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LINEAR STABILITY ANALYSIS OF MODEL ERROR CONTROL SYNTHESIS

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Abstract

Recently, a new approach for the robust control of nonlinear systems was presented. This approach, called Model Error Control Synthesis, employs an optimal real-time nonlinear estimator to determine model error corrections to the control input of a sliding mode controller using a one time-step ahead technique. Control compensation is achieved by using the estimated model error as a signal synthesis adaptive correction to the nominal control input so that maximum performance is achieved in the face of extreme model uncertainty and disturbance inputs. In this paper Model Error Control Synthesis is applied to a simple second order linear mass-spring-damper system and a higher order benchmark problem. A robustness analysis is performed for the mass-spring-damper system. The proof of the closed loop stability is derived using a Padé approximation for the time delay. It is shown that the weighting coefficient for the model error correction in the associated cost function cannot be set to zero since this gives a non-minimum phase system in the closed-loop response, and the control law itself is not asymptotically stable. Simulations are used to verify theoretical results.

Introduction

Robust control of dynamic systems is usually achieved using one of two schemes. The first scheme involves the design of a controller that is insensitive as possible to model uncertainty and/or disturbance inputs. The second scheme involves updating model parameters or control gains in real-time in order to achieve desired performance specifications. Adaptive control methods fall into this category. These control schemes can be used to provide robustness in a dynamic system with uncertainties, each with its own advantages and disadvantages.

Model Error Control Synthesis (MECS) is a signal synthesis adaptive control method.[1] Robustness is achieved by applying a correction control to eliminate the effect of model error at the output. The error is estimated by a one step ahead prediction technique. The main advantage of this technique is that the model error is determined during the estimation process using a predictive filter approach.[2] More details about predictive filter can be found in Refs [2] and [3]. In Ref. [1], MECS was first applied to suppress wing rock motion of a slender delta wing, which is described by a highly nonlinear differential equation. Also, a simple robustness analysis was provided; however, the analysis did not take into account the effects of the time delay on the overall system. Since the Laplace transform of a time delay is an exponential function, the analysis of the effect on the overall system is difficult. In this paper, we use a simple approach to test the stability of the system using a Padé approximation. The time delay caused by the controller or actuators is usually very short. Hence, it is very natural to use some approximation method.[4] Using a Padé approximation, the exponential function is approximated with a polynomial function.

To simplify the analysis we assume that the system is a second order mass-damper-spring system, which is linear, as follows:

$$m\ddot{x} + c\dot{x} + kx = u \tag{1}$$

where m is the mass, c is the damping coefficient, k is the spring constant, and u is the control input force. The state-space form is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u \tag{2a}$$

$$y = x \tag{2b}$$

where $\alpha \equiv k/m$, $\beta \equiv c/m$, $\gamma \equiv 1/m$, and $\mathbf{x} = [x\ \dot{x}]^T$. The output information is the displacement only, which is typical in actual applications. The as-

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sumed model in the estimator is given as follows:

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -\hat{\alpha} & -\hat{\beta} \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 0 \\ \hat{\gamma} \end{bmatrix} \bar{u} \tag{3a}$$

$$\hat{y} = \hat{x} \tag{3b}$$

where $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ are the estimated values for α, β, and γ, respectively, $\hat{\mathbf{x}}$ is the estimated value of $\hat{\mathbf{x}}$, and $\bar{u}$ is the control input from the nominal controller, which is sliding mode control (SMC) in this paper. Using the estimator, we estimate not only the displacement but also the velocity.

The benchmark problem is described in Ref. [5]. The model consists of a two mass-spring system, which is a generic model of an uncertain dynamical system with one rigid body mode and one vibration mode. The system of the benchmark problem is not stable and the relative degree is four. The application of the standard predictive estimator scheme shown in Ref. [2] results in unstable estimator in this case. To overcome this problem, an approximated receding-horizon control scheme is used.[6] The robustness and the stability analysis are analogous to the one for the mass-spring-damper system.

The organization of this paper is as follows. First, a summary of the nonlinear version of predictive filter is given, followed by the design process for the SMC with MECS of the linear system. Then, a robustness and stability analysis is shown for the MECS approach. Next, a robust control design is derived for the benchmark problem. Finally, simulation results are presented.

Control Design

In this section we introduce the concept of predictive filtering for nonlinear systems and design a SMC for tracking control with an assumed model.

Predictive Filtering

In the nonlinear predictive filter it is assumed that the state and output estimates are given by a preliminary model and a to-be-determined model error vector, given by[2]

$$\dot{\hat{x}}(t) = \mathbf{f}[\hat{\mathbf{x}}(t), t] + G(t)\hat{\mathbf{u}}(t) \tag{4a}$$

$$\hat{\mathbf{y}} = \mathbf{c}[\hat{\mathbf{x}}(t), t] \tag{4b}$$

where $\mathbf{f} \in \Re^n$ is the model vector, $\hat{\mathbf{x}}(t) \in \Re^n$ is the state estimate vector, $\hat{\mathbf{u}}(t) \in \Re^q$ is the model error vector, $G(t) \in \Re^{n\times q}$ is the model error distribution matrix, $\mathbf{c} \in \Re^m$ is the output vector, and $\hat{\mathbf{y}}(t) \in \Re^m$ is the estimated output vector. State-observable measurements are assumed for Eq. (4a) in the following form:

$$\tilde{\mathbf{y}}(t) = \mathbf{c}[\hat{\mathbf{x}}(t), t] + \mathbf{v}(t) \tag{5}$$

where $\tilde{\mathbf{y}}(t) \in \Re^m$ is the measurement vector at time t, $\mathbf{x}(t) \in \Re^n$ is the true state vector, and $\mathbf{v}(t) \in \Re^m$ is the measurement noise vector, which is assumed to be a zero-mean, stationary, Gaussian noise distributed process with

$$E\left\{\mathbf{v}(t)\right\} = \mathbf{0} \tag{6a}$$

$$E\left\{\mathbf{v}(t)\mathbf{v}^T(t+\Delta t)\right\} = R\,\delta(\Delta t) \tag{6b}$$

where $R \in \Re^{m\times m}$ is a positive-definite covariance matrix.

