by

$$s_i = \left\{ L_{g_1} \left[L_f^{p_i - 1}(c_i) \right], \ldots, L_{g_q} \left[L_f^{p_i - 1}(c_i) \right] \right\}, \quad i = 1, 2, \ldots, m \tag{37}$$

where g_j is the j^{th} column of $G(t)$, and the Lie derivative in equation (37) is defined by

$$L_{g_j} \left[L_f^{p_i - 1}(c_i) \right] \equiv \frac{\partial L_f^{p_i - 1}(c_i)}{\partial \hat{\mathbf{x}}} g_j, \qquad j = 1, 2, \ldots, q \tag{38}$$

Equation (37) is in essence a generalized sensitivity matrix for nonlinear systems.

A loss function consisting of the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square of the model correction is minimized, given by

$$J = \frac{1}{2} \left[\tilde{\mathbf{y}}(t_{k+1}) - \hat{\mathbf{y}}(t_{k+1}) \right]^T R^{-1} \left[\tilde{\mathbf{y}}(t_{k+1}) - \hat{\mathbf{y}}(t_{k+1}) \right] + \frac{1}{2} \mathbf{d}^T(t_k) W \mathbf{d}(t_k) \tag{39}$$

where $W \in \Re^{q \times q}$ is a positive semi-definite weighting matrix. Substituting equation (33) into equation (39), the necessary conditions for the minimization of equation (39) lead to the following model error solution [8]:

$$\begin{aligned} \mathbf{d}(t_k) = &- \left\{ \left[\Lambda(\Delta t) S(\hat{\mathbf{x}}_k) \right]^T R^{-1} \Lambda(\Delta t) S(\hat{\mathbf{x}}_k) + W \right\}^{-1} \\ &\times \left[\Lambda(\Delta t) S(\hat{\mathbf{x}}_k) \right]^T R^{-1} \left[\mathbf{z}(\hat{\mathbf{x}}_k, \Delta t) - \tilde{\mathbf{y}}(t_{k+1}) + \hat{\mathbf{y}}(t_k) \right] \end{aligned} \tag{40}$$

Therefore, given a state estimate at time t_k, then equation (40) is used to process the measurement at time t_{k+1} to find $\mathbf{d}(t_k)$ to be used in $[t_k, t_{k+1}]$ to propagate the state estimate to time t_{k+1} using equation (30). The weighting matrix W serves to weight the relative importance between the propagated model and measured quantities. If this matrix is set to zero, then no weight is placed on minimizing the model corrections so that a memoryless estimator is given.

Attitude and Position Determination

In this section the predictive filter is used to determine attitude and position from LOS measurements using the colinearilty equations. The attitude matrix A in equation (2) is parameterized by the quaternion, defined as [11]

$$\mathbf{q} \equiv \begin{bmatrix} \boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{41}$$

with

$$\boldsymbol{\varrho} \equiv \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T = \mathbf{e}\sin(\theta/2) \tag{42a}$$

$$q_4 = \cos(\theta/2) \tag{42b}$$

where $\mathbf{e}$ is a unit vector corresponding to the axis of rotation and θ is the angle of rotation. The quaternion satisfies a single constraint given by $\mathbf{q}^T\mathbf{q} = 1$. The attitude matrix is related to the quaternion by

$$A(\mathbf{q}) = \Xi^T(\mathbf{q})\Psi(\mathbf{q}) \tag{43}$$

with

$$\Xi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{44a}$$

$$\Psi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{44b}$$

The states in the predictive filter are given by the quaternion $\mathbf{q}$ and the position vector $\mathbf{p}$ so that $\mathbf{x} = [\mathbf{q}^T\ \mathbf{p}^T]^T$. The propagation model for attitude determination is given by the quaternion kinematics model [11], and the propagation model for position determination is assumed to be given by a simple first-order process:

