ROBUST SPACECRAFT ATTITUDE CONTROL USING MODEL-ERROR CONTROL SYNTHESIS

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ABSTRACT

Model-error control synthesis is a nonlinear robust control approach that uses an approximate receding-horizon estimation algorithm to cancel the effects of modelling errors and external disturbances on a system. In this paper the state prediction equations in the approximate receding-horizon algorithm are modified so that the solution provides better performance than the original approach. To verify the results the new approach is applied to the spacecraft attitude control problem with attitude-angle measurements only, i.e., without any angular-velocity measurements. Also, an optimal design scheme is presented to determine the weighting factor and receding-horizon time-length. In addition the closed-loop system is shown to be globally quadratically stable for a norm bounded nonlinear uncertainty. Simulation results are provided to show the performance of the new control approach.

INTRODUCTION

Model-Error Control Synthesis (MECS) is a signal synthesis adaptive control method.[1] Robustness is achieved by applying a correction control, which is determined during the estimation process, to the nominal control vector thereby eliminating the effects of modelling errors at the system output.[2] The model-error vector is estimated by using either a one-step ahead prediction approach,[1,3] or an Approximate Receding-Horizon (ARH) approach.[4] As shown by the benchmark problem example in Ref. [3], the one-step ahead prediction approach inherent in MECS could not stabilize the system, which has one pole at the origin and two poles on the imaginary axis. When using the ARH approach the closed-loop system can be stabilized and the system can tolerate relatively large uncertainties. However, the one-step ahead prediction approach may be easier to design for complicated systems than the ARH approach. Therefore, choosing between the one-step ahead prediction approach or the ARH approach to determine the model error depends on the particular properties and required robustness in the system to be controlled.

In Ref. [1] MECS with the one-step ahead prediction approach is first applied to suppress the wing rock motion of a slender delta wing, which is described by a highly nonlinear differential equation. Results indicated that this approach provides adequate robustness for this particular system. In Ref. [3] a simple study to test the stability of the closed-loop system is presented using a Padé approximation for the time delay, which showed the relation between the system zeros and the weighting in the cost function. The analysis proved that some systems may not be stabilized using the original model-error estimation algorithm, which lead to the ARH approach in the MECS design to determine the model-error vector in the system.[4]

The closed-form solution of the ARH approach using Quadratic Programming (QP) is first presented by Lu.[5] Although the problem is solved from a control standpoint, the algorithm can be reformulated as a filter and estimator problem.[2] The model-error vector is determined by the ARH optimal solution.[4] Using the ARH approach, the capability of MECS is expanded so that unstable non-minimum phase systems can be stabilized. Furthermore, Ref. [4] shows a method to calculate the stable regions with respect to the weighting and the length of receding-horizon step-time using the Hermite-Biehler theorem.[6] After the stable region is found, the weighting and the length of receding-horizon step-time are chosen to minimize the ∞-norm of the sensitivity function.[4]

The ARH solution for an r^{th}-order relative degree system shows that the model-error solution is zero before the end of receding-horizon step-time is reached.

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Some parts of the model-error vector are separated completely from the constraints, so that the optimal solution for those parts are automatically zero. In this paper an extension to the ARH approach is shown. For all model-error elements of each constraint at the time before the end of receding-horizon step-time, the state prediction is substituted by an r^{th}-order Taylor series expansion instead of a repeated first-order expansion in the ARH approach. We call this the Modified Approximate Receding-Horizon (MARH) approach, which leads to an even more robust MECS law than with the ARH solution.

In this paper the MECS approach with the MARH solution is applied to the spacecraft attitude control problem for the case where the only available information is attitude-angle measurements, i.e., with no angular-velocity measurements. In Ref. [7] an adaptive control approach using attitude, based on the Modified Rodrigues Parameters (MRPs), and angular-velocity information has been developed. This approach provides robustness in the system by estimating the inertia matrix and external disturbances through a linear closed-loop dynamics expression. In this paper the same basic non-adaptive portion of the controller in Ref. [7] is used as nominal controller, however, the angular-velocity information is provided using a Kalman filter with attitude measurements only. Furthermore, instead of estimating each element of the inertia matrix and the external disturbance separately, the whole effect of both uncertainties is estimated by the MARH approach through a model-error vector in the dynamics. The MECS approach uses this estimate to subtract the model error from the nominal control input in order to track the desired dynamics in the face of severe inertia and external disturbance errors.

The organization of this paper is as follows. First, the ARH approach to estimate the model-error vector in a system is summarized. Second, the state prediction in the ARH approach is modified using a Taylor series expansion, and the solution is generalized for a standard nonlinear system form. Next, the new approach is applied to the spacecraft attitude control problem. An optimal design scheme is presented to determine the weighting factor and receding-horizon time-length. Also, globally quadratic stability is provided for a norm bounded nonlinear uncertainty. Finally, the results are verified through several simulated cases.

MODIFIED ARH (MARH)

In this section the ARH approach to estimate the model-error vector in a system is first summarized, followed by a motivation for the modified algorithm. Finally, the modified algorithm is generalized for a standard nonlinear system form.

ARH APPROACH

The receding-horizon optimization problem is set up as follows:[5]

$$\min_{\hat{\mathbf{u}}} J\left[\hat{\mathbf{x}}(t),\, t,\, \hat{\mathbf{u}}(t)\right] = \frac{1}{2}\int_t^{t+T} \left[\mathbf{e}^T(\xi)\, R^{-1}(\xi)\, \mathbf{e}(\xi) + \hat{\mathbf{u}}^T(\xi)\, W(\xi)\, \hat{\mathbf{u}}(\xi)\right] d\xi \tag{1}$$

subject to the following:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{f}}\left[\hat{\mathbf{x}}(t)\right] + \hat{B}\left[\hat{\mathbf{x}}(t)\right]\mathbf{u}(t) + \hat{G}\left[\hat{\mathbf{x}}(t)\right]\hat{\mathbf{u}}(t) \tag{2a}$$

$$\hat{\mathbf{y}}(t) = \hat{\mathbf{c}}\left[\hat{\mathbf{x}}(t)\right] \tag{2b}$$

where $\hat{\mathbf{x}}(t) \in X \subset \Re^n$ is the state estimate vector of $\mathbf{x}(t)$, $R^{-1}(\xi)$ and $W(\xi)$ are positive-definite and symmetric weighting matrices for all $\xi \in [t,\, t+T]$, $\hat{\mathbf{f}}\left[\hat{\mathbf{x}}(t)\right] \in \Re^n$ is the assumed model vector, $\hat{B}\left[\hat{\mathbf{x}}(t)\right] \in \Re^{n\times q_u}$ is the assumed control input distribution matrix, $\hat{G}\left[\hat{\mathbf{x}}(t)\right] \in \Re^{n\times q_w}$ is the model-error distribution matrix, $\mathbf{u}(t) \in \Omega_u \subset \Re^{q_u}$ is the control input, $\hat{\mathbf{u}}(t) \in \Omega_{\hat{u}} \subset \Re^{q_w}$ is the to-be-determined model error, which also includes external disturbances, $\hat{\mathbf{c}}\left[\hat{\mathbf{x}}(t)\right] \in \Re^m$ is the measurement vector ($m \leq n$ in general), and $\hat{\mathbf{y}}(t) \in \Re^m$ is the estimated output vector.[5] Also, we assume that a unique solution for $\hat{\mathbf{x}}(t)$ exists, and $\mathbf{e}(t+T) = 0$ where the residual error is defined by

$$\mathbf{e}(t) = \tilde{\mathbf{y}}(t) - \hat{\mathbf{y}}(t) \tag{3}$$

where $\tilde{\mathbf{y}}(t)$ is the measurement. Note that T is the receding-horizon optimization-interval, which is not the sampling interval in general.

For most mechanical systems $\Omega_u \subset \Omega_{\hat{u}}$, i.e., the system is under-actuated or fully actuated at the maximum, so that $q_u \leq q_w$, where q_w is the dimension of the dynamics parts. The admissible sets X and $\Omega_u \subset \Omega_{\hat{u}}$ are compact and $X \times \Omega_{\hat{u}}$ contains a neighborhood around the origin. One important assumption is $m \geq q_w$, i.e., the dimension of the measurement vector is at least the dimension of the dynamics. Also, we assume that the rank of $\hat{G}\left[\hat{\mathbf{x}}(t)\right]$ is q_w, i.e., full rank. In addition controllability, observability, stable zero dynamics, and well-defined relative degree with respect to $\hat{\mathbf{u}}(t)$ are presumed, and the assumptions about continuity and $\hat{\mathbf{f}}\left(\mathbf{0}\right) = \mathbf{0}$ hold. Finally, we assume that each element of the model-error vector affects the output.