A Taylor series expansion using small Δt of the output estimate in Eq. (4b) is given by

$$\hat{\mathbf{y}}(t+\Delta t) = \hat{\mathbf{y}}(t) + \mathbf{z}(\hat{\mathbf{x}}(t), \Delta t) + \Lambda(\Delta t) S[\hat{\mathbf{x}}(t)]\hat{\mathbf{u}}(t) \tag{7}$$

where Δt is the measurement sampling interval, $S[\hat{\mathbf{x}}(t)] \in \Re^{m\times q}$ is a generalized sensitivity matrix, and $\Lambda(\Delta t) \in \Re^{m\times m}$ is a diagonal matrix with elements given by

$$\lambda_{ii} = \frac{\Delta t}{p_i!}, \quad i = 1, 2, \ldots, m \tag{8}$$

where p_i, $i = 1, 2, \ldots, m$ is the lowest order of the time derivative of $c_i[\hat{\mathbf{x}}(t), t]$ in which any component of $\mathbf{d}(t)$ first appears due to successive differentiation and substitution for $\dot{\hat{x}}_i(t)$ on the right side. The i^{th} component of $\mathbf{z}(\hat{\mathbf{x}}(t), \Delta t)$ is given by

$$z_i(\hat{\mathbf{x}}(t), \Delta t) = \sum_{k=1}^{p_i} \frac{\Delta t^k}{k!} L_f^k(c_i) \tag{9}$$

where $L_f^k(c_i)$ is the k^{th} Lie derivative, defined by

$$L_f^0(c_i) = c_i$$
$$\text{Ł}_f^k(c_i) = \frac{\partial L_f^{k-1}(c_i)}{\partial \hat{\mathbf{x}}}\mathbf{f}, \quad \text{for } k \geq 1 \tag{10}$$

The i^{th} row of $S(\hat{\mathbf{x}})$ is given by

$$s_i = \left\{ L_{g_1}\left[L_f^{p_i-1}(c_i)\right], \ldots, L_{g_q}\left[L_f^{p_i-1}(c_i)\right] \right\}, \quad i = 1, 2, \ldots, m \tag{11}$$

where g_j is the j^{th} column of $G(t)$, and the Lie derivative in Eq. (11) is defined by

$$L_{g_j}\left[L_f^{p_i-1}(c_i)\right] \equiv \frac{\partial L_f^{p_i-1}(c_i)}{\partial \hat{\mathbf{x}}} g_j, \qquad j = 1, 2, \ldots, q \tag{12}$$

Equation (11) is in essence a generalized sensitivity matrix for nonlinear systems.

A loss function consisting of the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square of the model correction is minimized, given by

$$J = \frac{1}{2}\left[\tilde{\mathbf{y}}(t+\Delta t) - \hat{\mathbf{y}}(t+\Delta t)\right]^T R^{-1} \times \left[\tilde{\mathbf{y}}(t+\Delta t) - \hat{\mathbf{y}}(t+\Delta t)\right] + \frac{1}{2}\hat{\mathbf{u}}(t) W \hat{\mathbf{u}}(t) \tag{13}$$

where $W \in \Re^{q\times q}$ is a positive semi-definite weighting matrix. Substituting Eq. (7) into Eq. (13), the necessary conditions for the minimization of Eq. (13) lead to the following model error solution:

$$\hat{\mathbf{u}}(t) = -\left\{[\Lambda(\Delta t)S[\hat{\mathbf{x}}(t)]\ ]^T R^{-1}\Lambda(\Delta t)S[\hat{\mathbf{x}}(t)] + W\right\} \times [\Lambda(\Delta t)S[\hat{\mathbf{x}}(t)]\ ]^T R^{-1}\left[\mathbf{z}(\hat{\mathbf{x}}(t),\Delta t) - \tilde{\mathbf{y}}(t+\Delta t) + \hat{\mathbf{y}}(t)\right] \tag{14}$$

Therefore, given a state estimate at time t, then Eq. (14) is used to process the measurement at time $t+\Delta t$ to find $\hat{\mathbf{u}}(t)$ to be used in $[t, t+\Delta t]$ to propagate the state estimate to time $t+\Delta t$ using Eq. (4). The weighting matrix W serves to weight the relative importance between the propagated model and measured quantities. If this matrix is set to zero, then no weight is placed on minimizing the model corrections so that a memoryless estimator is given.

Sliding Mode Control

The sliding surface for the tracking case is defined by[7]

$$s = \dot{\tilde{x}} + \lambda\tilde{x} \tag{15}$$

where $\tilde{x} = \hat{x} - x_d$, $\hat{x}$ is an output of the estimator to be designed, and x_d is the desired command response. Using the system given by Eq. (1), the equivalent control input is calculated as follows:

$$\dot{s} = -\hat{\alpha}\hat{x} - \hat{\beta}\dot{\hat{x}} + \hat{\gamma}u_{eq} - \ddot{x}_d + \lambda\dot{\hat{x}} - \lambda\dot{x}_d = 0 \tag{16}$$

To satisfy the sliding condition, $s\dot{s} < 0$, after substituting $\bar{u} = u_{eq} + u_{cr}$ into the dynamics of the sliding function, the correction control input is given by

$$u_{cr} = -\frac{\eta}{\hat{\gamma}}\text{sat}\left(\frac{s}{\rho}\right) \tag{17}$$

where sat is the saturation function. Finally, the sliding mode control input becomes

$$\bar{u} = \frac{\hat{\alpha}}{\hat{\gamma}}\hat{x} + \frac{\hat{\beta}-\hat{\lambda}}{\hat{\gamma}}\dot{\hat{x}} + \frac{1}{\hat{\gamma}}\left(\ddot{x}_d + \lambda\dot{x}_d\right) - \frac{\eta}{\hat{\gamma}}\text{sat}\left(\frac{s}{\rho}\right) \tag{18}$$

When the model is perfect, the SMC control input guarantees that the closed loop system is asymptotically stable. However, if model errors and/or estimation errors exist, the system may have a steady state error or the response may diverge.

Model Error Control Synthesis

In this section we derive the control input to correct the model error and consider the stability of the estimator. In the MECS approach Eq. (14) is used to compensate the control input for any general model error. Adding the model error into the control signal is a valid since we are free to choose where to place the model error correction in the system (the system is implicitly implied to be both controllable and observable). A block diagram of a overall control system with MECS is shown in Figure 1. First, a controller is designed using nominal system parameters, given by the sliding mode controller in Eq. (18). The predictive estimator is used for both state estimation and model error determination. The model error now represents a correction to the control. In order to compensate the effects of the model errors and disturbances on the control signal, the actual control input is now given by

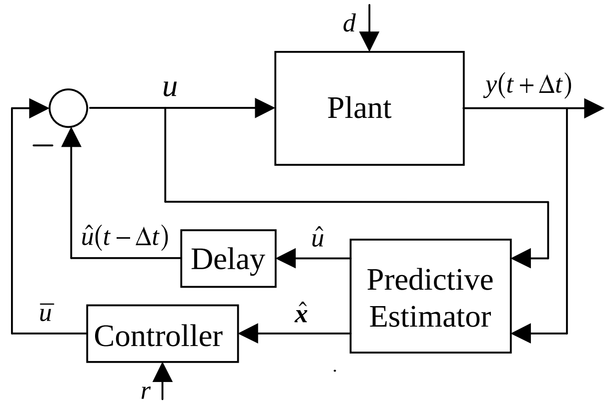

Fig. 1 Model Error Control Synthesis

$$u(t) = \bar{u}(t) - \hat{u}(t-\Delta t) \tag{19}$$

with $\hat{u}(t_0) = 0$. The model error is taken at time $t-\Delta t$ since a response from the plant must be given before the model error can be determined. The ramifications of this time delay in the control law will be shown later. The main advantage of this update law is that the system parameters need not be updated since only their effects on the nominal system are used to update the control input, which can provide closed-loop robustness with respect to the model errors. The same concept holds true for the effects of disturbances on the closed-loop system.

Model Error Determination

Using Eq. (7), the Taylor series expansion of the estimator output is as follows:

$$\begin{aligned}\hat{x}_1(t+\Delta t) &\approx \hat{x}_1(t) + \Delta t\ \dot{\hat{x}}_1(t) + \frac{\Delta t^2}{2}\ddot{\hat{x}}_1(t)\\ &\approx \hat{x}_1(t) + \Delta t\ \hat{x}_2(t) + \frac{\Delta t^2}{2}\\ &\quad\times\left[-\hat{\alpha}\hat{x}_1(t) - \hat{\beta}\hat{x}_2(t) + \hat{\gamma}u(t) + \hat{\gamma}\hat{u}(t)\right]\end{aligned} \tag{20}$$

where $\bar{u} = u + \hat{u}$. The cost function to be minimized is

$$J(t+\Delta t) = \frac{1}{2}\left[\tilde{y}(t+\Delta t) - \hat{y}(t+\Delta t)\right]^T R^{-1} \times \left[\tilde{y}(t+\Delta t) - \hat{y}(t+\Delta t)\right] + \frac{1}{2}\hat{u}^T(t) W \hat{u}(t) \tag{21}$$

Taking the partial derivative of Eq. (21) with respect to $\hat{u}(t)$ yields

$$\frac{\partial J(t+\Delta t)}{\partial \hat{u}(t)} = -\frac{1}{R}\left[\tilde{x}_1(t+\Delta t) - \hat{x}_1(t+\Delta t)\right] \times \frac{\partial \hat{x}_1(t+\Delta t)}{\partial \hat{u}(t)} + W\hat{u}(t) = 0 \tag{22}$$

The partial derivative of $\hat{x}(t+\Delta t)$ with respect to $\hat{u}(t)$ is calculated as follows:

$$\frac{\partial \hat{x}_1(t+\Delta t)}{\partial \hat{u}(t)} = \frac{\hat{\gamma}\Delta t^2}{2} \tag{23}$$

Substituting Eqs. (20) and (23) into Eq. (22) gives

$$\frac{\partial J(t+\Delta t)}{\partial \hat{u}(t)} = -\frac{\hat{\gamma}\Delta t^2}{2R}[x_1^{\Delta} - \hat{x}_1 - \Delta t\ \hat{x}_2 - \frac{\Delta t^2}{2}(-\hat{\alpha}\hat{x}_1 - \hat{\beta}\hat{x}_2 + \hat{\gamma}u + \hat{\gamma}\hat{u})] + W\hat{u} = 0 \tag{24}$$

where x_1^{Δ} is the sensor output at time $t+\Delta t$; all other quantities are given at time t. Then, the model error is determined by

$$\hat{u}(t) = \xi\left(R, W, \Delta t\right)\left[\ x_1^{\Delta} - \hat{x}_1 - \Delta t\ \hat{x}_2 - \frac{\Delta t^2}{2}\left(-\hat{\alpha}\hat{x}_1 - \hat{\beta}\hat{x}_2 + \hat{\gamma}u\right)\ \right] \tag{25}$$

where

$$\xi\left(R, W, \Delta t\right) \equiv \frac{2\hat{\gamma}\Delta t^2}{\hat{\gamma}^2\Delta t^4 + 4RW}$$

MECS Estimator

Substituting the determined model error and the control input into the state estimator, we obtain the following:

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -\hat{\alpha} - \hat{\gamma}\xi + \frac{\hat{\alpha}\hat{\gamma}\xi\Delta t^2}{2} & -\hat{\beta} - \hat{\gamma}\xi\Delta t + \frac{\hat{\beta}\hat{\gamma}\xi\Delta t^2}{2} \end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ \hat{\gamma} - \frac{\hat{\gamma}^2\xi\Delta t^2}{2} & \hat{\gamma}\xi \end{bmatrix}\begin{bmatrix} u \\ x_1^{\Delta} \end{bmatrix} \tag{26}$$

To have a stable estimator, the following inequalities by the Routh-Hurwitz stability criterion must be satisfied:

$$-\hat{\alpha} - \hat{\gamma}\xi + \frac{\hat{\alpha}\hat{\gamma}\xi\Delta t^2}{2} < 0 \tag{27a}$$

$$-\hat{\beta} - \hat{\gamma}\xi\Delta t + \frac{\hat{\beta}\hat{\gamma}\xi\Delta t^2}{2} < 0 \tag{27b}$$

Therefore, from Eqs. (27a) and (27b)

$$\text{sign}(\phi)\xi < \hat{\alpha}|\phi|^{-1} \tag{28a}$$

$$\text{sign}(\psi)\xi < \hat{\beta}|\psi|^{-1} \tag{28b}$$

where $\phi \equiv \left(\frac{\hat{\alpha}\hat{\gamma}\Delta t^2}{2} - \hat{\gamma}\right)$ and $\psi \equiv \left(\frac{\hat{\beta}\hat{\gamma}\Delta t^2}{2} - \hat{\gamma}\Delta t\right)$. If Δt is chosen to be small enough (i.e., within Nyquist's upper bound) then without loss of generality we have $\phi < 0$ and $\psi < 0$. Then the following inequality is given:

$$\xi > \max\left[-\hat{\alpha}|\phi|^{-1}, -\hat{\beta}|\psi|^{-1}\right]$$

Substituting the definition of ξ into the above inequalities gives

$$\frac{\hat{\gamma}\Delta t^2}{2R}\left[\frac{\hat{\gamma}^2\Delta t^4}{4R} + W\right]^{-1} > max\left[-\hat{\alpha}|\phi|^{-1}, -\hat{\beta}|\psi|^{-1}\right]$$

In addition, since $\gamma > 0$, then the following W always satisfies the above inequality:

$$W \geq 0$$

Analysis

In this section we apply the robust analysis process provided in Ref. [1] to the linear system. Also, a stability analysis is performed using a Padé approximation for the time delay.

Robustness

The measurement at $t+\Delta t$ can be approximated as follows:

$$x_1^{\Delta} \approx x_1 + \Delta t x_2 + \frac{\Delta t^2}{2}\left(-\alpha x_1 - \beta x_2 + \gamma u\right) \tag{29}$$

Substituting Eq. (29) into the determined model error gives

$$\hat{u}(t) = \xi[x_1 + \Delta t x_2 + \frac{\Delta t^2}{2}(-\alpha x_1 - \beta x_2 + \gamma u) - \hat{x}_1 - \Delta t\ \hat{x}_2 - \frac{\Delta t^2}{2}(-\hat{\alpha}\hat{x}_1 - \hat{\beta}\hat{x}_2 + \hat{\gamma}u)] \tag{30}$$

Assuming that the estimated states converge to the real state yields

$$\hat{u}(t) = \frac{\xi\Delta t^2}{2}\left[\left(-\alpha x_1 - \beta x_2 + \hat{\alpha}\hat{x}_1 + \hat{\beta}\hat{x}_2\right) + \left(\gamma - \hat{\gamma}\right)u\right] \tag{31}$$

Since a response from the plant must be given before the model error can be determined, for practical purposes the control input in MECS is given by

$$u(t) = \bar{u}(t) - \hat{u}(t - \Delta t) \tag{32}$$

With a one time-step delay we have

$$-\hat{\alpha}\hat{x}_1(t-\Delta t) - \hat{\beta}\hat{x}_2(t-\Delta t) = -\hat{\gamma}\bar{u}(t-\Delta t) - \lambda\tilde{x}_2(t-\Delta t) - \eta\text{sat}\left[\frac{s(t-\Delta t)}{\rho}\right] \tag{33}$$

and also from Eq. (29) with a one time-step delay we have

$$\begin{aligned} & -\alpha x_1(t-\Delta t)-\beta x_2(t-\Delta t) \\ & =\frac{2}{\Delta t^2}\left[x_1(t)-\hat{x}_1(t-\Delta t)\right] \\ & -\frac{2}{\Delta t} x_2(t-\Delta t)-\gamma u(t-\Delta t)\end{aligned} \tag{34}$$

Substituting Eqs. (33) and (34) into Eq. (31) with a one time-step delay gives

$$\begin{aligned} \hat{u}(t-\Delta t)= & \frac{\hat{\gamma}\xi\Delta t^2}{2}\hat{u}(t-2\Delta t) \\ & +\xi\left[\,x_1(t)-\hat{x}_1(t-\Delta t)-\Delta t x_2(t-\Delta t)\,\right] \\ & +\frac{\xi\Delta t^2}{2}\left[\lambda\tilde{x}_2(t-\Delta t)+\eta\,\mathrm{sat}\left(\frac{s(t-\Delta t)}{\rho}\right)\right]\end{aligned} \tag{35}$$

From Eq. (34), the following equation is given:

$$\begin{aligned} x_2(t-\Delta t)= & \frac{\Delta t}{2}\left[\alpha x_1(t-\Delta t)+\beta x_2(t-\Delta t)\right] \\ & +\frac{1}{\Delta t}\left[x_1(t)-\hat{x}_1(t-\Delta t)\right]-\frac{\Delta t}{2}\gamma u(t-\Delta t)\end{aligned} \tag{36}$$

Substituting Eq. (36) into Eq. (35) and taking one-step ahead yields

$$\begin{aligned} \hat{u}(t) \;\; = & \tfrac{\hat{\gamma}\xi\Delta t^2}{2}\hat{u}(t-\Delta t)+\tfrac{\xi\Delta t^2}{2}\left[\,-\alpha x_1(t)-\beta x_2(t)\right. \\ & \left.+\gamma u(t)\,\right]+\tfrac{\xi\Delta t^2}{2}\left[\lambda\tilde{x}_2(t)+\eta\,\mathrm{sat}\left(\tfrac{s(t)}{\rho}\right)\right]\end{aligned} \tag{37}$$

Therefore, the following control input is obtained:

$$\begin{aligned} u(t)= & \frac{\hat{\gamma}}{\gamma}\left[\frac{2}{\hat{\gamma}\xi\Delta t^2}\hat{u}(t)-\hat{u}(t-\Delta t)\right] \\ & -\frac{1}{\gamma}\left[-\alpha x_1(t)-\beta x_2(t)+\lambda\hat{x}_2(t)+\eta\ \mathrm{sat}\left(\frac{s(t)}{\rho}\right)\right]\end{aligned} \tag{38}$$

Applying Eq. (38) to the dynamic system in Eq. (1) gives

$$\begin{aligned} \ddot{x}(t) & =-\alpha x(t)-\beta\dot{x}(t)+\gamma u(t) \\ & =-g(t)+\hat{\gamma}\left[\frac{2}{\hat{\gamma}\xi\Delta t^2}\hat{u}(t)-\hat{u}(t-\Delta t)\right]\end{aligned} \tag{39}$$

where $g(t)\equiv\lambda\dot{\tilde{x}}+\eta\,\mathrm{sat}\,(s/\rho)$. Substituting the definition of ξ into the above equation leads to

$$\begin{aligned} \ddot{x}(t)= & -g(t)+\hat{\gamma}\left[\frac{4R}{\hat{\gamma}^2\Delta t^4}\left(\frac{\hat{\gamma}^2\Delta t^4}{4R}+W\right)\hat{u}(t)\right] \\ & -\hat{\gamma}\,\hat{u}(t-\Delta t)\end{aligned} \tag{40}$$

Finally, we obtain the following equation:

$$\ddot{x}(t)=-g(t)+\hat{\gamma}\left[(1+\delta)\,\hat{u}(t)-\hat{u}(t-\Delta t)\right] \tag{41}$$

where

$$\delta\left(R,W,\hat{\gamma},\Delta t\right)\equiv\frac{4RW}{\hat{\gamma}^2\Delta t^4}\geq 0 \tag{42}$$

and $R>0$, $W\geq 0$, $\Delta t>0$. From Eq. (41), a feedback linearization occurs in the closed-loop system, but with an extra term given by the difference in the control input from time t to $t-\Delta t$. This leads to a second-order effect, as opposed to a first-order effect without the MECS control correction (see [1] for details). If $W=0$ or with no time delay, the desired dynamics are obtained in spite of the uncertainty. However, in the next section we will show that W cannot be set to zero, which leads to a tradeoff between robustness in the overall system and adequate response characteristics.

Stability

Taking the Laplace transform of Eq. (41) when the trajectory is inside the boundary layer yields

$$\begin{aligned} s^2X(s)= & -\lambda sX(s)-\frac{\eta}{\rho}\left(s+\lambda\right)X(s) \\ & +\hat{\gamma}\left(1+\delta-e^{-s\Delta t}\right)\hat{U}(s)\end{aligned} \tag{43}$$

where s is the Laplace transform parameter. Collecting terms gives

$$\begin{aligned} & \left[s^2+\left(\lambda+\frac{\eta}{\rho}\right)s+\frac{\lambda\eta}{\rho}\right]X(s) \\ & =\hat{\gamma}\left(1+\delta-e^{-s\Delta t}\right)\hat{U}(s)\end{aligned} \tag{44}$$

Using a Padé approximation, the exponential term in the above equation can be approximated by[3]

$$\begin{aligned} e^{-s\Delta t} & \approx\frac{\Delta t^4s^4-20\Delta t^3s^3+180\Delta t^2s^2-840\Delta ts+1680}{\Delta t^4s^4+20\Delta t^3s^3+180\Delta t^2s^2+840\Delta ts+1680} \\ & \equiv\frac{N(s\Delta t)}{D(s\Delta t)}\end{aligned} \tag{45}$$

Substituting this approximation into the closed loop dynamics, we obtain the following transfer function:

$$G(s)\equiv\frac{X(s)}{\hat{U}(s)}=\hat{\gamma}\frac{(1+\delta)\,D(s\Delta t)-N(s\Delta t)}{\left[s^2+\left(\lambda+\frac{\eta}{\rho}\right)s+\frac{\lambda\eta}{\rho}\right]D(s\Delta t)}$$

If we assume that the following equation has only negative real parts poles then

$$s^2+\left(\lambda+\frac{\eta}{\rho}\right)s+\frac{\lambda\eta}{\rho}=0$$

Therefore, the closed loop dynamics is always stable. The zeros are determined by the roots of the following equation:

$$(1+\delta)D(s\Delta t)-N(s\Delta t)=0$$

i.e.,

$$\begin{aligned}&\delta\Delta t^4 s^4 + 20(2+\delta)\Delta t^3 s^3 + 180\delta\Delta t^2 s^2\\&+840(2+\delta)\Delta t s + 1680\delta = 0\end{aligned} \tag{46}$$

If $\delta = 0$, i.e., $W = 0$, this equation becomes

$$40\Delta t^3 s^3 + 1680\Delta t s = 0 \tag{47}$$

then, one zero is the origin and two zeros are on the imaginary axis as follows:

$$z_1 = 0 \tag{48}$$

$$z_{2,3} = \pm\frac{\sqrt{42}}{\Delta t} j \tag{49}$$

We can apply Routh-Hurwitz stability criterion to find the range of δ for the closed loop system to be a minimum phase one. Substituting Eq. (42) into Eq. (46) gives

$$\begin{aligned}&s^4 + \left(\frac{10\hat{\gamma}^2\Delta t^3}{RW} + \frac{20}{\Delta t}\right)s^3 + \frac{180}{\Delta t^2}s^2\\&+\left(\frac{420\hat{\gamma}^2\Delta t}{RW} + \frac{840}{\Delta t^3}\right)s + \frac{1680}{\Delta t^4} = 0\end{aligned} \tag{50}$$

Applying the Routh-Hurwitz stability criterion for the above equation with the fact that R and W are always greater than zero gives

$$\begin{aligned}\hat{\gamma}^4\Delta t^8 + 4R\hat{\gamma}^2 W\Delta t^4 + 4R^2W^2 &> 0\\ \hat{\gamma}^2\Delta t^4 + 2RW &> 0\end{aligned}$$

Finally, we obtain the following inequality:

$$W > 0$$

However, if W is set to a very small positive number, some zeros are very close to the imaginary axis. In addition, the zeros are the poles of the controller, so the control law itself is not asymptotically stable. As W become smaller, the poles of the controller are getting closer to the imaginary axis. To avoid this dangerous case, we need to set W as large as possible. In actuality W is determined by applying an output covariance condition of the measurement residual.[2] A value that is too large neglects the current measurement, which in turn will over correct model error predictions leading to a suboptimal control design in MECS. Therefore, a tradeoff between robustness and response characteristics must be weighted in the MECS design.

The Benchmark Problem

In this section, a MECS design is given for the benchmark problem using an approximated receding-horizon control scheme.[6] The benchmark problem, which is two masses linked by spring, is as follows:[5]

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & 0 & 0\\ k/m_2 & -k/m_2 & 0 & 0\end{bmatrix}\mathbf{x} + \begin{bmatrix}0\\0\\1/m_1\\0\end{bmatrix}u \tag{51}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0\end{bmatrix}\mathbf{x} \tag{52}$$

Applying the one-step ahead predictive schme for this system results in the unstable estimator. Therefore, the receding-horizon scheme is used.

Approximate Receding-Horizon Control

Consider the nonlinear system, which is given by Eqs. (4a) and (4b). The receding-horizon problem is set up as follows:[6]

$$\begin{aligned}\min_{\hat{\mathbf{u}}} J\left[\hat{\mathbf{x}}(t), t, \hat{\mathbf{u}}\right] = \min_{\hat{\mathbf{u}}}\frac{1}{2}\int_t^{t+T}&\left[\mathbf{e}^T(\tau)R\mathbf{e}(\tau)\right.\\ &\left.+\hat{\mathbf{u}}^T(\tau)W\hat{\mathbf{u}}(\tau)\right]\end{aligned} \tag{53}$$

subject to the state model in Eqs. (4a) and (4b), and

$$\mathbf{e}(t+T) = 0 \tag{54}$$

where $\mathbf{e}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t)$. The measurement is assumed to be constant during $[t,\ t+T]$ and $\hat{\mathbf{y}}(t+kh)$ is approximated by a recursive first order Taylor series expansion, where $1 \le k \le N$, and N is some integer and $h \equiv T/N$. The above optimization problem is approximated using N approximation values of the estimation error. Combining the measurement and the predictions of $\hat{\mathbf{y}}(t+kh)$, we obtain the prediction of the tracking error

$$\begin{aligned}\mathbf{e}(t+kh) &= \mathbf{y}(t+kh) - \hat{\mathbf{y}}(t+kh)\\ &\approx \mathbf{e}(t) + h\sum_{i=0}^{k-1} C(I+hA)^i\mathbf{f}\\ &+ h\sum_{i=0}^{k-1}\{C(I+hA)^i G\hat{\mathbf{u}}(t+(k-1-i)h)\\ &-(1+hp)^i p\hat{\mathbf{y}}(t)\}\end{aligned} \tag{55}$$

where $C = \partial\mathbf{c}/\partial\hat{\mathbf{x}}$, $A = \partial\mathbf{f}/\partial\hat{\mathbf{x}}$ and $p \equiv d/dt$. Finally, the cost function in Eq. (53) is approximated by a trapezoidal or Simpson's rule, which is written as a quadratic function of $\mathbf{v}_0 = \text{col}\{\hat{\mathbf{u}}(t), \hat{\mathbf{u}}(t+h), \cdots, \hat{\mathbf{u}}(t+(N-1)h)\}$ as follows:

$$\bar{J} = \frac{1}{2}\mathbf{v}_0^T H_0(\hat{\mathbf{x}})\mathbf{v}_0 + \mathbf{g}_0^T(\hat{\mathbf{x}})\mathbf{v}_0 + q_0(\mathbf{e}, \mathbf{y}, \hat{\mathbf{y}}) \tag{56}$$

and is subject to the constraint $\mathbf{e}(t+Nh) = 0$. This approximated formulation leads to the following Quadratic Programming Problem (QPP)

$$M^T(\hat{\mathbf{x}})\mathbf{v}_0 = \mathbf{d}(\mathbf{e}, \hat{\mathbf{x}}, \hat{\mathbf{y}}) \tag{57}$$

where

$$M^T = C\left[(I+hA)^{N-1}G, \cdots, (I+hA)G, G\right] \tag{58a}$$

$$\mathbf{d} = -\frac{1}{h}\mathbf{e} - \sum_{i=0}^{N-1}\left[C(I+hA)^i\mathbf{f} - (1+hp)^i p\hat{y}(t)\right] \tag{58b}$$

and the solution is given by

$$\mathbf{v}_0 = -\left[H_0^{-1} - H_0^{-1}M\left(M^T H_0^{-1} M\right)^{-1} M^T H_0^{-1}\right]\mathbf{g}_0 + \left[H_0^{-1}M\left(M^T H_0^{-1} M\right)^{-1}\right]\mathbf{d} \tag{59}$$

The first q (the dimension of control) equations in Eq. (59) give a closed-loop output-tracking control law, which is $\hat{\mathbf{u}}(t)$. More details can be found in Ref. [6].

MECS with Approximate Receding-Horizon

Since the relative degree of the benchmark problem is four, the minimum number of N for the existence of the control is four.[6] The assumed model is given by

$$\dot{\hat{x}} = A\hat{\mathbf{x}} + B\bar{u} \tag{60a}$$

$$\hat{y} = C\hat{\mathbf{x}} \tag{60b}$$

where

$$A \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\hat{k}/\hat{m}_1 & \hat{k}/\hat{m}_1 & 0 & 0 \\ \hat{k}/\hat{m}_2 & -\hat{k}/\hat{m}_2 & 0 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 0 \\ 0 \\ 1/\hat{m}_1 \\ 0 \end{bmatrix}, \quad C \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

The design procedure is the same as the one for the mass-spring-damper system. The control input, $\bar{u}(t)$, is derived using SMC. The sliding mode control design for the benchmark problem is adopted from Ref. [8]. The equivalent control is given by

$$u_{eq} = -\left(CA^3B\right)^{-1} CA^4\hat{\mathbf{x}} \tag{61}$$

and the correction control of SMC is divided into two parts as follows:

$$u_{cr} = -\left(CA^3B\right)^{-1} \sum_{k=0}^{3} \alpha_k CA^k \hat{\mathbf{x}} + \tilde{V} \tag{62}$$

where α_k is determined by the following desired closed loop dynamics

$$(s+\lambda)^4 = s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 \tag{63a}$$

$$\tilde{V} = -\eta\left(CA^3B\right)^{-1} \mathrm{sat}\left(\frac{\hat{y}}{\rho}\right) \tag{63b}$$

As a result, the sliding mode control input, $\bar{u}$, is given by

$$\bar{u} = u_{eq} + u_{cr} \tag{64}$$

More details on the design procedure can be found in Ref. [8].