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\Xi(\hat{\mathbf{q}})\mathbf{d}_q \tag{45a}$$

$$\dot{\hat{\mathbf{p}}} = \mathbf{d}_p \tag{45b}$$

where the model error vector in this case is given by $\mathbf{d} = [\mathbf{d}_q^T\ \mathbf{d}_p^T]^T$. The true observation equation in the predictive filter is given by $\mathbf{y}_i = A\mathbf{r}_i$, $i = 1, 2, \ldots, N$. The lowest-order time derivative of both the quaternion and position in $\mathbf{y}_i$ in which any component of $\mathbf{d}$ first appears in equation (45) is one. So the S matrix in the predictive filter, formed by using equation (37), can be shown to be given by

$$S = \begin{bmatrix} [A(\hat{\mathbf{q}})\hat{\mathbf{r}}_1\times] & -A(\hat{\mathbf{q}})\hat{\zeta}_1^{-1/2}(I_{3\times 3} - \hat{\mathbf{r}}_1\hat{\mathbf{r}}_1^T) \\ [A(\hat{\mathbf{q}})\hat{\mathbf{r}}_2\times] & -A(\hat{\mathbf{q}})\hat{\zeta}_2^{-1/2}(I_{3\times 3} - \hat{\mathbf{r}}_2\hat{\mathbf{r}}_2^T) \\ \vdots & \vdots \\ [A(\hat{\mathbf{q}})\hat{\mathbf{r}}_N\times] & -A(\hat{\mathbf{q}})\hat{\zeta}_N^{-1/2}(I_{3\times 3} - \hat{\mathbf{r}}_N\hat{\mathbf{r}}_N^T) \end{bmatrix} \tag{46}$$

The remaining quantities in equation (40) can be shown to be given by

$$\tilde{\mathbf{y}} = [\tilde{\mathbf{b}}_1^T \; \tilde{\mathbf{b}}_2^T \; \ldots \; \tilde{\mathbf{b}}_N^T]^T \tag{47a}$$

$$\hat{\mathbf{y}} = [\mathbf{r}_1^T A^T(\hat{\mathbf{q}}) \;\; \mathbf{r}_2^T A^T(\hat{\mathbf{q}}) \;\; \ldots \;\; \mathbf{r}_N^T A^T(\hat{\mathbf{q}})]^T \tag{47b}$$

$$\Lambda = \Delta t \, I_{(3N)\times(3N)} \tag{47c}$$

$$R = \text{diag}[\sigma_1^2 I_{3\times3} \; \sigma_2^2 I_{3\times3} \; \ldots \; \sigma_N^2 I_{3\times3}] \tag{47d}$$

$$\mathbf{z}(\hat{\mathbf{x}}, \Delta t) = \mathbf{0} \tag{47e}$$

$$\boldsymbol{\nu} = [\boldsymbol{v}_1^T \; \boldsymbol{v}_2^T \; \ldots \; \boldsymbol{v}_N^T]^T \tag{47f}$$

Therefore, the following model error equation is developed:

$$\mathbf{d}_k \equiv \mathbf{d}(t_k) = -\frac{1}{\Delta t}(S^T R^{-1} S)^{-1} \sum_{i=1}^{N} \sigma_i^{-2} \begin{bmatrix} [A(\hat{\mathbf{q}})\hat{\mathbf{r}}_i \times] \\ \hat{\zeta}_i^{-1/2}(I_{3\times3} - \hat{\mathbf{r}}_i \hat{\mathbf{r}}_i^T) A^T(\hat{\mathbf{q}}) \end{bmatrix} (\tilde{\mathbf{b}}_i^{\Delta} - \hat{\mathbf{b}}_i) \tag{48}$$

where the superscript Δ denotes that the quantity is measured at time t_{k+1} (all other quantities are at time t_k). The determined quaternion can be found by integrating equation (45a) from time t_k to t_{k+1}. Because $\mathbf{d}_q$ is assumed to be constant over this interval, a discrete propagation for equation (45a) can be used, given by

$$\hat{\mathbf{q}}_{k+1} = [\chi_k I_{4\times4} + \mu_k \Omega(\boldsymbol{\omega}_k)]\hat{\mathbf{q}}_k \tag{49}$$

where

$$\chi_k = \cos\left(\frac{1}{2}\|\mathbf{d}_{q_k}\|\Delta t\right) \tag{50a}$$

$$\mu_k = \sin\left(\frac{1}{2}\|\mathbf{d}_{q_k}\|\Delta t\right) \tag{50b}$$

$$\boldsymbol{\omega}_k = \mathbf{d}_{q_k}/\|\mathbf{d}_{q_k}\| \tag{50c}$$

$$\Omega(\boldsymbol{\omega}_k) = \begin{bmatrix} -[\boldsymbol{\omega}_k \times] & \boldsymbol{\omega}_k \\ -\boldsymbol{\omega}_k^T & 0 \end{bmatrix} \tag{50d}$$

A discrete propagation can also be used for equation (45b), given by

$$\hat{\mathbf{p}}_{k+1} = \hat{\mathbf{p}}_k + \Delta t \, \mathbf{d}_{p_k} \tag{51}$$

For practical applications the sampling interval should be well below Nyquist's limit [12]. Equations (48)-(51) are used to determine the attitude and position given LOS measurements at time t_{k+1} and previous estimates at time t_k.