State-observable measurements are assumed for Eq. (2b) in the following form:

$$\tilde{\mathbf{y}}(t) = \mathbf{c}[\mathbf{x}(t)] + \mathbf{v}(t) \tag{4}$$

where $\tilde{\mathbf{y}}(t) \in \Re^m$ is the measurement vector at time t, and $\mathbf{v}(t) \in \Re^m$ is the measurement noise vector, which is assumed to be a zero-mean, stationary, Gaussian noise distributed process with $E\left\{\mathbf{v}(t)\right\} = \mathbf{0}$ and $E\left\{\mathbf{v}(t)\,\mathbf{v}^T(t+\Delta t)\right\} = R_v\,\delta(\Delta t)$, where $E\{\cdot\}$ is expectation, $R_v \in \Re^{m\times m}$ is a positive-definite symmetric covariance matrix, $\delta(\cdot)$ is Dirac delta function, and Δt is the sampling rate for the discrete-time measurement case.

At each time t, the model-error solution $\hat{\mathbf{u}}$ over a finite horizon $[t,\, t+T]$ is determined on-line. Define $h \equiv T/N$ for some integer $N \geq n/m$, where N is the number of sub-intervals on $[t,\, t+T]$. Now $\hat{\mathbf{y}}(t+kh)$ for each $k = 1,\, 2,\, \cdots,\, Nh = T$ is approximated by an iterative first-order Taylor series. For simplicity and avoiding the cross-product terms of $\hat{\mathbf{u}}(t+ih)$ and $\hat{\mathbf{u}}(t+jh)$, we assume that $\hat{G}[\hat{\mathbf{x}}(t+kh)] \approx \hat{G}[\hat{\mathbf{x}}(t)]$ and $\hat{F}[\mathbf{x}(t+kh)] \approx \hat{F}[\hat{\mathbf{x}}(t)]$, where $\hat{F} \equiv \partial\hat{\mathbf{f}}/\partial\hat{\mathbf{x}}$. In addition since the future values of $\tilde{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are unknown in general, $\tilde{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are assumed to remain constant over the finite horizon $[t,\, t+T]$. Then the following expression is obtained for $1 \leq k \leq N$:[5]

$$\begin{aligned}\hat{\mathbf{y}}(t+kh) \approx \hat{\mathbf{y}}(t) + h\,\hat{C}\left[\left\{\sum_{i=0}^{k-1}\left(I_{n\times n} + h\hat{F}\right)^i\right\}\hat{\mathbf{f}}\right.\\ + \sum_{i=0}^{k-1}\left(I_{n\times n} + h\hat{F}\right)^i\left\{\hat{B}\,\mathbf{u}(t)\right.\\ \left.\left. + \hat{G}\,\hat{\mathbf{u}}[t+(k-1-i)h]\right\}\right]\end{aligned} \tag{5}$$

where $I_{n\times n}$ is an $n\times n$ identity matrix, $\hat{C} \equiv \partial\hat{\mathbf{c}}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$ and $\hat{\mathbf{f}}$, $\hat{F}$, $\hat{B}$, and $\hat{G}$ are evaluated at $\hat{\mathbf{x}}(t)$. Define the following:

$$\begin{aligned}L(k\,h) \equiv\ & \mathbf{e}^T(t+k\,h)R_k^{-1}\mathbf{e}(t+k\,h)\\ & + \hat{\mathbf{u}}^T(t+k\,h)\,W_k\,\hat{\mathbf{u}}(t+k\,h)\end{aligned} \tag{6}$$

The cost function to be minimized is approximated using a trapezoidal formula or Simpson's rule as follows:[5] when N is odd,

$$\bar{J} = \frac{h}{2}\sum_{k=1}^{N}\left\{\frac{1}{2}\,L\left[(k-1)\,h\right] + \frac{1}{2}\,L\,(kh)\right\} \tag{7}$$

when N is even,

$$\begin{aligned}\bar{J} = \frac{h}{6}\sum_{k=0}^{(N/2)-1}&\left\{L\,(2kh) + 4L\left[(2k+1)\,h\right]\right.\\ &\left. + L\left[2\,(k+1)\,h\right]\right\}\end{aligned} \tag{8}$$

With the following definition:

$$\boldsymbol{\nu}_0 \equiv \left\{\hat{\mathbf{u}}^T(t),\quad \hat{\mathbf{u}}^T(t+h),\quad \cdots,\quad \hat{\mathbf{u}}^T[t+(N-1)h]\right\}^T \tag{9}$$

The approximate cost, $\bar{J}$, can be rewritten in quadratic form as

$$\bar{J} = \frac{1}{2}\,\boldsymbol{\nu}_0^T H_0\,\boldsymbol{\nu}_0 + \mathbf{g}_0^T(\hat{\mathbf{x}},\mathbf{u},\tilde{\mathbf{y}})\,\boldsymbol{\nu}_0 + q_0(\hat{\mathbf{x}},\mathbf{u},\tilde{\mathbf{y}}) \tag{10}$$

where H_0, $\mathbf{g}_0$ and q_0 are functions of $L(k\,h)$.[5] Also, the terminal constraint, $\mathbf{e}(t+T) = 0$, can be formulated as a constraint on $\boldsymbol{\nu}_0$ as follows:

$$M^T\boldsymbol{\nu}_0 = \mathbf{d}(t) \tag{11}$$

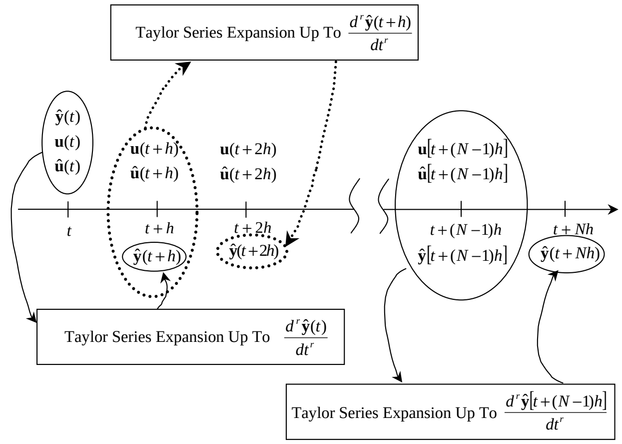

Fig. 1 Modified Approximate Receding-Horizon (MARH) Concept

where

$$M^T = C\left[(I_{n\times n} + h\hat{F})^{N-1}\hat{G},\ \cdots,\ (I_{n\times n} + h\hat{F})\hat{G},\ \hat{G}\right] \tag{12}$$

and

$$\mathbf{d}(t) = \frac{1}{h}\,\mathbf{e}(t) - \hat{C}\,\sum_{i=0}^{N-1}(I + h\hat{F})^i\left\{\hat{\mathbf{f}} + \hat{B}\mathbf{u}(t)\right\} \tag{13}$$

Finally, the solution of the QP problem is given by

$$\begin{aligned}\boldsymbol{\nu}_0 = &-\left[H_0^{-1} - H_0^{-1}M\left(M^TH_0^{-1}M\right)^{-1}M^TH_0^{-1}\right]\mathbf{g}_0(t)\\ &+ \left[H_0^{-1}M\left(M^TH_0^{-1}M\right)^{-1}\right]\mathbf{d}(t)\end{aligned} \tag{14}$$

where the rank of M is m. The first q_w equations give a current model error minimizing the cost function, which leads to a predictive filter structure:

$$\hat{\mathbf{u}}[t;\, \hat{\mathbf{x}}(t),\, \mathbf{u}(t),\, \tilde{\mathbf{y}}(t),\, h] = I_{q_w\times N}\ \boldsymbol{\nu}_0 \tag{15}$$

where $I_{q_w\times N}$ is a $\min(q_w,\, N)\times\min(q_w,\, N)$ identity matrix with zeros for the remaining elements.

MOTIVATION FOR THE MODIFIED ARH

In this section the motivation for a new ARH approach is shown. Consider the following linear system:

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + B\hat{\mathbf{u}}(t) \tag{16a}$$

$$\hat{\mathbf{y}}(t) = C\hat{\mathbf{x}}(t) \tag{16b}$$

and assume that the relative degree is r for each element of $\hat{\mathbf{y}}$. The quantity M^T is given by

$$\begin{aligned}M^T = \Big[\ & C\,(I_{n\times n} + hA)^{N-1}B,\ C\,(I_{n\times n} + hA)^{N-2}B,\\ & \vdots\\ & C\,(I_{n\times n} + hA)^{r}B,\ C\,(I_{n\times n} + hA)^{r-1}B,\\ & \vdots\\ & C\,(I_{n\times n} + hA)\,B,\ CB\ \Big]\\ = &\left[M_1^T\ \ \vdots\ \ 0_{m\times qr}\right]\end{aligned} \tag{17}$$

where $0_{m\times qr}$ is an $m \times qr$ zero matrix. Consider the case where the equality constraint is free from the last r model-error terms as follows:

$$\begin{bmatrix} M_1^T & \vdots & 0_{m\times qr} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_{01} \\ \cdots \\ \boldsymbol{\nu}_{02} \end{bmatrix} = \mathbf{d}(t)$$

$$M_1^T \boldsymbol{\nu}_{01} = \mathbf{d}(t) \tag{18}$$

where

$$\boldsymbol{\nu}_{01} = \left\{ \hat{\mathbf{u}}^T(t) \quad \hat{\mathbf{u}}^T(t+h) \quad \cdots \right.$$
$$\left. \cdots \quad \hat{\mathbf{u}}^T[t+(N-r-1)h] \right\}^T \tag{19a}$$

$$\boldsymbol{\nu}_{02} = \left\{ \hat{\mathbf{u}}^T[t+(N-r)h] \cdots \right.$$
$$\left. \cdots \quad \hat{\mathbf{u}}^T[t+(N-1)h] \right\}^T \tag{19b}$$