Table 1 Tracking Error

p	SMC [%]	SMC+MECS [%]
-0.9	∞	bounded
-0.7	543.478	85.403
-0.5	159.211	19.759
-0.3	51.555	6.570
0.0	3.139	3.261
0.3	17.811	3.697
0.5	67.028	3.886
0.7	122.585	3.956
0.9	187.171	3.968

The control input, $\hat{u}(t)$, used to correct the model errors is determined by Eq. (59). For $N = 4$ each term in Eqs. (58a), (58b) and (59) is given by

$$G = B \tag{65a}$$

$$M^T = C\left[\ (I+NA)^3B\ \cdots\ (I+NA)B\ B\right] \tag{65b}$$

$$\mathbf{f} = A\hat{\mathbf{x}} \tag{65c}$$

$$H_0 = \mathrm{diag}\left[\ R + \frac{T}{3N}\left(\frac{\hat{k}h^4}{\hat{m}_1\hat{m}_2}\right)^2 \quad R \quad R \quad R\ \right] \tag{65d}$$

$$\mathbf{g}_0 = \left[\ \frac{T}{3N}\Gamma(\hat{\mathbf{x}}) + W \quad W \quad W \quad W\right]^T \tag{65e}$$

where $\Gamma(\hat{\mathbf{x}})$ is the first order term of $\hat{u}$ in $e(t+4h)$. Finally, the model error correction control term is given by the first equation in $\mathbf{v}_0$. The estimator with model error correction is given by

$$\dot{\hat{x}} = A\hat{\mathbf{x}} + B(u + \hat{u}) \tag{66}$$

Also, the control input is given by the same equation as the mass-spring-damper system case, i.e., Eq. (19).

Simulations

Mass-Spring-Damper

The system parameters are given as follows: $m = 1$ kg, $c = 0.3$ N sec/m, and $k = 1$ N/m. Also, the estimated system parameters are $\hat{m} = m(1+p)$, $\hat{c} = c(1+p)$, and $\hat{k} = k(1+p)$, where p is a parameter for uncertainty. The control gains for the SMC are $\lambda = 0.2$, $\eta = 1$, and $\rho = 0.01$. In addition, the weights and the time delay are given by $R = 5 \times 10^{-7}$, $W = 1 \times 10^{-2}$, and $\Delta t = 0.0025$ sec, which is a 400Hz sampling rate. Simulations were performed using a Runge-Kutta fifth order integration routine in MATLAB with relative tolerance and absolute tolerance set to 1×10^{-6} and 1×10^{-9}, respectively.

The tracking error is defined by

$$\text{Tracking Error [\%]} \equiv \frac{\max|y_{ref} - y|}{\max(|y_{ref}|)} \times 100 \tag{67}$$

As shown in Table 1, when SMC is combined with MECS, the robustness is significantly improved. The simulations for $p = 0.5$ are shown in Figures 2-4. Figure 2 shows that the tracking error for SMC with

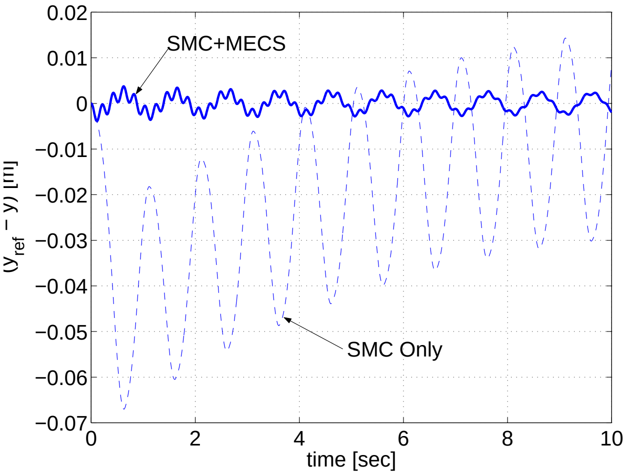

Fig. 2 Tracking Errors for $p = 0.5$

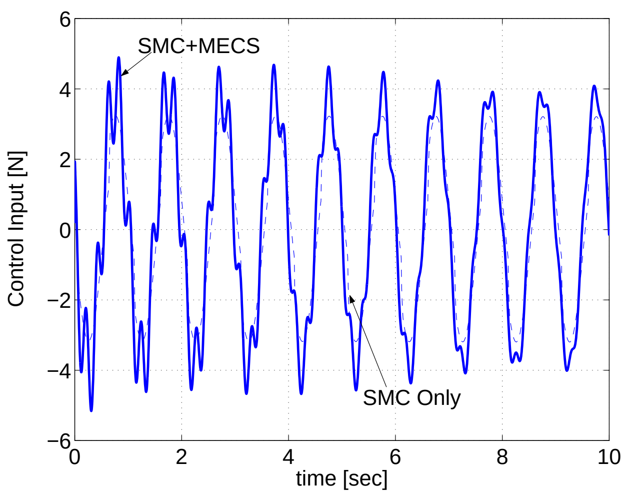

Fig. 3 Control Inputs for $p = 0.5$

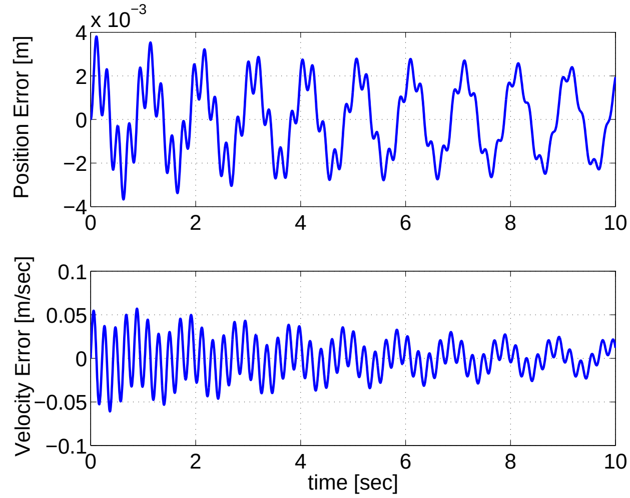

Fig. 4 Estimation Errors for $p = 0.5$

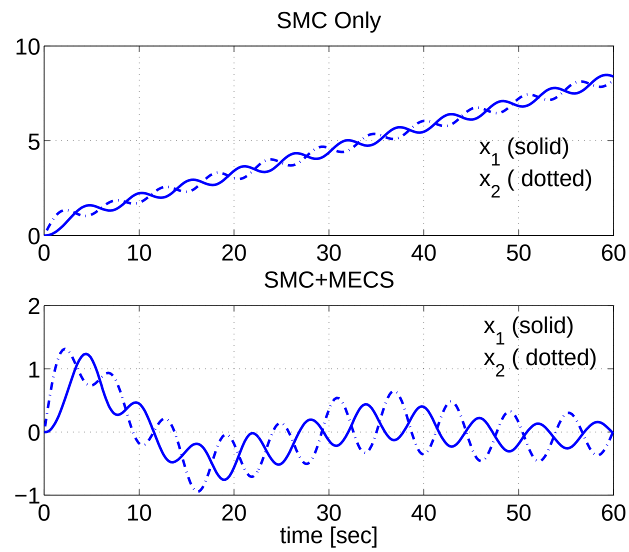

Fig. 5 Displacement Trajectories for Benchmark Problem

MECS is bounded with a smaller value than the error of the SMC only case. Control effort for SMC with MECS is shown in Figure 3. The control effort is not much larger than the one of SMC only. Also, the estimation errors are shown in Figure 4. The position and the velocity estimation errors are about 2.660% and 8.212% compared to each maximum value of the true state.

