Optimal Variable-Structure Control Tracking of Spacecraft Maneuvers

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Introduction

In recent years, much effort has been devoted to the closed-loop design of spacecraft with large angle slews. Vadali and Junkins[1] and Wie and Barba[2] derive a number of simple control schemes using quaternion and angular velocity (rate) feedback. Other full state feedback techniques have been developed that are based on variable-structure (sliding-mode) control, which uses a feedback linearizing technique and an additional term aimed at dealing with model uncertainty. A variable-structure controller has been developed for the regulation of spacecraft maneuvers using a Gibbs vector parameterization,[3] a modified-Rodrigues parameterization,[4] and a quaternion parameterization.[5] In both [2] and [5], a term was added so that the spacecraft maneuver follows the shortest path and requires the least amount of control torque. The variable-structure control approach using a quaternion parameterization has been recently expanded to the attitude tracking case.[6,7] However, these controllers do not take into account the shortest possible path as shown in Refs. [2] and [5].

This note expands upon the results in Ref. [5] to provide an optimal control law for asymptotic tracking of spacecraft maneuvers using variable-structure control. It also provides new insight using a simple term in the control law to produce a maneuver to the reference attitude trajectory in the shortest distance. Controllers are derived for either external torque inputs or reaction wheels.

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Background

In this section, a brief review of the kinematic and dynamic equations of motion for a three-axis stabilized spacecraft is shown. The attitude is assumed to be represented by the quaternion, defined as $\boldsymbol{q} \equiv \begin{bmatrix} \boldsymbol{q}_{13}^T & q_4 \end{bmatrix}^T$, with $\boldsymbol{q}_{13} \equiv \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T = \hat{\boldsymbol{n}} \sin(\Phi/2)$ and $q_4 = \cos(\Phi/2)$, where $\hat{\boldsymbol{n}}$ is a unit vector corresponding to the axis of rotation and Φ is the angle of rotation. The quaternion kinematic equations of motion are derived by using the spacecraft's angular velocity ($\boldsymbol{\omega}$), given by

$$\dot{\boldsymbol{q}} = \frac{1}{2}\Omega(\boldsymbol{\omega})\boldsymbol{q} = \frac{1}{2}\Xi(\boldsymbol{q})\boldsymbol{\omega} \tag{1}$$

where $\Omega(\boldsymbol{\omega})$ and $\Xi(\boldsymbol{q})$ are defined as

$$\Omega(\boldsymbol{\omega}) \equiv \begin{bmatrix} -[\boldsymbol{\omega}\times] & \vdots & \boldsymbol{\omega} \\ \cdots\cdots & \vdots & \cdots\cdots \\ -\boldsymbol{\omega}^T & \vdots & 0 \end{bmatrix}, \quad \Xi(\boldsymbol{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{q}_{13}\times] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -\boldsymbol{q}_{13}^T \end{bmatrix} \tag{2}$$

and $I_{n\times n}$ represents an $n \times n$ identity matrix. The 3×3 dimensional matrices $[\boldsymbol{\omega}\times]$ and $[\boldsymbol{q}_{13}\times]$ are referred to as cross product matrices since $\boldsymbol{a}\times\boldsymbol{b} = [\boldsymbol{a}\times]\boldsymbol{b}$ (see Ref. [8]).

Since a three degree-of-freedom attitude system is represented by a four-dimensional vector, the quaternion components cannot be independent. This condition leads to the following normalization constraint $\boldsymbol{q}^T\boldsymbol{q} = \boldsymbol{q}_{13}^T\boldsymbol{q}_{13} + q_4^2 = 1$. Also, the error quaternion between two quaternions, $\boldsymbol{q}$ and $\boldsymbol{q}_d$, is defined by

$$\boldsymbol{\delta q} \equiv \begin{bmatrix} \boldsymbol{\delta q}_{13} \\ \cdots\cdots \\ \delta q_4 \end{bmatrix} = \boldsymbol{q} \otimes \boldsymbol{q}_d^{-1} \tag{3}$$