Covariance

In this section an error covariance expression is derived for the new attitude and position determination algorithm. The attitude error propagation can be derived using a similar approach found in Ref. [13]. The small angle attitude-error perturbation ($\boldsymbol{\delta\alpha}$) and position-error perturbation ($\boldsymbol{\delta}\mathbf{p}$) equations are given by

$$\boldsymbol{\delta}\dot{\boldsymbol{\alpha}} = -[\mathbf{d}_q\times]\boldsymbol{\delta\alpha} + \boldsymbol{\delta}\mathbf{d}_q \tag{52a}$$

$$\boldsymbol{\delta}\dot{\mathbf{p}} = \boldsymbol{\delta}\mathbf{d}_p \tag{52b}$$

where $\boldsymbol{\delta}\mathbf{d}_q$ and $\boldsymbol{\delta}\mathbf{d}_p$ are model error perturbations. The discrete propagation equations are given by

$$\boldsymbol{\delta\alpha}_{k+1} = \Phi_k\boldsymbol{\delta\alpha}_k + \Gamma_k\boldsymbol{\delta}\mathbf{d}_{q_k} \tag{53a}$$

$$\boldsymbol{\delta}\mathbf{p}_{k+1} = \boldsymbol{\delta}\mathbf{p}_k + \Delta t\,\boldsymbol{\delta}\mathbf{d}_{p_k} \tag{53b}$$

where

$$\Phi_k = e^{-[\mathbf{d}_q\times]\Delta t} \tag{54a}$$

$$\Gamma_k = \int_0^{\Delta t} e^{-[\mathbf{d}_q\times]t}\,dt \tag{54b}$$

The true output is given by a first-order expansion in the predictive filter output [8], so that

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t\bar{S}_k\bar{\mathbf{d}}_k \tag{55}$$

where $\bar{S}_k$ and $\bar{\mathbf{d}}_k$ correspond to the true quantities of S_k and $\mathbf{d}_k$, respectively. Therefore, using equations (31), (55) and (47) in equation (40) leads to the following model error vector:

$$\mathbf{d}_k = (1/\Delta t)K_k\left(\mathbf{y}_k - \hat{\mathbf{y}}_k + \boldsymbol{\nu}_{k+1} + \Delta t\bar{S}_k\bar{\mathbf{d}}_k\right) \tag{56}$$

where

$$K_k \equiv \left(S_k^T R^{-1} S_k\right)^{-1} S_k^T R^{-1} \tag{57}$$

The true position vector is given by $\mathbf{p} = \boldsymbol{\delta}\mathbf{p} + \hat{\mathbf{p}}$. Next, using a small-angle approximation, similar to equation (8), and making use of the following small perturbation in the reference vector

$$\mathbf{r}_i - \hat{\mathbf{r}}_i \approx \frac{\partial \hat{\mathbf{r}}_i}{\partial \hat{\mathbf{p}}}\,\boldsymbol{\delta}\mathbf{p}, \quad i = 1,\, 2,\, \ldots\,,\, N \tag{58}$$

leads to

$$A(\mathbf{q})\mathbf{r}_i - A(\hat{\mathbf{q}})\hat{\mathbf{r}}_i \approx [A(\hat{\mathbf{q}})\hat{\mathbf{r}}_i\times]\,\boldsymbol{\delta\alpha} - A(\hat{\mathbf{q}})\hat{\zeta}_i^{-1/2}(I_{3\times 3} - \hat{\mathbf{r}}_i\hat{\mathbf{r}}_i^T)\boldsymbol{\delta}\mathbf{p}, \quad i = 1,\, 2,\, \ldots\,,\, N \tag{59}$$

Therefore, from the definitions of $\mathbf{y}_k$, $\hat{\mathbf{y}}_k$ and S_k, equation (59) can be re-written in compact form as

$$\mathbf{y}_k - \hat{\mathbf{y}}_k \approx S_k \begin{bmatrix} \boldsymbol{\delta\alpha} \\ \boldsymbol{\delta}\mathbf{p} \end{bmatrix} \tag{60}$$