Benchmark Problem

The system parameters are given by $m_1 = 1.2$ kg, $m_2 = 1.2$ kg, and $k = 0.8$ N/m. Also, the estimated system parameters are $\hat{m_1} = \hat{m_2} = 1$ kg, $\hat{k} = 1$ N/m, which is the worst case scenario.[9] The control gains for the SMC are $\lambda = 1$, $\eta = 0.1$, and $\rho = 0.02$. In addition, the weights and the time interval are given by $R = 1$, $W = 1 \times 10^{-5}$, $T = 5$ sec, $N = 4$ and the initial condition is $\mathbf{x} = [\ 0\ 0\ 0\ 1]^T$. As shown in Figure 5, for the SMC only case the trajectories diverge, which is due to the inadequacy of the SMC to handle the model errors. However, for the SMC with MECS case the trajectories remain around the zero position even though significant model error exists. The control inputs for each case are shown in Figure 6. The control input for SMC with MECS is larger than the SMC only case; however, it is still well within acceptable limits.[5] Next, an external disturbance of $0.3\sin(0.2\pi t)$ is applied at the first mass. The results are shown in Figures 7 and and 8. Clearly, MECS with SMC is able to effectively reject external disturbances, while the SMC only case diverges.

Conclusions

A Model Error Control Synthesis approach was applied to the second order mass-damper-spring system. A robustness analysis was performed and stability was shown using a Padé approximation for the time delay in the control design. As a result, we showed that the

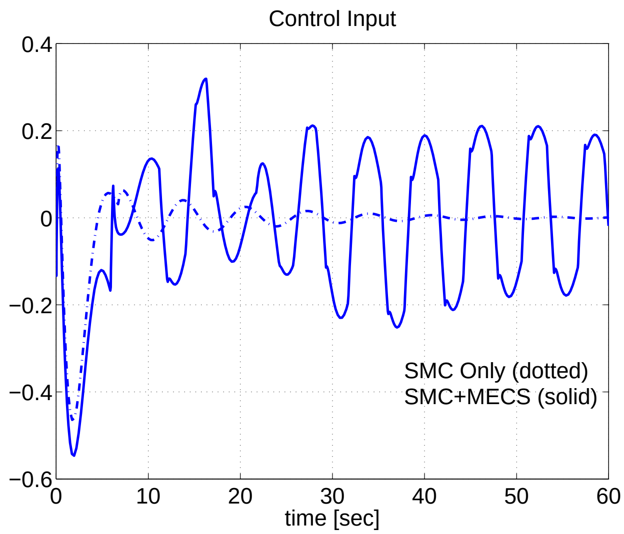

Fig. 6 Control Inputs for Benchmark Problem

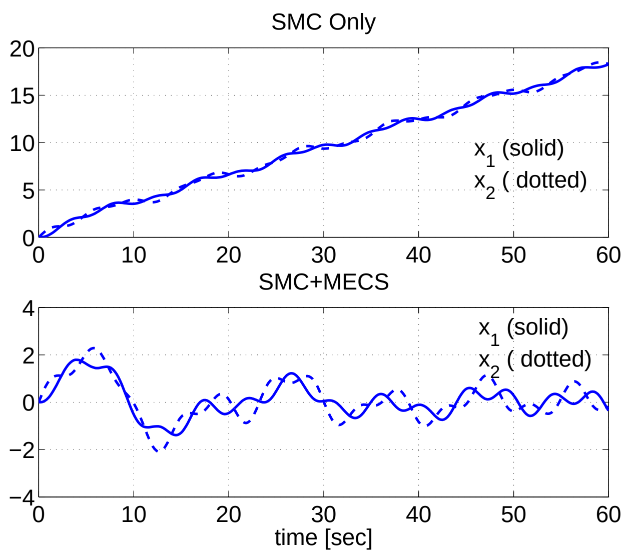

Fig. 7 External Disturbance Responses for Benchmark Problem

weighting matrix for the model error correction cannot be set to zero because of the system zeros and an instability in the control law. Also, using an approximate receding-horizon control scheme, a MECS control system has been designed for the benchmark problem. Simulation results showed that the performance of MECS with SMC was robust with respect to a wide range of system parameter variations and external disturbance inputs.

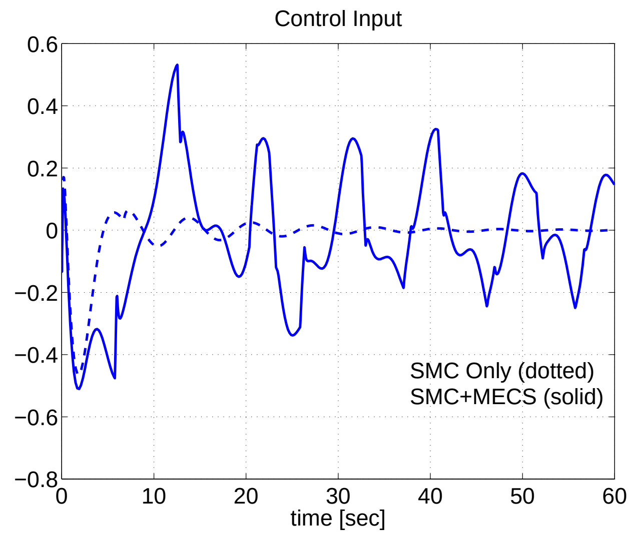

Fig. 8 Control Inputs with External Disturbance for Benchmark Problem

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