Also, the last (qr)-columns of the first-order term of $\boldsymbol{\nu}_0$ in $\bar{J}$, i.e., $\mathbf{g}_0(\hat{\mathbf{x}})$, are all zeros and the coupling terms for $\boldsymbol{\nu}_{01}$ and $\boldsymbol{\nu}_{02}$ in H_0 are zeros. Therefore, the original minimization problem is written as

$$\bar{J} = \frac{1}{2}\boldsymbol{\nu}_{01}^T H_{01}(\hat{\mathbf{x}})\,\boldsymbol{\nu}_{01} + \frac{1}{2}\boldsymbol{\nu}_{02}^T H_{02}(\hat{\mathbf{x}})\,\boldsymbol{\nu}_{02}$$
$$+ \mathbf{g}_{01}^T(\hat{\mathbf{x}})\,\boldsymbol{\nu}_{01} + \mathbf{q}_0(\hat{\mathbf{x}}) \tag{20a}$$

$$M_1^T \boldsymbol{\nu}_{01} = \mathbf{d}(t) \tag{20b}$$

where

$$H_0 = \begin{bmatrix} H_{01} & 0 \\ 0 & H_{02} \end{bmatrix} \tag{21a}$$

$$\mathbf{g}_0(\hat{\mathbf{x}}) = \begin{bmatrix} \mathbf{g}_{01}(\hat{\mathbf{x}})^T & \mathbf{0}_{1\times qr} \end{bmatrix}^T \tag{21b}$$

Hence, the optimal value of $\boldsymbol{\nu}_{02}$ to minimize $\bar{J}$ is $\mathbf{0}_{qr\times 1}$, i.e., the model error is already zero before the end of the receding horizon $(t+Nh)$ is reached. To avoid the model-error separation from the equality constraint, the state prediction in Eq. (5) has to be modified.

In this paper an r^{th}-order Taylor series expansion is used to predict the future state. The basic concept is shown in Fig. 1. Using the given $\hat{\mathbf{y}}(t)$, $\mathbf{u}(t)$, and $\hat{\mathbf{u}}(t)$, the states at $t+h$ are approximated by a Taylor series expansion. The order of the expansion of each predicted state is given when $\hat{\mathbf{u}}(t)$ first appears due to successive differentiation of the output. For the states at time $t+2h$, the expansion is similar to the previous case, using the states at time $t+h$ when $\hat{\mathbf{u}}(t+h)$ first appears. Hence after this expansion is given, the states at time $t+2h$ are functions of the states, the control, and the model error at time $t+h$. Then the states at time $t+h$ in the predicted states at $t+2h$ are substituted by the approximated ones at the first stage. This process is repeated up to time $t+Nh$.

GENERALIZATION OF MODIFIED ARH

The output prediction at $t+(k+1)h$ is given by

$$\hat{\mathbf{y}}\,[t+(k+1)\,h] \approx \hat{\mathbf{y}}\,(t+kh) + \mathbf{z}\,[\hat{\mathbf{x}}\,(t+kh)\,,\,h]$$
$$+ \Lambda(h)\,S_u\,[\hat{\mathbf{x}}\,(t+kh)]\;\mathbf{u}(t)$$
$$+ \Lambda(h)\,S_{\hat{u}}\,[\hat{\mathbf{x}}\,(t+kh)]\;\hat{\mathbf{u}}\,(t+kh) \tag{22}$$

for $k = 1, 2, \ldots, N$, where $\hat{\mathbf{y}}\,(t+kh)$ and $\hat{\mathbf{x}}\,(t+kh)$ are given by the predictions from the previous stage. This process is repeated up to all $\hat{\mathbf{x}}\,(t+kh)$ written in terms of $\hat{\mathbf{x}}\,(t)$. The i^{th} component of $\mathbf{z}\,[\hat{\mathbf{x}}(t),\,h]$ is given by

$$z_i\,[\hat{\mathbf{x}}(t),\,h] = \sum_{k=1}^{p_i} \frac{h^k}{k!}\;L_{\hat{\mathbf{f}}}^k(c_i) \tag{23}$$

where $L_{\mathbf{f}}^k(c_i)$ is the k^{th} Lie derivative, defined by

$$L_{\hat{\mathbf{f}}}^0(c_i) = c_i \tag{24a}$$

$$L_{\hat{\mathbf{f}}}^k(c_i) = \left[\frac{\partial L_{\hat{\mathbf{f}}}^{k-1}(c_i)}{\partial \hat{\mathbf{x}}}\right]^T \hat{\mathbf{f}}, \quad \text{for } k \geq 1 \tag{24b}$$

where the gradient is represented by a column vector with elements given by $(\partial c_i/\partial \mathbf{x})_k = \partial c_i/\partial x_k$. The i^{th} rows of $S_u\,[\hat{\mathbf{x}}(t)]$ and $S_{\hat{u}}\,[\hat{\mathbf{x}}(t)]$ are given by

$$\mathbf{s}_{u_i} = \left\{ L_{\hat{\mathbf{b}}_1}\left[L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)\right],\,\cdots,\,L_{\hat{\mathbf{b}}_{q_u}}\left[L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)\right] \right\} \tag{25a}$$

$$\mathbf{s}_{\hat{u}_i} = \left\{ L_{\hat{\mathbf{g}}_1}\left[L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)\right],\,\cdots,\,L_{\hat{\mathbf{g}}_{q_w}}\left[L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)\right] \right\} \tag{25b}$$

for $i = 1, 2, \ldots, m$, where $\hat{\mathbf{b}}_j$ is the j^{th} column of $\hat{B}\,[\hat{\mathbf{x}}(t)]$, $\hat{\mathbf{g}}_j$ is the j^{th} column of $\hat{G}\,[\hat{\mathbf{x}}(t)]$, and the Lie derivative in Eq. (25) is defined by

$$L_{\hat{\mathbf{b}}_j}\left[L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)\right] \equiv \left[\frac{\partial L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)}{\partial \hat{\mathbf{x}}}\right]^T \hat{\mathbf{b}}_j \tag{26}$$

for $j = 1, 2, \ldots, q_u$, and

$$L_{\hat{\mathbf{g}}_j}\left[L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)\right] \equiv \left[\frac{\partial L_{\hat{\mathbf{f}}}^{p_i-1}(c_i)}{\partial \hat{\mathbf{x}}}\right]^T \hat{\mathbf{g}}_j \tag{27}$$

for $j = 1, 2, \ldots, q_w$.

Finally, the approximated cost function $\bar{J}$ is obtained through the same steps as in the ARH approach, and the optimal solution is obtained by Eqs. (14) and (15). As a result, the MARH approach is derived by combining the one-step ahead state prediction with the approximate receding-horizon cost function.

MECS CONCEPT

The block diagram with MECS is shown in Fig. 2, where $\mathbf{r}(t)$ is the reference command. The model error is determined using the estimated states, $\hat{\mathbf{x}}(t)$, the control input, $\mathbf{u}(t)$, and the current measurement, $\tilde{\mathbf{y}}(t)$. The determined model error, $\hat{\mathbf{u}}(t)$, corrects not only the nominal control input, $\bar{\mathbf{u}}(t)$, but also the filter model. After the model error is determined, any state estimator or observer can be implemented, including

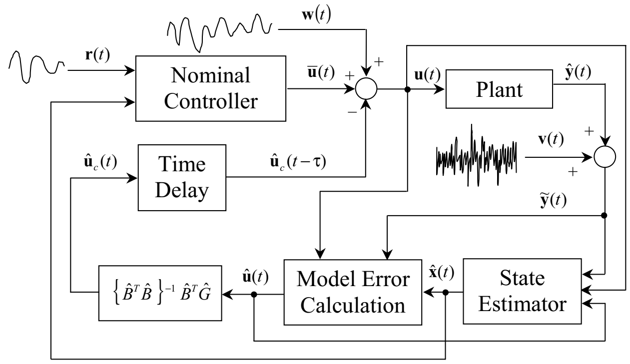

Fig. 2 Overall Block Diagram with MECS

a Kalman filter. The total control input $\mathbf{u}(t)$ with model-error correction is given by

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) - \hat{\mathbf{u}}_c(t-\tau) \tag{28}$$

where $\bar{\mathbf{u}}(t)$ is the nominal control input at time t, which can be any controller, i.e., Proportional-Integrate-Derivative (PID) Control, Lead-Lag Compensator, Sliding Mode Control, H_∞ Control, Linear Quadratic Regulator (LQR) Control, Linear Quadratic Gaussian (LQG) Control, etc. The time delay τ is always present in the overall MECS design because the measurement $\tilde{\mathbf{y}}(t)$ must be given before the error in the system can be corrected. The term $\hat{\mathbf{u}}_c\,(t-\tau)$ is used to cancel the estimated model error at time $t-\tau$, determined by the current information using a Pseudo-Inverse ($n \geq q_u$, i.e., under-actuated) as follows:

$$\hat{\mathbf{u}}_c(t) = [\hat{B}^T\,\hat{B}]^{-1}\,\hat{B}^T\,\hat{G}\,\hat{\mathbf{u}}(t) \tag{29}$$

When $\hat{B}\,[\hat{\mathbf{x}}(t)] = \hat{G}\,[\hat{\mathbf{x}}(t)]$, i.e., separate actuators are installed for each dynamics part, $\hat{\mathbf{u}}_c(t)$ is equal to $\hat{\mathbf{u}}(t)$, which will be the case for the spacecraft attitude control problem.