Now, if $\boldsymbol{\delta\alpha}$ is small, using equation (15), and ignoring second-order terms the following approximation is given:

$$\bar{S}_k = S_k \begin{bmatrix} I_{3\times 3} + [\boldsymbol{\delta\alpha}\times] & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} \tag{61}$$

Hence, since $K_k S_k = I_{6\times 6}$, and using equations (56), (60) and (61), with $\boldsymbol{\delta}\mathbf{d}_q = \bar{\mathbf{d}}_q - \mathbf{d}_q$ and $\boldsymbol{\delta}\mathbf{d}_p = \bar{\mathbf{d}}_p - \mathbf{d}_p$, the model error perturbation is now given by

$$\begin{bmatrix} \boldsymbol{\delta}\mathbf{d}_{q_k} \\ \boldsymbol{\delta}\mathbf{d}_{p_k} \end{bmatrix} = -\frac{1}{\Delta t} K_k \boldsymbol{\nu}_{k+1} - \frac{1}{\Delta t} \begin{bmatrix} \left(I_{3\times 3} - \Delta t[\bar{\mathbf{d}}_{q_k}\times]\right)\boldsymbol{\delta\alpha}_k \\ \boldsymbol{\delta}\mathbf{p}_k \end{bmatrix} \tag{62}$$

Substituting equation (62) into equation (53) leads directly to

$$\begin{bmatrix} \boldsymbol{\delta\alpha}_{k+1} \\ \boldsymbol{\delta}\mathbf{p}_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k - (1/\Delta t)\Gamma_k + \Gamma_k[\bar{\mathbf{d}}_{q_k}\times] \\ 0_{3\times 3} \end{bmatrix} \boldsymbol{\delta\alpha}_k - \begin{bmatrix} (1/\Delta t)\Gamma_k & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} K_k \boldsymbol{\nu}_{k+1} \tag{63}$$

If Δt is small, as assumed in this approach, i.e. the sampling interval is within Nyquist's limit, then the quantities in equation (54) can be approximated adequately by

$$\Phi_k \approx \left(I_{3\times 3} - \Delta t[\bar{\mathbf{d}}_{q_k}\times]\right) \tag{64a}$$

$$\Gamma_k \approx \Delta t\, I_{3\times 3} \tag{64b}$$

Substituting these quantities into equation (63) leads to

$$\begin{bmatrix} \boldsymbol{\delta\alpha}_{k+1} \\ \boldsymbol{\delta}\mathbf{p}_{k+1} \end{bmatrix} = -K_k \boldsymbol{\nu}_{k+1} \tag{65}$$

The cancellation of the terms in $\boldsymbol{\delta\alpha}_k$ reflects that setting $W = 0$ in equation (40) gives a memoryless estimator. The estimate error covariance is now clearly given by

$$P_{k+1} \equiv E\left\{ \left[\boldsymbol{\delta\alpha}_{k+1}^T \; \boldsymbol{\delta}\mathbf{p}_{k+1}^T \right]^T \left[\boldsymbol{\delta\alpha}_{k+1}^T \; \boldsymbol{\delta}\mathbf{p}_{k+1}^T \right] \right\} = K_k R K_k^T \tag{66}$$

This can adequately be approximated by using the covariance at time t_k with

$$P_k \approx K_k R K_k^T \tag{67}$$

since the errors introduced by this approximation are of second-order in $\Delta t \|\mathbf{d}\|$, which can be neglected. From the definitions of S_k, K_k and R, the error covariance is equivalent within first-order to equation (10). Therefore, the estimator given by equations (48)-(51) is an *efficient* estimator since it achieves the Cramér-Rao lower bound [9].