where the operator $\otimes$ denotes quaternion multiplication (see Ref. [8] for details), and the inverse quaternion is defined by $\boldsymbol{q}_d^{-1} = \begin{bmatrix} -q_{d_1} & -q_{d_2} & -q_{d_3} & q_{d_4} \end{bmatrix}^T$. Other useful identities are given by $\boldsymbol{\delta q}_{13} = \Xi^T(\boldsymbol{q}_d)\boldsymbol{q}$ and $\delta q_4 = \boldsymbol{q}^T\boldsymbol{q}_d$. Also, if Equation (3) represents a small rotation then $\delta q_4 \approx 1$, and $\boldsymbol{\delta q}_{13}$ corresponds to the half-angles of rotation.

The dynamic equations of motion, also known as Euler's equations, for a rotating spacecraft are given by

$$J\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times (J\boldsymbol{\omega}) + \boldsymbol{u} \tag{4}$$

where J is the inertia matrix of the spacecraft, and $\boldsymbol{u}$ is the total external torque input. If the spacecraft is equipped with 3 orthogonal reaction or momentum wheels, then Euler's equations become:

$$(J - \bar{J})\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times (J\boldsymbol{\omega} + \bar{J}\bar{\boldsymbol{\omega}}) - \bar{\boldsymbol{u}} \tag{5a}$$

$$\bar{J}(\dot{\bar{\boldsymbol{\omega}}} + \dot{\boldsymbol{\omega}}) = \bar{\boldsymbol{u}} \tag{5b}$$

where $\bar{J}$ is the diagonal inertia matrix of the wheels, J now includes the mass of the wheels, $\bar{\omega}$ is the wheel angular velocity vector relative to the spacecraft, and $\bar{\boldsymbol{u}}$ is the wheel torque vector.

Selection of Switching Surfaces

Optimal Control Analysis

We first consider a sliding manifold for a purely kinematic relationship, with $\boldsymbol{\omega}$ as the control input. The variable-structure control design is used to track a desired quaternion $\boldsymbol{q}_d$ and corresponding angular velocity $\boldsymbol{\omega}_d$. As shown previously for regulation,[5] under ideal sliding conditions, the trajectory in the state-space moves on the sliding manifold. For tracking, the following loss function is minimized to determine the optimal switching surfaces:

$$\Pi(\boldsymbol{\omega}) = \frac{1}{2}\int_{t_s}^{\infty} \left[\rho\, \boldsymbol{\delta q}_{13}^T \boldsymbol{\delta q}_{13} + (\boldsymbol{\omega} - \boldsymbol{\omega}_d)^T (\boldsymbol{\omega} - \boldsymbol{\omega}_d) \right] dt \tag{6}$$

subject to the bilinear system constraint given in Equation (1). Note that ρ is a scalar gain and t_s is the time of arrival at the sliding manifold. Minimization of Equation (6) subject to Equation (1) leads to the following two-point-boundary-value-problem:

$$\dot{\boldsymbol{q}} = \frac{1}{2}\Xi(\boldsymbol{q})\boldsymbol{\omega} \tag{7a}$$

$$\dot{\boldsymbol{\lambda}} = -\rho\,\Xi(\boldsymbol{q}_d)\Xi^T(\boldsymbol{q}_d)\boldsymbol{q} + \frac{1}{2}\Xi(\boldsymbol{\lambda})\boldsymbol{\omega} \tag{7b}$$

where $\boldsymbol{\lambda}$ is the co-state vector. The optimal $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = -\frac{1}{2}\Xi^T(\boldsymbol{q})\boldsymbol{\lambda} + \boldsymbol{\omega}_d \tag{8}$$

Next, the following sliding vector is chosen:

$$(\boldsymbol{\omega}-\boldsymbol{\omega}_d)+k\,\Xi^T(\boldsymbol{q}_d)\boldsymbol{q}=\mathbf{0} \tag{9}$$

where k is a scalar gain. The sliding vector is optimal if the solution of Equation (9) minimizes Equation (6). This can be proven by first substituting Equation (9) into Equation (8), yielding

$$\boldsymbol{\lambda}=-2k\,\boldsymbol{q}_d \tag{10}$$

Next, using the fact that the desired quaternion can be obtained from the following

$$\dot{\boldsymbol{q}}_d=\frac{1}{2}\Xi(\boldsymbol{q}_d)\omega_d \tag{11}$$

leads directly to

$$\dot{\boldsymbol{\lambda}}=-k\,\Xi(\boldsymbol{q}_d)\boldsymbol{\omega}_d \tag{12}$$

Comparing Equation (12) to Equation (7b), and using Equation (9) now leads to the following relationship:

$$-k\,\Xi(\boldsymbol{q}_d)\boldsymbol{\omega}_d=-\rho\,\Xi(\boldsymbol{q}_d)\Xi^T(\boldsymbol{q}_d)\boldsymbol{q}-k\,\Xi(\boldsymbol{q}_d)\boldsymbol{\omega}_d+k^2\Xi(\boldsymbol{q}_d)\Xi^T(\boldsymbol{q}_d)\boldsymbol{q} \tag{13}$$

Equation (13) is satisfied for $k=\pm\sqrt{\rho}$. Therefore, the sliding condition in Equation (9) leads to an optimal solution (i.e., minimum Π in Equation (6)).

For this special case, it can be shown that the value of the loss function in Equation (6) is given by

$$\Pi^*=2k\left[1-\delta q_4(t_s)\right] \tag{14}$$

where k must now be strictly positive. Since δq_4 corresponds directly to the cosine of half the angle error of rotation, both $\boldsymbol{\delta q}$ and $-\boldsymbol{\delta q}$ represent the same rotation; however, the value of the loss function in Equation (14) is significantly different for each rotation. One rotation $(\boldsymbol{\delta q})$ gives the shortest distance to the sliding manifold, while the other $(-\boldsymbol{\delta q})$ gives the longest distance. In order to give the shortest possible distance the following sliding vector is chosen:

$$(\boldsymbol{\omega}-\boldsymbol{\omega}_d)+k\,\mathrm{sgn}\left[\delta q_4(t_s)\right]\Xi^T(\boldsymbol{q}_d)\boldsymbol{q}=\mathbf{0} \tag{15}$$

For a discussion on using $\left[\delta q_4(t_s)\right]$ in a control law see Refs. [2] and [5]. Using this sliding condition leads to the following value for the loss function:

$$\Pi^* = 2k\left[1 - \left|\delta q_4(t_s)\right|\right] \tag{16}$$

which yields a minimal value for any rotation.

Lyapunov Analysis

The sliding vector shown in Equation (15) can also be shown to be stable using a Lyapunov analysis. The time derivative of $\boldsymbol{\delta q}_{13}$ can be shown to be given by

$$\boldsymbol{\delta \dot{q}}_{13} = \frac{1}{2}\delta q_4(\boldsymbol{\omega} - \boldsymbol{\omega}_d) + \frac{1}{2}[\boldsymbol{\delta q}_{13}\times](\boldsymbol{\omega} + \boldsymbol{\omega}_d) \tag{17}$$

Next, the following candidate Lyapunov function is chosen:

$$V_s = \frac{1}{2}\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13} \tag{18}$$

Using the sliding vector in Equation (15), the time derivative of Equation (18) is given by

$$\dot{V}_s = -\frac{1}{2}k\left|\delta q_4\right|\boldsymbol{\delta q}_{13}^T\boldsymbol{\delta q}_{13} \le 0 \tag{19}$$

Hence, V_s is indeed a Lyapunov function for $k > 0$. This analysis generalizes the results shown in Ref. [7], where the spacecraft's attitude is restricted in the workspace defined with $q_4 > 0$.

Variable-Structure Tracking

The previous section showed the effectiveness of using $\text{sgn}\left[\delta q_4(t_s)\right]$ in the sliding manifold. In this section, we consider a variable-structure controller for the complete system (i.e., including the dynamics). The goal of the variable-structure controller is to track a desired quaternion $\boldsymbol{q}_d$ and corresponding angular velocity $\boldsymbol{\omega}_d$. The variable-structure control design with external torques only is given by (note that $\text{sgn}\left[\delta q_4(t)\right]$ is now being used instead of $\text{sgn}\left[\delta q_4(t_s)\right]$)

$$\boldsymbol{u} = [\boldsymbol{\omega}\times]J\,\boldsymbol{\omega} + J\left\{\frac{1}{2}k\,\text{sgn}(\delta q_4)\left[\Xi^T(\boldsymbol{q})\Xi(\boldsymbol{q}_d)\boldsymbol{\omega}_d - \Xi^T(\boldsymbol{q}_d)\Xi(\boldsymbol{q})\boldsymbol{\omega}\right] + \dot{\boldsymbol{\omega}}_d - G\boldsymbol{\vartheta}\right\} \tag{20}$$

where G is a 3×3 positive definite, diagonal matrix, and the i^{th} component of $\boldsymbol{\vartheta}$ is given by

$$\vartheta_i = \text{sat}(s_i, \varepsilon_i), \qquad i = 1, 2, 3 \tag{21}$$

As stated previously, the term $\operatorname{sgn}(\delta q_4)$ is used to drive the system to the desired trajectory in the shortest distance. The saturation function is used to minimize chattering in the control torques. This function is defined by

$$\operatorname{sat}(s_i, \varepsilon_i) \equiv \begin{cases} 1 & \text{for} \quad s_i > \varepsilon_i \\ \dfrac{s_i}{\varepsilon} & \text{for} \quad |s_i| \le \varepsilon_i \\ -1 & \text{for} \quad s_i < -\varepsilon_i \end{cases} \qquad i = 1, 2, 3 \tag{22}$$

where ε is a small positive quantity. The sliding manifold is given by

$$\boldsymbol{s} = (\boldsymbol{\omega} - \boldsymbol{\omega}_d) + k \operatorname{sgn}(\delta q_4) \Xi^T (\boldsymbol{q}_d) \boldsymbol{q} \tag{23}$$

The stability of the closed-loop system using this controller can be evaluated using the following candidate Lyapunov function

$$V = \frac{1}{2} \boldsymbol{s}^T \boldsymbol{s} \tag{24}$$

Using Equations (4), (20) and (23) the time-derivative of Equation (24) can be shown to be given by $\dot{V} = -\boldsymbol{s}^T G \boldsymbol{\vartheta}$, which is always less than or equal to zero as long as G is positive definite. Hence, stability is proven.

If wheels are used to control the spacecraft, then the sliding mode controller is given by the following:

$$\begin{aligned} \bar{\boldsymbol{u}} = (J - \bar{J}) \left\{ \frac{1}{2} k \operatorname{sgn}(\delta q_4) \left[\Xi^T (\boldsymbol{q}_d) \Xi(\boldsymbol{q}) \boldsymbol{\omega} - \Xi^T (\boldsymbol{q}) \Xi(\boldsymbol{q}_d) \boldsymbol{\omega}_d \right] - \dot{\boldsymbol{\omega}}_d + G \boldsymbol{\vartheta} \right\} \\ - [\boldsymbol{\omega} \times] (J \boldsymbol{\omega} + \bar{J} \bar{\boldsymbol{\omega}}) \end{aligned} \tag{25}$$

The stability of the control system with wheels in Equations (5) and (25) can also be easily proven using the Lyapunov function in Equation (24).

Analysis

In this section an analysis of using $\operatorname{sgn}(\delta q_4)$ for all times in the control law is shown. We first assume that the desired angular velocity is zero $(\boldsymbol{\omega}_d = \mathbf{0})$ and that the matrix G is given by a scalar times the identity matrix $(g\, I_{3\times 3})$. We further assume that the thickness of the boundary layer ε and the gain g are sufficiently large so that

$$G\boldsymbol{\vartheta} = \beta\boldsymbol{\omega} + \beta k \,\mathrm{sgn}(\delta q_4)\boldsymbol{\delta q}_{13} \tag{26}$$

where $\beta = g/\varepsilon$. Using Equations (4), (17) and (20), the closed-loop dynamics for $\boldsymbol{\omega}$ now become

$$\dot{\boldsymbol{\omega}} = -\frac{1}{2}k\,\mathrm{sgn}(\delta q_4)\left\{\delta q_4 I_{3\times3} + [\boldsymbol{\delta q}_{13}\times]\right\}\boldsymbol{\omega} - \beta\boldsymbol{\omega} - \beta k\,\mathrm{sgn}(\delta q_4)\boldsymbol{\delta q}_{13} \tag{27}$$

Taking two time derivatives of $\delta q_4 = \boldsymbol{q}^T\boldsymbol{q}_d$, and using both Equation (27) and the quaternion constraint equation $\boldsymbol{\delta q}^T\boldsymbol{\delta q} = 1$ yields

$$\delta\ddot{q}_4 + \left(\frac{1}{2}k|\delta q_4| + \beta\right)\delta\dot{q}_4 + \left(\frac{1}{2}\beta k|\delta q_4| + \frac{1}{4}\boldsymbol{\omega}^T\boldsymbol{\omega}\right)\delta q_4 = \frac{1}{2}\beta k\,\mathrm{sgn}(\delta q_4) \tag{28}$$

Equation (28) represents a second-order nonlinear spring-mass-damper type system with an exogenous step input. The stability of Equation (28) can be evaluated by considering the following candidate Lyapunov function

$$V_{\delta q_4} = \frac{1}{2}\delta\dot{q}_4^2 + \frac{1}{2}\left(\frac{1}{4}\boldsymbol{\omega}^T\boldsymbol{\omega}\right)\delta q_4^2 + \frac{1}{2}\beta k\left[1 - \delta q_4\,\mathrm{sgn}(\delta q_4)\right] \tag{29}$$

The last term in Equation (29) is always greater or equal to zero since $\beta > 0$, $k > 0$, and $0 \le \delta q_4\,\mathrm{sgn}(\delta q_4) \le 1$. Taking the time-derivative of Equation (29), and using Equations (27) and (28) gives

$$\dot{V}_{\delta q_4} = -\left(\frac{1}{2}k|\delta q_4| + \beta\right)\delta\dot{q}_4^2 - \frac{1}{4}\left(\beta + \frac{1}{2}k|\delta q_4|\right)\left(\boldsymbol{\omega}^T\boldsymbol{\omega}\right)\delta q_4^2 \tag{30}$$

Hence, since $\beta > 0$ and $k > 0$, Equation (29) is indeed a Lyapunov function. The advantage of using $\mathrm{sgn}(\delta q_4)$ in the control law at all times (even before the sliding manifold is reached) now becomes clear. The step input in Equation (28) is a function of $\mathrm{sgn}(\delta q_4)$. Therefore, the response for δq_4 will approach $\mathrm{sgn}(\delta q_4)$ for any initial condition. This tends to drive the system to the desired location in the shortest distance. Furthermore, this inherently takes into account the rate errors as well. For example say that $\delta q_4(t_0)$ is positive, and that a high initial rate is given which tends to drive the system away from $\delta q_4 = 1$. The control law will automatically begin to null the rate. But, if the initial rate is large enough and the control dynamics are relatively slow, then δq_4 may become negative. Since $\mathrm{sgn}(\delta q_4)$ is used in the control system, then from Equation (28) the control law will subsequently drive the system towards $\delta q_4 = -1$. Therefore, using $\operatorname{sgn}(\delta q_4)$ at all times produces an optimal response for any type of initial condition error.

Example

A variable-structure controller has been used to control the attitude of the Microwave Anisotropy Probe (MAP) spacecraft from quaternion observations and gyro measurements. Details of the spacecraft parameters and desired attitude and rates can be found in Ref. [9]. The control system has been designed to bring the actual attitude to the desired attitude in less than 20 minutes. For the simulations the spacecraft has been controlled using reaction wheels. The gains used in the control law given by Equation (25) are: $k = 0.015$, $G = 0.0015 I_{3\times 3}$, and $\varepsilon = 0.01$. To illustrate the importance of using $k \operatorname{sgn}[\delta q_4(t)]$, a number of simulations have been run. The initial quaternion is given by $\boldsymbol{q}(t_0) = [0 \quad 0 \quad \sin(\Phi/2) \quad \cos(\Phi/2)]^T \otimes \boldsymbol{q}_d(t_0)$ and the actual velocity has been set to zero. Test cases have been executed using $\Phi = 210^\circ$, 240°, 270°, 300°, and 330°. Table 1 summarizes the results of using $k \operatorname{sgn}[\delta q_4(t)]$ and k only in Equation (25). The final time for the simulation runs is given by $t_f = 20$ minutes. Clearly, by using $k \operatorname{sgn}[\delta q_4(t)]$ in the control law, better performance is achieved in the closed-loop system than using just k.