SPACECRAFT ATTITUDE CONTROL

In this section the nominal control design in Ref. [7] is first summarized, and then the model-error correction input using the MARH approach is derived. Next, a method is derived to choose the optimal weighting and length of receding-horizon step-time. Then, the quadratic stability of the closed-loop system for a norm bounded uncertainty is derived. Finally, simulation results are shown to verify the new control design approach.

NOMINAL CONTROLLER DESIGN

The spacecraft attitude kinematics and dynamics can be written as follows:[8]

$$\dot{\hat{\boldsymbol{\sigma}}}(t) = \frac{1}{4}\mathcal{B}\,[\hat{\boldsymbol{\sigma}}(t)]\,\hat{\boldsymbol{\omega}}(t) \tag{30a}$$

$$\dot{\hat{\boldsymbol{\omega}}}(t) = -\hat{I}^{-1}\,[\hat{\boldsymbol{\omega}}(t)\times]\,I\hat{\boldsymbol{\omega}}(t) + \hat{I}^{-1}\,[\mathbf{u}(t)+\hat{\mathbf{u}}(t)] \tag{30b}$$

where $\hat{\boldsymbol{\sigma}}(t)$ represents the estimated Modified Rodrigues Parameter (MRP) vector, $\hat{\boldsymbol{\omega}}(t)$ is the angular-velocity vector, $\hat{I}$ is the nominal spacecraft inertia matrix, $\hat{\mathbf{u}}(t)$ is the model-error vector to be determined (which is a function of the unknown external disturbances, spacecraft moment of inertia, and the angular velocity), and $\mathbf{u}(t)$ is the total control input defined by Eq. (28). The matrix $\mathcal{B}\,[\hat{\boldsymbol{\sigma}}(t)]$ is given by[8]

$$\mathcal{B}\,[\hat{\boldsymbol{\sigma}}] \equiv \left[1-\hat{\sigma}^2\right] I_{3\times 3} + 2\,[\hat{\boldsymbol{\sigma}}\times] + 2\,\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}^T \tag{31}$$

where $\hat{\sigma}^2 = \hat{\boldsymbol{\sigma}}^T\hat{\boldsymbol{\sigma}}$, and the inverse is given by

$$\mathcal{B}^{-1}\,[\hat{\boldsymbol{\sigma}}] = \frac{1}{\left\{1+\hat{\sigma}^2\right\}^2}\,\mathcal{B}^T\,[\hat{\boldsymbol{\sigma}}] \tag{32}$$

For $\mathbf{a} = [a_1,\, a_2,\, a_3]^T$, the cross product operator $[\mathbf{a}\times]$ is defined by

$$[\mathbf{a}\times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{33}$$

By the control design in Ref. [7], the nominal control input is given by

$$\bar{\mathbf{u}}(t) = [\hat{\boldsymbol{\omega}}(t)\times]\,\hat{I}\hat{\boldsymbol{\omega}}(t) + \hat{I}\boldsymbol{\phi}(t) \tag{34}$$

where

$$\begin{aligned}\boldsymbol{\phi}(t) = &-P\hat{\boldsymbol{\omega}}(t) - \Big\{\hat{\boldsymbol{\omega}}(t)\,\hat{\boldsymbol{\omega}}^T(t) \\ &+ \left(\frac{4K}{1+\hat{\sigma}^2(t)} - \frac{\hat{\omega}^2(t)}{2}\right) I_{3\times 3}\Big\}\,\hat{\boldsymbol{\sigma}}(t) \\ &- 4K_I\mathcal{B}^{-1}\,[\hat{\boldsymbol{\sigma}}(t)]\int_0^t \hat{\boldsymbol{\sigma}}(\xi)d\xi \end{aligned} \tag{35}$$

where P, K, and K_I are the control gain matrices, and $\hat{\omega}^2(t) = \hat{\boldsymbol{\omega}}^T(t)\,\hat{\boldsymbol{\omega}}(t)$. After substituting this control input into the dynamics in Eq. (30), the closed-loop dynamics become

$$\begin{aligned}\ddot{\hat{\boldsymbol{\sigma}}}(t) = &-P\dot{\hat{\boldsymbol{\sigma}}}(t) - K\hat{\boldsymbol{\sigma}}(t) - K_I\int_0^t \hat{\boldsymbol{\sigma}}(\xi)\,d\xi \\ &+ \frac{1}{4}\,\mathcal{B}\,[\hat{\boldsymbol{\sigma}}(t)]\;\hat{I}^{-1}\,\{\hat{\mathbf{u}}\,(t) - \hat{\mathbf{u}}\,(t-\tau)\}\end{aligned} \tag{36}$$

In Ref. [7] $\hat{I}$ and the external disturbances are estimated by an adaptive scheme where the model parameters are updated on-line in the control law, so that $\hat{\mathbf{u}}(t)$ approaches zero as time increases. In this paper instead of using the adaptive scheme, the total model-error vector $\hat{\mathbf{u}}(t)$ is estimated by the MARH solution and the control input is corrected using the MECS approach shown in Fig. 2.

MODEL-ERROR ESTIMATION USING MARH

Choosing $P = pI_{3\times 3}$, $K = kI_{3\times 3}$, and $K_I = k_I I_{3\times 3}$ (where p, k, and k_I are positive constants), then

$$\begin{aligned}\ddot{\hat{\sigma}}_i(t) = &-p\,\dot{\hat{\sigma}}_i - k\,\hat{\sigma}_i(t) - k_I\int_0^t \hat{\sigma}(\xi)\,d\xi \\ &+ \hat{\nu}_i(t) - \hat{\nu}_i(t-\tau)\end{aligned} \tag{37}$$

for $i = 1, 2, 3$, where

$$\hat{\boldsymbol{\nu}}(t) = \left[\ \hat{\nu}_1(t) \quad \hat{\nu}_2(t) \quad \hat{\nu}_3(t)\ \right]^T \tag{38a}$$

$$\hat{\boldsymbol{\nu}}(t) = \frac{1}{4}\,\mathcal{B}\left[\hat{\boldsymbol{\sigma}}(t)\right]\,\hat{I}^{-1}\hat{\mathbf{u}}(t) \tag{38b}$$

The state-space form for each axis is given by

$$\dot{\mathbf{x}}_i(t) = \hat{A}_i\mathbf{x}_i(t) + \hat{B}_i\nu_i(t) + \hat{B}_i\,\hat{\nu}_i(t) \tag{39a}$$

$$\hat{y}_i(t) = \hat{C}_i\,\hat{\mathbf{x}}_i(t), \quad \text{for } i = 1, 2, 3 \tag{39b}$$

where

$$\nu_i(t) = \bar{\nu}_i(t) - \hat{\nu}_i(t-\tau) \tag{40a}$$

$$\bar{\nu}_i(t) = -k_I\int_0^t \sigma_i(\xi)\,d\xi \tag{40b}$$

$$\hat{\mathbf{x}}_i(t) = \left[\hat{x}_{i1}(t) \quad \hat{x}_{i2}(t)\right]^T = \left[\hat{\sigma}_i(t) \quad \dot{\hat{\sigma}}_i(t)\right]^T \tag{40c}$$

and

$$\hat{A}_i = \begin{bmatrix} 0 & 1 \\ -k & -p \end{bmatrix},\ \hat{B}_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \hat{C}_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{41}$$

Usually k_I is chosen to be as small as possible so that the integral control action does not significantly affect the transient response, while also reducing the steady-state error. Therefore, in order to keep the order of the model equal to two (which is required to simplify the analysis), the integral control term is not put into the model (for small k_I this approximation is valid). Note, integral control will still be used in the final control form. The measurement is attitude-angle only, so that

$$\tilde{y}_i(t) = \tilde{\sigma}_i(t) = \sigma_i(t) + v_i(t) \quad \text{for } i = 1, 2, 3 \tag{42}$$

where $v_i(t)$ is the measurement noise with known covariance.

As shown in the above equations the dynamics is completely linearized and decoupled without any approximation. After the vector $\hat{\boldsymbol{\nu}}$ is determined, the actual model-error correction input, $\hat{\mathbf{u}}(t)$, is set to the following:

$$\hat{\mathbf{u}}(t) = \frac{4}{\left\{1+\hat{\sigma}^2(t)\right\}^2}\,\hat{I}\,\mathcal{B}^T\left[\hat{\boldsymbol{\sigma}}(t)\right]\hat{\boldsymbol{\nu}}(t) \tag{43}$$

Since the relative degree is two, we choose the value of the subinterval N be equal to two. From the steps of the MARH approach, the following terms are obtained:

$$h_{11} = \frac{h}{32\,r_2}\left\{k^2h^8 + 4\,k\,p\,h^7 + \left(4p^2 - 12k\right)h^6 \right. \\ \left. -24\,p\,h^5 + \left(8r_1^{-1} + 36\right)h^4 + 16\,w_0\,r_2\right\} \tag{44a}$$

$$h_{12} = h_{21} = -\frac{h^5\left(k\,h^2 + 2\,p\,h - 6\right)}{16\,r_2} \tag{44b}$$

$$h_{22} = \frac{h\left(h^4 + 8\,w_1\,r_2\right)}{8\,r_2} \tag{44c}$$

where h_{ij} is the i^{th}-row and j^{th}-column element of H_0, $h = T/2$, and

$$M^T = \begin{bmatrix} -\dfrac{k\,h^4}{4} - \dfrac{p\,h^3}{2} + \dfrac{3\,h^2}{2} \\ \\ \dfrac{h^2}{2} \end{bmatrix} \tag{45}$$

and

$$g_1(t) = -\frac{h^3}{32\,r_1\,r_2} \times \left\{k_{g11}\hat{x}_{i1}(t) + k_{g12}\hat{x}_{i2}(t) \right. \\ \left. + k_{g13}\nu_i(t) + k_{g14}\tilde{y}_i(t)\right\} \tag{46a}$$

$$g_2(t) = \frac{h^3}{16\,r_2} \times \left\{k_{g21}\hat{x}_{i1}(t) + k_{g22}\hat{x}_{i2}(t) \right. \\ \left. + k_{g23}\nu_i(t) + k_{g24}\tilde{y}_i(t)\right\} \tag{46b}$$

where

$$k_{g11} = k^3\,r_1\,h^6 + 4\,k^2\,p\,r_1\,h^5 \\ + \left(4\,p^2\,k\,r_1 - 14\,k^2\,r_1\right)h^4 - 28\,p\,k\,r_1\,h^3 \\ + \left(52\,kr_1 + 8kr_2\right)h^2 + 8\,p\,r_1\,h - 24\,r_1 - 16\,r_2 \tag{47a}$$

$$k_{g12} = k^2\,p\,r_1\,h^6 + \left(-2\,k^2\,r_1 + 4\,p^2\,kr_1\right)h^5 \\ + \left(4\,p^3\,r_1 - 18\,k\,p\,r_1\right)h^4 + \left(20\,k\,r_1 - 28\,p^2\,r_1\right) \\ + \left(64p\,r_1 + 8\,p\,r_2\right)h^2 + \left(-48\,r_1 - 16\,r_2\right)h \tag{47b}$$

$$k_{g13} = -k^2\,r_1\,h^6 - 4\,k\,p\,r_1\,h^5 + \left(14\,k\,r_1 - 4\,p^2\,r_1\right)h^4 \\ + 28\,p\,r_1\,h^3 - \left(8\,r_2 + 48\,r_1\right)h^2 \tag{47c}$$

$$k_{g14} = 24\,r_1 - 8\,h\,p\,r_1 - 4\,h^2\,k\,r_1 + 16\,r_2 \tag{47d}$$

$$k_{g21} = k^2\,h^4 + 2\,p\,k\,h^3 - 8\,k\,h^2 + 4 \tag{47e}$$

$$k_{g22} = k\,p\,h^4 + \left(2\,p^2 - 2\,k\right)h^3 - 8\,p\,h^2 + 8\,h \tag{47f}$$

$$k_{g23} = -k\,h^4 - 2\,p\,h^3 + 8\,h^2 \tag{47g}$$

$$k_{g24} = -4 \tag{47h}$$

with $\mathbf{g}_0(t) = \left[\,g_1,\ g_2\,\right]^T$. Also, $d_i(t)$ in Eq. (11) is given by

$$d_i(t) = -\left(\frac{k^2\,h^4}{4} + \frac{k\,p\,h^3}{2} - 2\,k\,h^2 + 1\right)\hat{x}_{i1}(t) \\ - \left\{\frac{k\,p\,h^4}{4} + \left(p^2 - k\right)\frac{h^3}{2} - 2\,p\,h^2 + 2\,h\right\}\hat{x}_{i2}(t) \\ + \left(\frac{k\,h^4}{4} + \frac{p\,h^3}{2} - 2\,h^2\right)\nu_i(t) + \tilde{y}_i(t) \tag{48}$$

Using the above equations, the current estimated model error is determined by the first element of Eq. (14).

The output is assumed to be constant during the given time interval. However, this assumption becomes less accurate as the receding-horizon step-time

T increases and/or the speed of response increases. Therefore, the weights have to be adjusted accordingly. To accomplish this task, the following exponential functions are used:

$$r(t_k) = e^{r_p}\, r(t_{k-1}) \tag{49a}$$
$$w(t_k) = e^{w_p}\, w(t_{k-1}) \tag{49b}$$

where $r_0 \equiv r(t_0)$ and $w_0 \equiv w(t_0)$ are given, and r_p and w_p are non-negative real values. More details of this concept can be found in Ref. [4]. Finally, the estimated model-error correction input is simply given by

$$\hat{\nu}_i(t) = a_1\, \hat{x}_{i1}(t) + a_2\, \hat{x}_{i2}(t) + a_3\, \nu_i(t) + a_4\, \tilde{y}_i(t) \tag{50}$$

for $i = 1, 2, 3$, where a_1, a_2, a_3 and a_4 are the functions of w_i, r_i, h and τ (given in the Appendix).

After $\hat{\nu}_i(t)$ in Eq. (39a) is determined, a Kalman filter is designed for state estimation. For the simulations the noise variance for the MRP measurement is given by 6.67×10^{-4}, which corresponds to the standard deviation of a Fine Sun Sensor angle measurement, $0.5°/\sqrt{12}$. The state estimation errors and 3σ bounds are shown in Fig. 3. As shown in the figure without any angular-velocity sensor, such as a three-axis gyro, the rate estimation error bound is about $\pm 1.24°$/sec. If this error is too large, then gyro measurements should be employed.

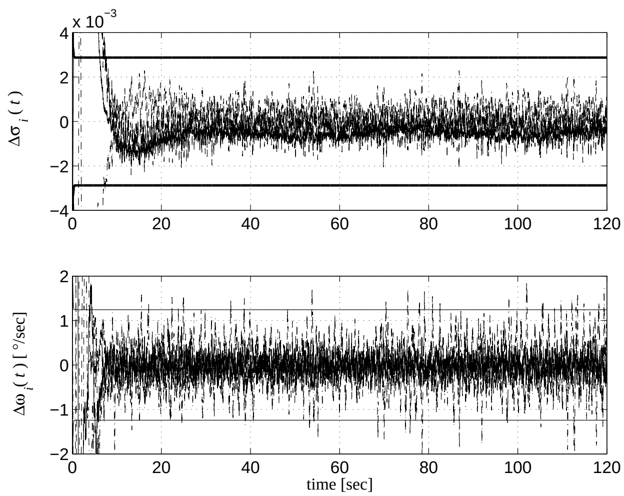

Fig. 3 Estimation Errors and 3σ Bounds

OPTIMAL DESIGN

In this section the optimal weighting and length of receding-horizon step-time are determined. Our goal is to determine w_p and/or r_p and h that minimizes the ∞-norm of sensitivity function for the system given by Eq. (39). To find a stable region the Hermite-Biehler theorem is used, which gives the necessary and the sufficient conditions for a system to be Hurwitz stable.[6]

Theorem 1 *Hermite-Biehler Theorem*
Consider the following polynomial:

$$d_{cl}(s) = c_n\, s^n + c_{n-1}\, s^{n-1} + \cdots + c_1\, s + c_0 \tag{51}$$

where $c_n \neq 0$ can be decomposed as

$$d_{cl}(s) = p(s) + s q(s) \tag{52}$$

where $p(s)$ contains even power terms and $q(s)$ contains odd power terms of $d_{cl}(s)$. Then $d_{cl}(s)$ is Hurwitz stable if and only if c_n and c_{n-1} are the same sign with all roots of $p(j\omega)$ and $q(j\omega)$ real, and the nonnegative roots satisfy the following interlacing property:

$$0 < \omega_{e1} < \omega_{o1} < \omega_{e2} < \omega_{o2} < \cdots \tag{53}$$

where ω_{ei} and ω_{oi} are the roots of $p(j\omega)$ and $q(j\omega)$, respectively. ■

From the Hermite-Biehler theorem the following is deduced:

Corollary 1 *Consider the following 6^{th}-order polynomial:*

$$d_{cl}(s) = c_6\, s^6 + c_5\, s^5 + c_4\, s^4 + c_3\, s^3 + c_2\, s^2 + c_1\, s + c_0 \tag{54}$$

where $c_6 > 0$, then $d_{cl}(s)$ is Hurwitz stable if and only if the stability index,

$$\varepsilon \equiv \text{sgn}(\kappa)\, \log_{10}(|\kappa| + 1) \tag{55}$$

is greater than zero, where

$$\kappa \equiv \min(\text{I, II-a, II-b, II-c, III-a, III-b, III-c, III-d}) \tag{56}$$

and

$$\text{I}: \ \underline{c}_0 > 0 \tag{57a}$$
$$\begin{aligned} &\text{II-a}: \ \textit{min}\ (c_3\, c_5) > 0, \\ &\text{II-b}: \ \textit{min}\ (c_5\, c_6) > 0, \\ &\text{II-c}: \ \textit{min}\ (c_1\, c_5) > 0 \end{aligned} \tag{57b}$$
$$\begin{aligned} &\text{III-a}: \bar{c}_6,\ \underline{c}_5,\ \bar{c}_4,\ \bar{c}_3,\ \underline{c}_2,\ \underline{c}_1,\ \bar{c}_0, \\ &\text{III-b}: \underline{c}_6,\ \bar{c}_5,\ \underline{c}_4,\ \underline{c}_3,\ \bar{c}_2,\ \bar{c}_1,\ \underline{c}_0, \\ &\text{III-c}: \underline{c}_6,\ \bar{c}_5,\ \bar{c}_4,\ \underline{c}_3,\ \underline{c}_2,\ \bar{c}_1,\ \bar{c}_0, \\ &\text{III-d}: \bar{c}_6,\ \underline{c}_5,\ \bar{c}_4,\ \bar{c}_3,\ \underline{c}_2,\ \underline{c}_1,\ \underline{c}_0 \end{aligned} \tag{57c}$$

are substituted into III

where $\underline{c}_i$ and $\bar{c}_i$ are the lower and the upper bounds of each c_i, for $i = 1, 2, \ldots, 6$, and

$$\text{III}: \ -\left(4\, c_1\, c_5\, A^2 + 2\, c_3\, A\, B + B^2\right) > 0 \tag{58}$$

where

$$A \equiv c_1\, c_5\, c_6 - c_2\, c_5^2 + c_3\, c_4\, c_5 - c_3^2\, c_6 \tag{59a}$$
$$B \equiv 2\, c_0\, c_5^3 - 2\, c_1\, c_4\, c_5^2 + 2\, c_1\, c_3\, c_5\, c_6 \tag{59b}$$

■

Proof: the proof can be found in Ref. [4].

We assume that no estimation errors are present, i.e., the estimator transients have sufficiently decayed (the estimator is also assumed to provide unbiased estimates). Then the following closed-loop transfer function is obtained:

$$y_i(t) = \frac{N_v(s)}{D_{cl}(s)}\,[v_i(t)] + \frac{N_w(s)}{D_{cl}(s)}\,[\hat{\nu}_i(t)] \tag{60a}$$

$$\equiv S_\nu(s)\,[\nu_i(t)] + S(s)\,[\hat{\nu}_i(t)] \tag{60b}$$

where $S_\nu(s)$ is the measurement-noise transfer function and $S(s)$ is sensitivity function, with

$$D_{cl}(s) = \{d_t(s) + a_3 n_t(s)\}\,D_k(s)D_s(s) + \{(1-a_3)\,d_t(s) + a_3 n_t(s)\}\,N_k(s)N_s(s) + d_t(s)\,(a_1 + a_4 + s a_2)\,D_k(s)N_s(s) \tag{61a}$$

$$N_v(s) = -a_4 d_t(s) D_k(s) N_s(s) \tag{61b}$$

$$N_w(s) = \{d_t(s) + a_3 n_t(s)\}\,D_k(s)D_s(s) \tag{61c}$$

The term $N_k(s)/D_k(s)$ is the transfer function of integral control action given by Eq. (40b). Note that the nominal controller in Eq. (34) is now embedded in the system model through Eq. (41). Also, $n_t(s)/d_t(s)$ is a Padé approximation of $e^{-\tau s}$ (from the time-delay in the MECS design). The following (3, 3) Padé approximation is used:[9]

$$e^{-\tau s} \approx \frac{-\tau^3 s^3 + 12\,\tau^2 s^2 - 60\,\tau\,s + 120}{\tau^3 s^3 + 12\,\tau^2 s^2 + 60\,\tau\,s + 120} \equiv \frac{n_t(s)}{d_t(s)} \tag{62}$$

and h, r_p and/or w_p are chosen so that the following H_∞ norm is minimized:

$$\min\,||S(j\omega)||_\infty \tag{63}$$

To narrow down the searching space, $k = 1.0$, $p = 3.0$ and $k_I = 0.090$ are adopted from Ref. [7], and $w_1 = 1$, $w_p = 0.1$, $r_1 = 0.5$ and $\tau = 0.0025$ sec. Then, the parameter space for the optimal values is now 2-dimensional (r_p and h).

By calculating ε and $\|S(j\omega)\|_\infty$ for various values of h and r_p, we find that the stability index and the norm are more sensitive to h than r_p. Figure 4 depicts h versus the normalized values of $\|S(j\omega)\|_\infty$, ε, settling time and maximum overshoot for an impulse $\hat{\nu}(t)$ input, with r_p set to 0.1 (chosen by trial and error). To minimize the sensitivity norm ($\|S(j\omega)\|_\infty$) the value of h has to be chosen as small as possible. However, the settling time increases as h decreases and the control input may saturate. Therefore, the optimal value of h is in the range of $1.48 \le h^* \le 1.58$. By trial and error $h^* = 1.5$ sec is selected. Finally, the determined model error for $i = 1, 2, 3$ is given by

$$\hat{\nu}_i(t) \approx 0.72\,\hat{\sigma}_i(t) + 2.03\,\dot{\hat{\sigma}}_i(t) - 0.66\,\nu_i(t) - 0.06\,\tilde{y}_i(t) \tag{64}$$

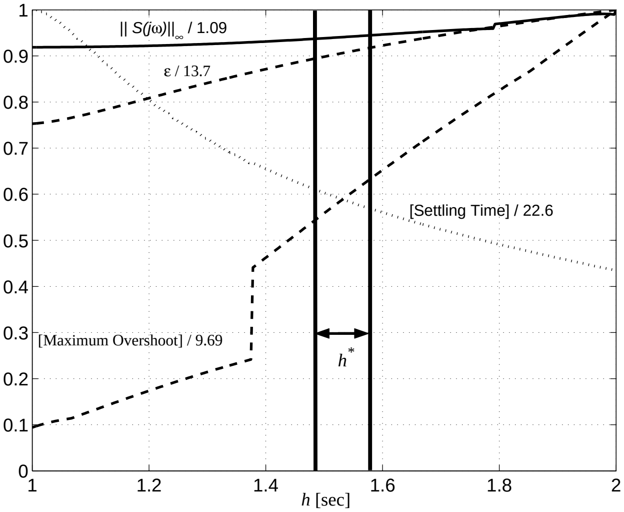

Fig. 4 h **vs.** $\|S(j\omega)\|_\infty$**,** ε**, Settling Time, and Maximum Overshoot**

QUADRATIC STABILITY

To provide a stability proof, the following are summarized from Ref. [10] and the proof of each of the following can also be found in Ref. [10]:

Definition 1 *Quadratically Stable*
Consider the following system with nonlinear uncertainty $\Delta\mathbf{f}[\mathbf{x}(t)]$*:*

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + \Delta\mathbf{f}[\mathbf{x}(t)] \tag{65}$$

where $\mathbf{x}(t) \in \Re^n$ *and the nonlinear uncertainty* $\Delta\mathbf{f}[\mathbf{x}(t)] \equiv E_f\,\boldsymbol{\delta}\,[\mathbf{x}(t)]$ *is a* C^0 *function, and* $\boldsymbol{\delta}\,[\mathbf{x}(t)]$ *is a element of the following set:*

$$\Omega \equiv \{\,\boldsymbol{\delta}\,[\mathbf{x}(t)]\ |\ \|\boldsymbol{\delta}\,[\mathbf{x}(t)]\,\|_\infty \le \|N_f\,\mathbf{x}(t)\|_\infty,\ \forall\ \mathbf{x}(t)\} \tag{66}$$

where E_f *and* N_f *are some constant matrices. The system, Eq.* (65)*, is said to be quadratically stable if there exists a positive-definite symmetric matrix* $P_q > 0$ *such that*

$$\{A\mathbf{x}(t) + \Delta\mathbf{f}[\mathbf{x}(t)]\}^T\ P_q\,\mathbf{x}(t) + \mathbf{x}^T(t)\,P_q\ \{A\mathbf{x}(t) + \Delta\mathbf{f}[\mathbf{x}(t)]\} < 0 \tag{67}$$

for all nonzero $\mathbf{x}(t) \in \Re^n$ *and all admissible nonlinear uncertainty,* $\Delta\mathbf{f}[\mathbf{x}(t)]$*.* ■

Definition 2 *Quadratic Cost Matrix,* P_q
A positive definite matrix $P_q > 0$ *is said to be a quadratic cost matrix for Eq.* (65) *and the following cost function:*

$$J_q = \int_0^\infty \mathbf{x}^T(t)\,Q_q\,\mathbf{x}(t)\,dt \tag{68}$$

where $Q_q \ge 0$*, if*

$$\{A\mathbf{x}(t) + \Delta\mathbf{f}[\mathbf{x}(t)]\}^T\ P_q\,\mathbf{x}(t) + \mathbf{x}^T(t)\,P_q\ \{A\mathbf{x}(t) + \Delta\mathbf{f}[\mathbf{x}(t)]\} < -\mathbf{x}^T(t)\,Q_q\,\mathbf{x}(t) \tag{69}$$

Table 1 **Simulation Scenarios**

Case Scenario	Inertia Uncertainty	External Disturbance	Full State Information	Sensor Noise	MECS On/Off
(1)	No	No	Yes	N/A	Off
(2)	Yes	Yes	Yes	N/A	Off
(3)	Yes	Yes	No	Yes	On

for all nonzero $\mathbf{x}(t) \in \Re^n$ *and all admissible nonlinear uncertainty,* $\Delta\mathbf{f}[\mathbf{x}(t)]$.

■

Theorem 2 *The Cost Function Bound*
If $P_q > 0$ *is a quadratic cost matrix of Eq.* (65)*, then the cost function is bounded by*

$$J_q \leq \mathbf{x}^T(0)\, P_q\, \mathbf{x}(0) \tag{70}$$

and if the system is quadratically stable, then there exists a quadratic cost matrix. ■

Lemma 1 H_∞ *Norm Bound Condition*
For the system, Eq. (65)*, there exists a quadratic cost matrix,* $P_q > 0$*, if and only if the following conditions hold:*

1. *A is a stable matrix.*
2. *The following* H_∞ *norm bound is satisfied for some* $\epsilon > 0$*:*

$$\left\| \begin{bmatrix} N_f \\ \sqrt{\epsilon}\, Q_q \end{bmatrix} (s\, I_n - A)^{-1}\, E_f \right\|_\infty < 1 \tag{71}$$

Then, for such ϵ*, the Riccati equation*

$$A^T P_q + P_q A + \epsilon\, P_q E_f E_f^T P_q + \frac{1}{\epsilon} N_f^T N_f = -Q_q \tag{72}$$

has a solution.

■

For the state-space form of Eq. (60), the following values are obtained:

$$E_f = \{\, 0,\, 0,\, 1,\, 0,\, 0,\, 0 \,\}^T \tag{73a}$$
$$N_f = \left[\mathrm{diag}\,[0.248,\ 2.250,\ 8.160] \quad 0_{3\times3}\right] \tag{73b}$$
$$Q_q = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} \tag{73c}$$
$$\epsilon = 0.001 \tag{73d}$$

where the first three diagonal terms of N_f and ϵ are the maximum values to satisfy Eq. (71) with the given matrices (the ∞-norm of Eq. (71) is 0.9982) and P_q is given in the Appendix. Therefore, for the norm bounded uncertainty by Definition 1, the closed-loop system is globally quadratically stable.

SIMULATION

The initial MRP is $\hat{\boldsymbol{\sigma}}(t_0) = \{-0.3,\, -0.4,\, 0.2\,\}^T$ and the initial angular velocity is $\hat{\boldsymbol{\omega}}(t_0) = \{11.46,\, 11.46,\, 11.46\,\}^T$ [°/sec]. The true and assumed inertia matrices are given by (consistent with Ref. [7])

$$I = \begin{bmatrix} 30 & 10 & 5 \\ 10 & 20 & 3 \\ 5 & 3 & 15 \end{bmatrix},\ \hat{I} = \mathrm{diag}\,\{\,5,\ 5,\ 5\}\ \left[\mathrm{kg{\cdot}m^2}\right] \tag{74}$$

and the external disturbance, $\mathbf{F}_e(t)$, is given by

$$\mathbf{F}_e(t) = \begin{bmatrix} 2 + \frac{1}{5}\sin\left(\frac{t}{7}\right) \\ 1 + \frac{1}{10}\sin\left(\frac{t}{7} + \frac{\pi}{4}\right) \\ -1 - \frac{1}{10}\cos\left(\frac{t}{7}\right) \end{bmatrix} \ [\mathrm{N{\cdot}m}] \tag{75}$$

The model-error upper bounds in an ∞-norm sense are as follows:

$$\|\boldsymbol{\nu}\|_\infty \leq 3.605 \quad \text{and} \quad \|\hat{I}^{-1}\,\hat{\mathbf{u}}\|_\infty \leq 4.120 \tag{76}$$

In addition since $\hat{I}$ is a diagonal matrix, then the upper bound of $\hat{\mathbf{u}}$ is given by

$$\|\hat{\mathbf{u}}\|_\infty \leq 5 \times 4.120 = 20.60 \ \ [\mathrm{N{\cdot}m}] \tag{77}$$

A simulation result of the actual calculated model error compared to the upper bound given by Eq. (77) is shown in Fig. 5.

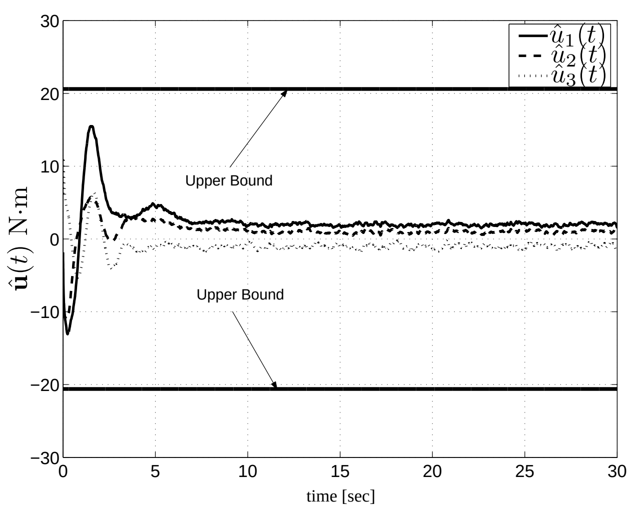

Fig. 5 True Model Error and Upper Bound

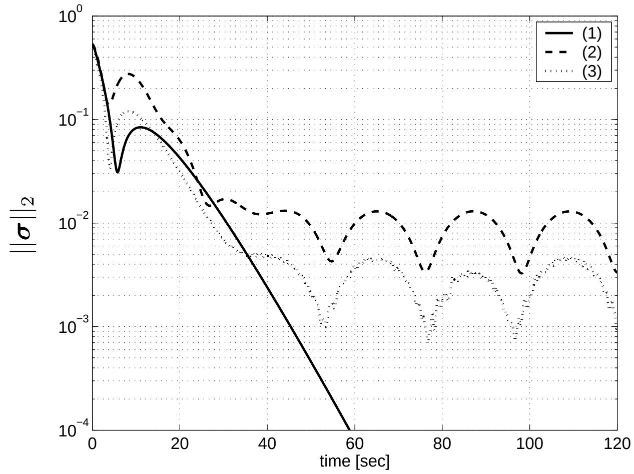

Fig. 6 $\|\boldsymbol{\sigma}\|_2$ History for Each Case

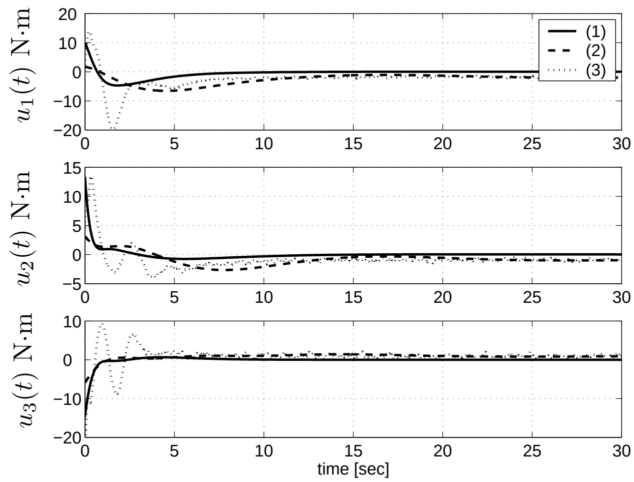

Fig. 8 Control History for Each Case

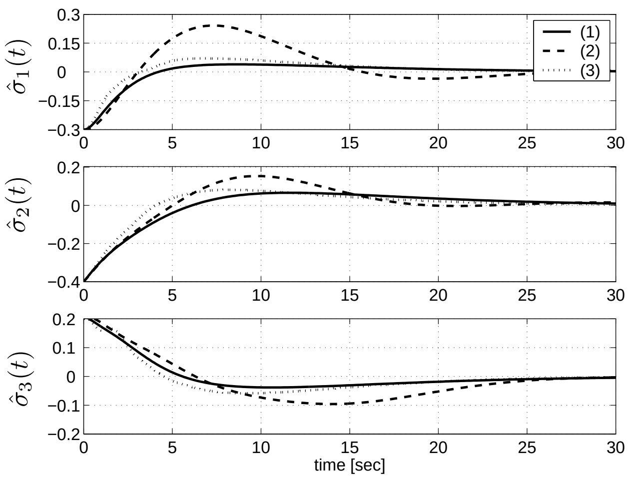

Fig. 7 Time History of $\boldsymbol{\sigma}(t)$

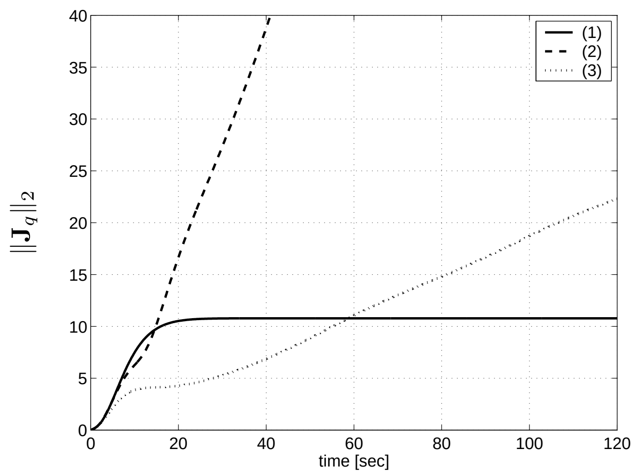

Fig. 9 Time History of the Cost Function Norm for Each Case

The simulation scenarios are given in Table 1. The MRP norm histories for each case are shown in Fig. 6. After the transient response settles, the mean value of the norm for Case 3 is 0.002 and the one for Case 2 is 0.007. This represents a 71% performance improvement in the sense of the 2-norm of $\hat{\boldsymbol{\sigma}}(t)$. Also, the time histories of the MRPs for each case are shown in Fig. 7. MECS provides the best transient response, i.e., less overshoot and closer to the response of Case 1. The control histories for each case are shown in Fig. 8. The MECS controller reacts more to the modelling and external disturbance errors.

The norm of the cost function, Eq. (68), for each case shows the significant performance improvement of MECS. The norm is defined as

$$\|\mathbf{J}_q\|_2 = \sqrt{J_{q1}^2 + J_{q2}^2 + J_{q3}^2} \tag{78}$$

where

$$J_{qi} = \int_0^\infty \mathbf{x}_{ie}^T(t)\,\mathbf{x}_{ie}(t)dt, \quad \text{for } i = 1,\, 2,\, 3 \tag{79}$$

with

$$\mathbf{x}_{ie}^T = \left\{ \int_0^t \hat{\sigma}_i(\xi)d\xi,\ \hat{\sigma}_i(t),\ \dot{\hat{\sigma}}_i(t) \right\} \tag{80}$$

As shown in Fig. 9, the slope of Case 2 is very steep compared to the one of Case 3. MECS decreases the increasing speed of the norm $\|\mathbf{J}_q\|_2$, especially when $t < 60$ sec, the norm for Case 3 is even less than the one for Case 1, the perfect case (given by using the nominal controller with no model errors or external disturbances with full state measurement information). As shown in Fig. 8, at the beginning of the simulation the control torque for each axis of Case 3 is relatively larger than the ones for the other two cases. Since the initial value of the rate is not zero, the rate dependent part of the true model error dominates. At the beginning of the simulation MECS not only cancels this initial model error but also makes the system response faster than the one for the perfect case. As shown in Fig. 6, the first minimum for Case 3 is 1.3 sec faster than the one for Case 1.

CONCLUSION

A new approach to determine modelling errors in a dynamical system was derived using a modified approximate receding-horizon expression with a Taylor series expansion at each instant of time. This new approach was used in the model-error control synthesis design to provide robustness with respect to extreme modelling errors. An application was shown for the spacecraft attitude control problem using attitude-angle information only. A Kalman filter was designed to estimate the angular velocity, which was subsequently used in the overall controller. Simulation results indicated that a nominal controller combined with the model-error control synthesis approach produced robust transient response behaviors, and the steady-state attitude errors were much smaller than nominal controller only design case. In addition the closed-loop system is globally quadratically stable for a norm bounded nonlinear uncertainty.

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APPENDIX

The parameters in Eq. (50) are given by

$$\begin{aligned} a_1 = \big[& k\left(r_0\, w_0\, e^{r_p+w_p}\, k^2+1\right) h^6 + 4\, r_0\, w_0\, e^{r_p+w_p}\, p\, k^2\, h^5 \\ & - 2\left(7\, r_0\, w_0\, e^{r_p+w_p}\, k^2 - 2\, r_0\, w_0\, e^{r_p+w_p}\, p^2\, k + 1\right) h^4 \\ & - 28\, r_0\, w_0\, e^{r_p+w_p}\, p\, k\, h^3 + 52\, r_0\, w_0\, e^{r_p+w_p}\, k\, h^2 \\ & + 8\, r_0\, w_0\, e^{r_p+w_p}\, p\, h - 24 r_0\, w_0\, e^{r_p+w_p} \big] / d_a \end{aligned} \tag{81a}$$

$$\begin{aligned} a_2 = \big[& p\left(r_0\, w_0\, e^{r_p+w_p}\, k^2+1\right) h^6 \\ & + 2\left(2\, r_0\, w_0\, e^{r_p+w_p}\, p^2\, k - r_0\, w_0\, e^{r_p+w_p}\, k^2 - 1\right) h^5 \\ & + 2\left(2\, r_0\, w_0\, e^{r_p+w_p}\, p^3 - 9\, r_0\, w_0\, e^{r_p+w_p}\, p\, k\right) h^4 \\ & + 4\left(5\, r_0\, w_0\, e^{r_p+w_p}\, k - 7\, r_0\, w_0\, e^{r_p+w_p}\, p^2\right) h^3 \\ & + 64\, r_0\, w_0\, e^{r_p+w_p}\, p\, h^2 - 48\, r_0\, w_0\, e^{r_p+w_p}\, h \big] / d_a \end{aligned} \tag{81b}$$

$$\begin{aligned} a_3 = \big[& -\left(r_0\, w_0\, e^{r_p+w_p}\, k^2+1\right) h^6 - 4\, r_0\, w_0\, e^{r_p+w_p}\, p\, k\, h^5 \\ & + 2\left(7\, r_0\, w_0\, e^{r_p+w_p}\, k - 2\, r_0\, w_0\, e^{r_p+w_p}\, p^2\right) h^4 \\ & + 28\, r_0\, w_0\, e^{r_p+w_p}\, p\, h^3 - 48\, r_0\, w_0\, e^{r_p+w_p}\, h^2 \big] / d_a \end{aligned} \tag{81c}$$

$$\begin{aligned} a_4 = 2\, \big[& h^4 - 2\, r_0\, w_0\, e^{r_p+w_p}\, k\, h^2 - 4\, r_0\, w_0\, e^{r_p+w_p}\, p\, h \\ & + 12\, r_0\, w_0\, e^{r_p+w_p} \big] / d_a \end{aligned} \tag{81d}$$

where

$$\begin{aligned} d_a = & \left(r_0\, w_0\, e^{r_p+w_p}\, k^2+1\right) h^6 + 4\, r_0\, w_0\, e^{r_p+w_p}\, p\, k\, h^5 \\ & + 4\, r_0\, w_0\, e^{r_p+w_p}\left(p^2 - 3\, k\right) h^4 - 24\, r_0\, w_0\, e^{r_p+w_p}\, p\, h^3 \\ & + 2\, r_0\, w_0\, e^{r_p}\left(18\, e^{w_p} + e^{r_p}\right) h^2 \end{aligned} \tag{82}$$

The matrix P_q used in Eq. (72) is given by

$$P_q \approx 1 \times 10^7 \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{83}$$

where

$$P_{11} = \begin{bmatrix} 0.0001 & 0.0002 & 0 \\ 0.0002 & 0.0027 & 0.0003 \\ 0 & 0.0003 & 0.0006 \end{bmatrix} \tag{84a}$$

$$P_{22} = \begin{bmatrix} 1.1646 & 0.0009 & 0.2543 \\ 0.0009 & 0.0001 & 0.0002 \\ 0.2543 & 0.0002 & 0.0556 \end{bmatrix} \tag{84b}$$

$$P_{12} = \begin{bmatrix} -0.0007 & 0 & -0.0002 \\ -0.0131 & 0 & -0.0029 \\ -0.0254 & 0 & -0.0055 \end{bmatrix} \tag{84c}$$