Simulation Results

In the following simulation the VISNAV system is used for navigation during a $45m$ to zero approach of a vehicle in 30 minutes (see Fig. 3). The vehicle also performs a 10 degree roll and pitch maneuver in the 30 minute timespan (the orientation is given by a 1-2-3 sequence). The attitude and position data update rate is $100Hz$, and for sake of simplicity the camera image frame is assumed to be the same as that of the vehicle body frame. Six beacons are used within a volume of $1m \times 0.5m \times 1m$, with locations depicted in Fig. 4. The PSD measurement error is assumed to be Gaussian with a standard deviation of 1/5000 of the focal plane dimension, which for a 90 degree field-of-view corresponds to an angular resolution of $90/5000 \simeq 0.02$ degrees. The initial conditions for this simulation are set to their respective true values. Attitude and position determination is accomplished using equations (48)-(51). A plot of the (roll, pitch, yaw) attitude errors with 3σ outliers using equation (10) is shown in Fig. 5. The attitude errors at the beginning of the maneuver are relatively large at around 1 degree, and reduce to values less than $4/100^{\text{th}}$ of a degree at rendezvous. As the vehicle approaches the beacons they more completely span the PSD active area so that the attitude becomes more observable. A plot of the (X, Y, Z) position errors is shown in Fig. 6. The position errors at the beginning of the maneuver are around 1 meter, and reduce to errors of a few millimeters as the vehicle reaches its target destination just in front of the beacons. As in the case of the attitude errors, the position errors decrease as the vehicle approaches the beacons since the triangulation problem becomes better conditioned. It should be noted that these results are not counterintuitive to the two vector observation study shown previously since multiple beacons are now used for attitude and position determination.

To test the robustness of the new algorithm poor initial conditions are introduced. As mentioned previously, because the new algorithm is sequential and noniterative unlike the GLSDC process, convergence is given over the sampled intervals. A plot of the attitude and position errors is shown in Fig. 7. The new algorithm converges very quickly (in less than 0.1 seconds). This has significant advantages over the GLSDC approach in Ref. [2], which may take several iterations at each time stop to converge.

Conclusions

A new optimal and efficient algorithm has been developed for attitude and position determination from line-of-sight measurements. The new noniterative algorithm provides sequential estimates using a recursive one-time step ahead approach. Attitude and position determination is accomplished by calculating the angular velocity and linear velocity components which are used to propagate simple kinematic models. An error covariance expression has been derived using a maximum likelihood approach of the associated cost function. Furthermore, an observability analysis using two line-of-sight observations indicated that the beacon that is closest to the target provides the most attitude information but has the least position information, and the beacon that is farthest to the target provides the most position information but has the least attitude information. An error covariance expression has also been derived for the new algorithm using perturbation techniques. This covariance has been shown to be equivalent to the covariance derived from maximum likelihood if the sample interval is small enough (which poses no problem for most applications). Simulation results indicate that the new algorithm provides optimal attitude and position estimates, and is robust with respect to initial condition errors.