Conclusions

A new variable-structure controller for optimal spacecraft tracking maneuvers has been shown. The new controller was formulated for both external torque inputs and reaction wheel inputs. Global asymptotic stability was shown using a Lyapunov analysis. New insight was shown for the advantage of using a simple term in the control law (i.e., producing a maneuver to the reference attitude trajectory in the shortest distance). The sliding motion was also shown to be optimal in the sense of a quadratic loss function in the multiplicative error quaternions and angular velocities. Simulation results indicated that the addition of the simple term in the control law always provides an optimal response, so that the reference attitude motion is achieved in the shortest possible distance.

References

[1] Vadali, S.R., and Junkins, J.L., “Optimal Open-Loop and Stable Feedback Control of Rigid Spacecraft Attitude Maneuvers,” *The Journal of the Astronautical Sciences*, Vol. 32, No. 2, April-June 1984, pp. 105-122.

[2] Wie, B., and Barba, P.M., “Quaternion Feedback for Spacecraft Large Angle Maneuvers,” *Journal of Guidance, Control and Dynamics*, Vol. 8, No. 3, May-June 1985, pp. 360-365.

[3] Dwyer, T.A.W., and Sira-Ramirez, H., “Variable Structure Control of Spacecraft Reorientation Maneuvers,” *Journal of Guidance, Control and Dynamics*, Vol. 11, No. 3, May-June 1988, pp. 262-270.

[4] Crassidis, J.L., and Markley, F.L., “Sliding Mode Control Using Modified Rodrigues Parameters,” *Journal of Guidance, Control and Dynamics*, Vol. 19, No. 6, Nov.-Dec. 1996, pp. 1381-1383.

[5] Vadali, S.R., “Variable-Structure Control of Spacecraft Large-Angle Maneuvers,” *Journal of Guidance, Control and Dynamics*, Vol. 9, No. 2, March-April 1986, pp. 235-239.

[6] Robinett, R.D., and Parker, G.G., “Least Squares Sliding Mode Control Tracking of Spacecraft Large Angle Maneuvers,” *Journal of the Astronautical Sciences*, Vol. 45, No. 4, Oct.-Dec. 1997, pp. 433-450.

[7] Lo, S.-C., and Chen, Y.-P., “Smooth Sliding-Mode Control for Spacecraft Attitude Tracking Maneuvers,” *Journal of Guidance, Control and Dynamics*, Vol. 18, No. 6, Nov.-Dec. 1995, pp. 1345-1349.

[8] Shuster, M.D., “A Survey of Attitude Representations,” *The Journal of the Astronautical Sciences*, Vol. 41, No. 4, Oct.-Dec. 1993, pp. 439-517.

[9] Crassidis, J.L., Vadali, S.R., and Markley, F.L., “Optimal Tracking of Spacecraft Using Variable-Structure Control,” *Proceedings of the Flight Mechanics/Estimation Theory Symposium*, (Greenbelt, MD), NASA-Goddard Space Flight Center, May 1999, NASA/CP-1999-209235, pp. 201-214.

Table 1 Cost Function Values for Various Φ

	Value of $\frac{1}{2}\int_0^{t_f} \boldsymbol{s}^T \boldsymbol{s} \; dt$	
Φ (deg)	Gain = k	Gain = $k \operatorname{sgn}\left[\delta q_4(t)\right]$
210	1.0787	0.9313
240	0.8988	0.6543
270	0.6136	0.3566
300	0.3185	0.1300
330	0.1095	0.0194