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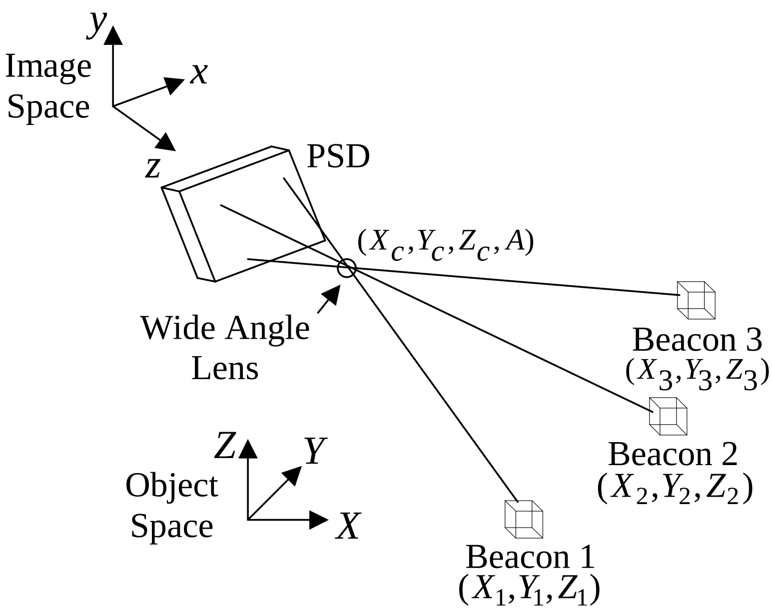
List of Figure Captions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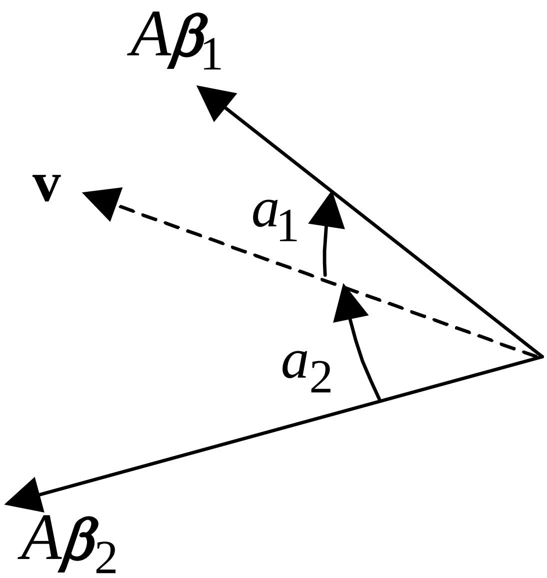
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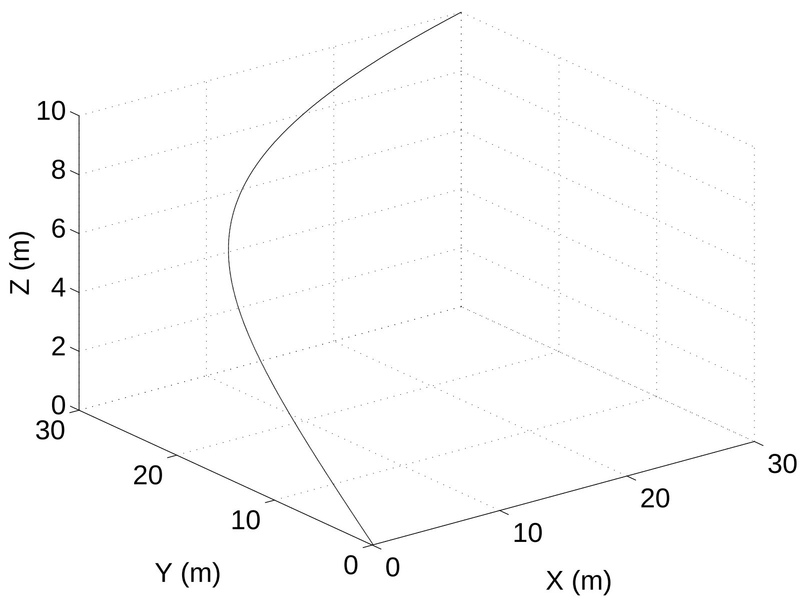
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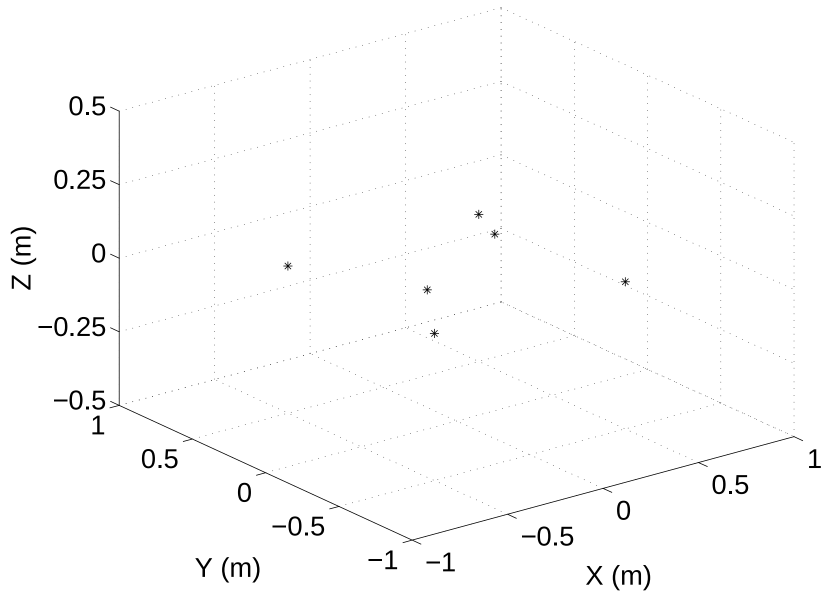

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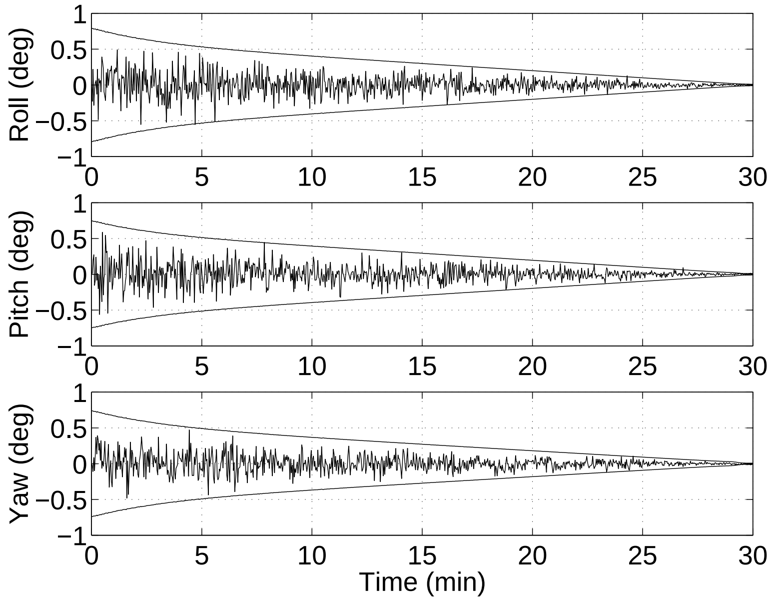

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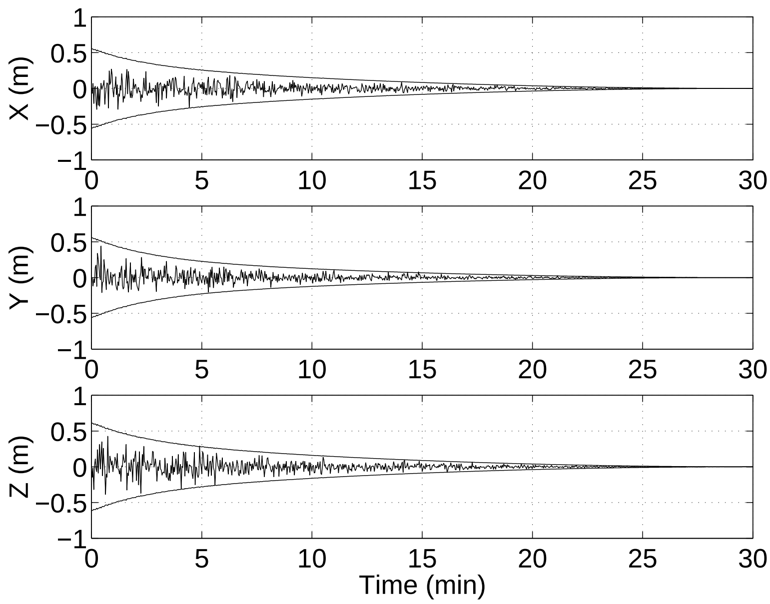
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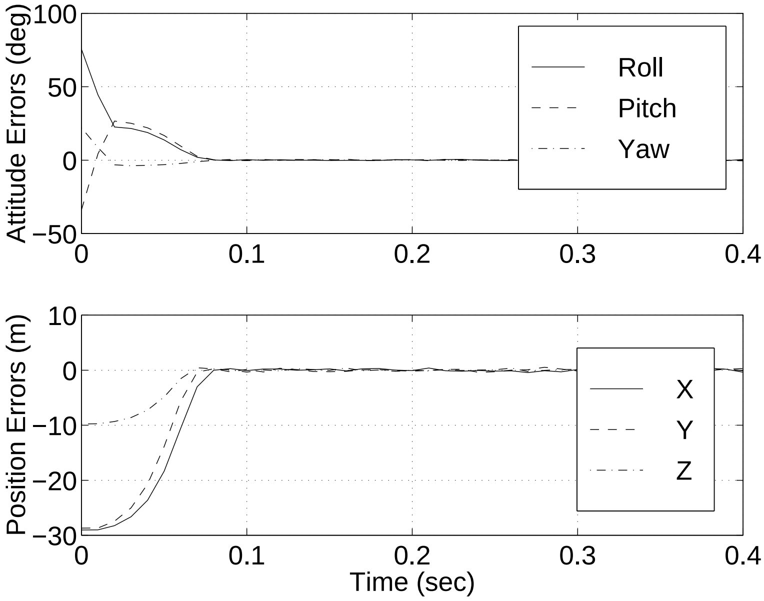
Time (min)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Roll
Pitch
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0
−10
−20
−30
0
0.1
0.2
0.3
0.4
Time (sec)
X
Y
Z

Figure 7: