FUNDAMENTAL RELATION OF APPROXIMATE RECEDING-HORIZON OPTIMIZATION TO THE SIMPLE MASS-SPRING-DAMPER SYSTEM

Jongrae Kim*
Department of Electrical & Computer Engineering
University of California
Santa Barbara, CA, 93106-9560

John L. Crassidis†
Department of Mechanical & Aerospace Engineering
University at Buffalo, The State University of New York
Amherst, NY 14260-4400

ABSTRACT

Model-error control synthesis is a nonlinear robust control approach that uses an optimal solution to cancel the effects of modelling errors and external disturbances on a system. The model error is determined by the approximated receding-horizon optimal solution. The future states are predicted by a repeated first-order Taylor series method or a higher-order Taylor series method. In this paper we show the fundamental relation between these two methods using the physical parameters, such as time constant and damping factor, of a simple mass-spring-damper system.

INTRODUCTION

Model-Error Control Synthesis (MECS) is a signal synthesis adaptive control method.[1] Robustness is achieved by applying a correction control, which is determined during the estimation process, to the nominal control vector thereby eliminating the effects of modelling errors at the system output.[2] The model-error vector is estimated by using either a one-step ahead prediction approach,[1,3] an Approximate Receding-Horizon (ARH) approach,[4] or a Modified Approximate Receding-Horizon (MARH) approach.[5] Choosing among the one-step ahead prediction approach, the ARH approach, or the MARH approach to determine the model error depends on the particular properties and required robustness in the system to be controlled.

In Ref. [1] MECS with the one-step ahead prediction approach is first applied to suppress the wing rock motion of a slender delta wing, which is described by a highly nonlinear differential equation. Results indicated that this approach provides adequate robustness for this particular system. In Ref. [3] a simple study to test the stability of the closed-loop system is presented using a Padé approximation for the time delay, and we showed the relation between the system zeros and the weighting in the cost function. The analysis proved that some systems may not be stabilized using the original model-error estimation algorithm, which leads to the ARH approach in the MECS design to determine the model-error vector in the system.[4] The closed-form solution of the ARH approach using Quadratic Programming (QP) is first presented by Lu.[6] The model-error vector is determined by the ARH optimal solution.[4]

Using the ARH approach, the capability of MECS is expanded so that unstable non-minimum phase systems can be stabilized. Furthermore, Ref. [4] shows a method to calculate the stable regions with respect to the weighting and the length of receding-horizon step-time using the Hermite-Biehler theorem.[7] After the stable region is found, the weighting and the length of receding-horizon step-time are chosen to minimize the H_∞-norm of the sensitivity function.[4]

The ARH solution for an r^{th}-order relative degree system shows that the model-error solution is zero before the end of receding-horizon step-time is reached. Some parts of the model-error vector are separated completely from the constraints, so that the optimal solution for those parts are automatically zero. To avoid this situation the state prediction is substituted by an r^{th}-order Taylor series expansion instead of a repeated first-order expansion in the ARH approach. We call this the Modified Approximate Receding-Horizon (MARH) approach, which leads to an even more robust MECS law than with the ARH solution.[5]

In Refs. [5] and [8] the MARH approach is used to the spacecraft attitude control problem for the case where the only available information is attitude-angle measurements, i.e., with no angular-velocity measurements, and the limit cycle oscillation control of aeroelastic system for the case when the pitch and the plunge displacement information are available. In this

*Post-Doctoral Fellow, Member AIAA, jongrae@ece.ucsb.edu
†Associate Professor, Associate Fellow AIAA, johnc@eng.buffalo.edu

paper we show the fundamental relation between the ARH and MARH approaches using the physical parameters, such as time constant and damping factor, of a simple mass-spring-damper system

The organization of this paper is as follows. First, the model-error representation of a nonlinear system is introduced. Second, model-error control synthesis is summarized. Third, a systematic approach for robust stability and optimal control design is presented. Finally, relations between the optimal model-error solutions using the physical parameters of the simple mass-spring-damper system a shown.

PROBLEM FORMULATION

Consider the following general form of a real plant:

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t)\right] + B\left[\mathbf{x}(t)\right]\mathbf{u}(t) + G_w\left[\mathbf{x}(t)\right]\mathbf{w}(t) \tag{1a}$$
$$\mathbf{y}(t) = \mathbf{c}\left[\mathbf{x}(t)\right] \tag{1b}$$

where $\mathbf{f}\left[\mathbf{x}(t)\right] \in \Re^n$, $B\left[\mathbf{x}(t)\right] \in \Re^{n\times q_u}$ is the control input distribution matrix, $G_w\left[\mathbf{x}(t)\right] \in \Re^{n\times q_w}$ is the external disturbance distribution matrix, $\mathbf{x}(t) \in \Re^n$ is the state vector, $\mathbf{u}(t) \in \Re^{q_u}$ is the control input, $\mathbf{w}(t) \in \Re^{q_w}$ is the external disturbance, $\mathbf{c}\left[\mathbf{x}(t)\right] \in \Re^m$ is the measurement vector, and $\mathbf{y}(t) \in \Re^m$ is the output vector. We assume $\mathbf{f}\left[\mathbf{x}(t)\right]$, $B\left[\mathbf{x}(t)\right]$, and $G\left[\mathbf{x}(t)\right]$ are C^2, i.e., the function itself, the first and the second derivatives are continuous, respectively, and $\mathbf{c}\left[\mathbf{x}(t)\right]$ is sufficiently differentiable. In addition, we assume $\mathbf{f}\left(\mathbf{0}\right) = \mathbf{0}$ (if not, we can theoretically transform the states $\mathbf{x}(t)$ to some new states so that this condition holds) and $q_u \le q_w$, i.e., in general the number of actuators is less than or equal to the dimension of external disturbances.

Define the following:

$$\mathbf{f}\left[\mathbf{x}(t)\right] \equiv \hat{\mathbf{f}}\left[\mathbf{x}(t)\right] + \Delta\mathbf{f}\left[\mathbf{x}(t)\right] \tag{2a}$$
$$B\left[\mathbf{x}(t)\right] \equiv \hat{B}\left[\mathbf{x}(t)\right] + \Delta B\left[\mathbf{x}(t)\right] \tag{2b}$$

where $\hat{\mathbf{f}}\left[\mathbf{x}(t)\right]$ is the assumed model and $\Delta\mathbf{f}\left[\mathbf{x}(t)\right]$ is a model error in $\mathbf{f}\left[\mathbf{x}(t)\right]$, respectively, and $\hat{B}\left[\mathbf{x}(t)\right]$ is the assumed model and $\Delta B\left[\mathbf{x}(t)\right]$ is a model error in $B\left[\mathbf{x}(t)\right]$, respectively. Substituting Eq. (2a) into Eq. (1a) gives

$$\dot{\mathbf{x}}(t) = \hat{\mathbf{f}}\left[\mathbf{x}(t)\right] + \hat{B}\left[\mathbf{x}(t)\right]\mathbf{u}(t) + \hat{G}\left[\mathbf{x}(t)\right]\hat{\mathbf{u}}(t) \tag{3}$$

with

$$\begin{aligned}\hat{G}\left[\mathbf{x}(t)\right]\hat{\mathbf{u}}(t) \equiv\ & \Delta\mathbf{f}\left[\mathbf{x}(t)\right] + \Delta B\left[\mathbf{x}(t)\right]\mathbf{u}(t) \\ & + G_w\left[\mathbf{x}(t)\right]\mathbf{w}(t)\end{aligned} \tag{4}$$

where $\hat{G} \in \Re^{n\times q_w}$ is a model-error distribution matrix and $\hat{\mathbf{u}}(t)$ is the model-error vector. Since the model error occurs only in the dynamics, acceleration or momentum, parts of $\mathbf{f}\left[\mathbf{x}(t)\right]$, at least first $n - q_w$ rows, i.e., the kinematics parts, of $\Delta\mathbf{f}\left[\mathbf{x}(t)\right]$, $\Delta\hat{B}\left[\hat{\mathbf{x}}(t)\right]$ and $G_w\left[\mathbf{x}(t)\right]$ are all zero as follows:

$$\dot{\mathbf{x}}_1(t) = \hat{\mathbf{f}}_1\left[\mathbf{x}(t)\right], \quad \text{(exact kinematics)} \tag{5a}$$
$$\begin{aligned}\dot{\mathbf{x}}_2(t) =\ & \hat{\mathbf{f}}_2\left[\mathbf{x}(t)\right] + \hat{B}_2\left[\mathbf{x}(t)\right]\mathbf{u}(t) \\ & + \hat{G}_2\left[\mathbf{x}(t)\right]\hat{\mathbf{u}}(t), \quad \text{(uncertain dynamics)}\end{aligned} \tag{5b}$$

where $\mathbf{x}_1(t) \in \Re^{n-q_w}$, $\mathbf{x}_2(t) \in \Re^{q_w}$,

$$\mathbf{x}(t) = \left\{\mathbf{x}_1^T(t), \quad \mathbf{x}_2^T(t)\right\}^T \tag{6a}$$
$$\hat{\mathbf{f}}\left[\mathbf{x}(t)\right] = \left\{\hat{\mathbf{f}}_1^T\left[\mathbf{x}(t)\right], \quad \hat{\mathbf{f}}_2^T\left[\mathbf{x}(t)\right]\right\}^T \tag{6b}$$
$$\hat{B}\left[\mathbf{x}(t)\right] = \begin{bmatrix} 0_{(n-q_w)\times q_u} \\ \hat{B}_2\left[\mathbf{x}(t)\right]\end{bmatrix} \tag{6c}$$
$$\hat{G}\left[\mathbf{x}(t)\right] = \begin{bmatrix} 0_{(n-q_w)\times q_w} \\ \hat{G}_2\left[\mathbf{x}(t)\right]\end{bmatrix}, \tag{6d}$$

and

$$\begin{aligned}\hat{G}_2\left[\mathbf{x}(t)\right]\hat{\mathbf{u}}(t) \equiv\ & \Delta\mathbf{f}_2\left[\mathbf{x}(t)\right] + \Delta B_2\left[\mathbf{x}(t)\right]\mathbf{u}(t) \\ & + G_{w2}\left[\mathbf{x}(t)\right]\mathbf{w}(t)\end{aligned} \tag{7}$$

where

$$\Delta\mathbf{f}\left[\mathbf{x}(t)\right] = \begin{bmatrix} 0_{(n-q_w)\times 1} \\ \Delta\mathbf{f}_2\left[\mathbf{x}(t)\right]\end{bmatrix} \tag{8a}$$
$$\Delta B\left[\mathbf{x}(t)\right] = \begin{bmatrix} 0_{(n-q_w)\times q_u} \\ \Delta B_2\left[\mathbf{x}(t)\right]\end{bmatrix} \tag{8b}$$
$$\Delta G_w\left[\mathbf{x}(t)\right] = \begin{bmatrix} 0_{(n-q_w)\times q_w} \\ \Delta G_{w2}\left[\mathbf{x}(t)\right]\end{bmatrix} \tag{8c}$$

and $0_{(n-q_w)\times 1}$ is an $(n - q_w) \times 1$ zero matrix, etc.

Assume that $\mathbf{x}(0) = \hat{\mathbf{x}}(0)$ and the model error in the output equation, $\mathbf{y}(t)$, is already compensated. Then, the following general form of system equations is obtained:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{f}}\left[\hat{\mathbf{x}}(t)\right] + \hat{B}\left[\hat{\mathbf{x}}(t)\right]\mathbf{u}(t) + \hat{G}\left[\hat{\mathbf{x}}(t)\right]\hat{\mathbf{u}}(t) \tag{9a}$$
$$\hat{\mathbf{y}}(t) = \hat{\mathbf{c}}\left[\hat{\mathbf{x}}(t)\right] \tag{9b}$$

where $\hat{\mathbf{f}}\left[\hat{\mathbf{x}}(t)\right] \in \Re^n$ is the assumed model vector, $\hat{B}\left[\hat{\mathbf{x}}(t)\right] \in \Re^{n\times q_u}$ is the assumed control input distribution matrix, $\hat{G}\left[\hat{\mathbf{x}}(t)\right] \in \Re^{n\times q_w}$ is the model-error distribution matrix, $\hat{\mathbf{x}}(t) \in X \subset \Re^n$ is the state estimate vector, $\mathbf{u}(t) \in \Omega_u \subset \Re^{q_u}$ is the control input, $\hat{\mathbf{u}}(t) \in \Omega_{\hat{u}} \subset \Re^{q_w}$ is the to-be-determined model error, $\hat{\mathbf{c}}\left[\hat{\mathbf{x}}(t)\right] \in \Re^m$ is the measurement vector ($m \le n$ in general), and $\hat{\mathbf{y}}(t) \in \Re^m$ is the estimated output vector.[6]

For most mechanical systems $\Omega_u \subset \Omega_{\hat{u}}$, i.e., the system is under-actuated or fully actuated at the maximum, so that $q_u \le q_w$, where q_w is the dimension of the dynamics parts. In this paper the case of redundant actuator systems is not included. The admissible sets X and $\Omega_u \subset \Omega_{\hat{u}}$ are compact and $X \times \Omega_{\hat{u}}$ contains a neighborhood around the origin. One important

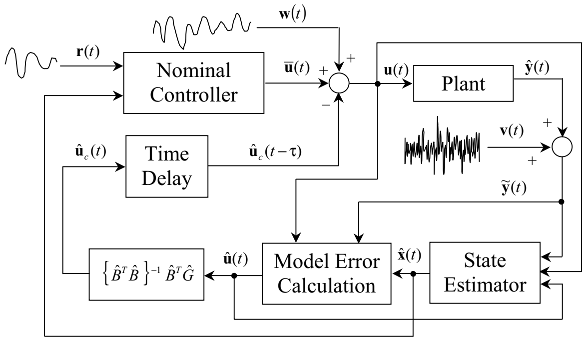

Fig. 1 Overall Block Diagram with MECS

assumption is $m \geq q_w$, i.e., the dimension of the measurement vector is at least the dimension of dynamics. Also, assume that the rank of $\hat{G}\,[\hat{\mathbf{x}}(t)]$ is q_w, i.e., full rank. In addition controllability, observability, stable zero dynamics, and well-defined relative degree with respect to $\hat{\mathbf{u}}(t)$ are presumed, and the assumptions about continuity and $\hat{\mathbf{f}}\,(\mathbf{0}) = \mathbf{0}$ hold. Also, we assume that each element of the model-error vector affects the output. From now on in this paper the real plant is represented by Eqs. (9a) and (9b).

State-observable measurements are assumed for Eq. (9b) in the following form:

$$\tilde{\mathbf{y}}(t) = \hat{\mathbf{c}}[\hat{\mathbf{x}}(t)] + \mathbf{v}(t) \tag{10}$$

where $\tilde{\mathbf{y}}(t) \in \Re^m$ is the measurement vector at time t, and $\mathbf{v}(t) \in \Re^m$ is the measurement noise vector, which is assumed to be a zero-mean, stationary, Gaussian noise distributed process with

$$E\left\{\mathbf{v}(t)\right\} = \mathbf{0} \tag{11a}$$

$$E\left\{\mathbf{v}(t)\,\mathbf{v}^T(t+\Delta t)\right\} = R\,\delta(\Delta t) \tag{11b}$$

where $E\{\cdot\}$ is expectation, $R \in \Re^{m\times m}$ is a positive-definite symmetric covariance matrix, $\delta(\cdot)$ is Dirac delta function, and Δt is the sampling rate for the discrete measurement case.[9]

MECS

The block diagram with MECS is shown in Fig. 1, where $\mathbf{r}(t)$ is the reference command. The model error is determined using the estimated state, $\hat{\mathbf{x}}(t)$, the control input, $\mathbf{u}(t)$, and the current measurement, $\tilde{\mathbf{y}}(t)$. The estimated model error, $\hat{\mathbf{u}}(t)$, corrects not only the nominal control input, $\bar{\mathbf{u}}(t)$, but also the filter model. After the model error is determined, any state estimator or observer can be implemented, including a Kalman filter. The total control input $\mathbf{u}(t)$ with model-error correction is given by

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) - \hat{\mathbf{u}}_c(t-\tau) \tag{12}$$

where $\bar{\mathbf{u}}(t)$ is the nominal control input at time t, which can be any controller, i.e., Proportional-Integral-Derivative (PID) Control, Lead-Lag Compensator, Sliding Mode Control, H_∞ Control, Linear Quadratic Regulator (LQR) Control, Linear Quadratic Gaussian (LQG) Control, etc. The time delay τ is always present in the overall MECS design because the measurement $\tilde{\mathbf{y}}(t)$ must be given before the error in the system can be corrected. The term $\hat{\mathbf{u}}_c\,(t-\tau)$ is used to cancel the estimated model error at time $t-\tau$, determined by the current information using a Pseudo-Inverse ($n \geq q_u$, i.e., under-actuated) or least square solution as follows:[10]

$$\hat{\mathbf{u}}_c(t) = \left\{\hat{B}^T\left[\hat{\mathbf{x}}(t)\right]\hat{B}\left[\hat{\mathbf{x}}(t)\right]\right\}^{-1} \times \hat{B}^T\left[\hat{\mathbf{x}}(t)\right]\hat{G}\left[\hat{\mathbf{x}}(t)\right]\hat{\mathbf{u}}(t) \tag{13}$$

When $\hat{B}\,[\hat{\mathbf{x}}(t)] = \hat{G}\,[\hat{\mathbf{x}}(t)]$, i.e., actuators are installed for each component of dynamics part independently, $\hat{\mathbf{u}}_c(t)$ is equal to $\hat{\mathbf{u}}(t)$.

The receding-horizon optimization problem is set up as follows:[6]

$$\min_{\hat{\mathbf{u}}} J\left[\hat{\mathbf{x}}(t),\, t,\, \hat{\mathbf{u}}(t)\right] = \frac{1}{2}\int_t^{t+T}\left[\mathbf{e}^T(\xi)\,R^{-1}(\xi)\,\mathbf{e}(\xi) + \hat{\mathbf{u}}^T(\xi)\,W(\xi)\,\hat{\mathbf{u}}(\xi)\right]d\xi \tag{14}$$

subject to the system given by Eq. (9) and the terminal constraint, $\mathbf{e}(t+T) = 0$, where the residual error is defined by

$$\mathbf{e}(t) \equiv \tilde{\mathbf{y}}(t) - \hat{\mathbf{y}}(t) \tag{15}$$

where $R^{-1}(\xi)$ and $W(\xi)$ are positive definite and symmetric weighting matrices for all $\xi \in [t,\, t+T]$. Note that T is the receding-horizon optimization interval, which in general is not the sampling time.

At each time t, the optimal model-error solution, $\hat{\mathbf{u}}^*$, over a finite horizon $[t,\, t+T]$ is determined online. Then, the current model error $\hat{\mathbf{u}}(t)$ is set equal to $\hat{\mathbf{u}}^*$ and this process is repeated for every instant of time t, continuously. Define $h \equiv T/N$ for some integer $N \geq n/m$, where N is the number of sub-intervals on $[t,\, t+T]$. Now, $\hat{\mathbf{y}}(t+kh)$ for each $k = 1,\, 2,\, \ldots,\, N$ is approximated by an Taylor series. MECS is divided into two approaches depending how $\hat{\mathbf{y}}(t+kh)$ to be approximated, which are discussed next.

ARH APPROACH

In the first approach $\hat{\mathbf{y}}(t+kh)$ for each $k = 1,\, 2,\, \ldots,\, N$ is approximated by an iterative first-order Taylor series. For simplicity and avoiding the cross-product terms of $\hat{\mathbf{u}}(t+ih)$ and $\hat{\mathbf{u}}(t+jh)$, $\hat{B}\,[\hat{\mathbf{x}}(t+kh)] \approx \hat{B}\,[\hat{\mathbf{x}}(t)]$,$\hat{G}\,[\hat{\mathbf{x}}(t+kh)] \approx \hat{G}\,[\hat{\mathbf{x}}(t)]$ and $\hat{F}\,[\hat{\mathbf{x}}(t+kh)] \approx \hat{F}\,[\hat{\mathbf{x}}(t)]$, where $\hat{F} \equiv \partial\hat{\mathbf{f}}/\partial\hat{\mathbf{x}}$. In addition since the future values of $\tilde{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are unknown, $\tilde{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are assumed to be constants over the finite horizon $[t,\, t+T]$. Then, the following expression for $1 \leq k \leq N$ is obtained:[6]

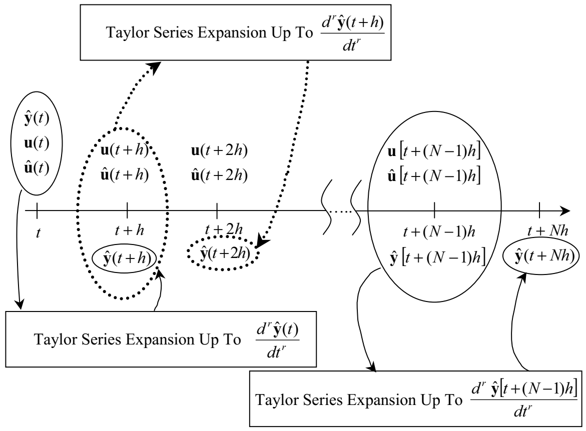

Fig. 2 State Prediction for MARH Approach

$$
\begin{aligned}
\hat{\mathbf{y}}(t+kh) \approx \hat{\mathbf{y}}(t) + h\,\hat{C} \sum_{i=0}^{k-1} \left(I_{n\times n} + h\hat{F}\right)^{i} \\
\times \Big\{ \hat{\mathbf{f}}\left[\hat{\mathbf{x}}(t)\right] + \hat{B}\left[\hat{\mathbf{x}}(t)\right]\,\mathbf{u}(t) \\
+ \hat{G}\left[\hat{\mathbf{x}}(t)\right]\,\hat{\mathbf{u}}\left[t+(k-1-i)h\right] \Big\}
\end{aligned}
\tag{16}
$$

where $\hat{C} \equiv \partial\hat{\mathbf{c}}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$ and assumed to be the rank m, $\hat{\mathbf{f}}$, $\hat{F}$, $\hat{B}$, and $\hat{G}$ are evaluated at $\hat{\mathbf{x}}(t)$, and $I_{n\times n}$ is an $n \times n$ identity matrix.

MARH APPROACH

In the first approach the basic concept of how $\hat{\mathbf{y}}(t+kh)$ is approximated using the MARH solution is shown in Fig. 2. Using the given $\hat{\mathbf{y}}(t)$, $\mathbf{u}(t)$, and $\hat{\mathbf{u}}(t)$, the states at $t+h$ are approximated by a Taylor series expansion. The order of the expansion of each predicted state is given when $\hat{\mathbf{u}}(t)$ first appears due to successive differentiation of the output. For the states at time $t+2h$, the expansion is similar to the previous case, using the states at time $t+h$ when $\hat{\mathbf{u}}(t+h)$ first appears. Hence after this expansion is given, the states at time $t+2h$ are functions of the states, the control, and the model error at time $t+h$. Then the states at time $t+h$ in the predicted states at $t+2h$ are substituted by the approximated ones at the first stage. This process is repeated up to time $t+Nh$, which is now used to replace Eq. (16). The output prediction at $t+(k+1)h$ is given by

$$
\begin{aligned}
&\hat{\mathbf{y}}\left[t+(k+1)\,h\right] \approx \hat{\mathbf{y}}\left(t+kh\right) + \mathbf{z}\left[\hat{\mathbf{x}}\left(t+kh\right),\, h\right] \\
&\quad + \Lambda(h)\, S_u \left[\hat{\mathbf{x}}\left(t+kh\right)\right]\,\mathbf{u}(t) \\
&\quad + \Lambda(h)\, S_{\hat{u}} \left[\hat{\mathbf{x}}\left(t+kh\right)\right]\,\hat{\mathbf{u}}\left(t+kh\right)
\end{aligned}
\tag{17}
$$

for $k = 1, 2, \ldots, N$, where $\hat{\mathbf{y}}\left(t+kh\right)$ and $\hat{\mathbf{x}}\left(t+kh\right)$ are given by the predictions from the previous stage. This process is repeated up to all $\hat{\mathbf{x}}\left(t+kh\right)$ written in terms of $\hat{\mathbf{x}}\left(t\right)$.

APPROXIMATE COST FUNCTION

The future states are approximated by either the ARH or MARH approaches. The next step is to approximate the cost function. Define the following:

$$
\begin{aligned}
L(k\,h) \equiv\ & \mathbf{e}^T(t+k\,h)\, R_k^{-1}\, \mathbf{e}(t+k\,h) \\
& + \hat{\mathbf{u}}^T(t+k\,h)\, W_k\, \hat{\mathbf{u}}(t+k\,h)
\end{aligned}
\tag{18}
$$

for $k = 1, 2, \ldots, N-1$, and

$$
L(Nh) \equiv \mathbf{e}^T(t+Nh)\, R_N^{-1}\, \mathbf{e}(t+Nh) \tag{19}
$$

where R_k^{-1}, W_k, and R_N^{-1} are positive definite.

The cost function, J, to be minimized is approximated using a trapezoidal formula or Simpson's rule. With the following definition:

$$
\boldsymbol{\nu}_0 \equiv \left\{ \hat{\mathbf{u}}^T(t), \cdots, \hat{\mathbf{u}}^T\left[t+(N-1)h\right] \right\}^T \tag{20}
$$

the approximated cost, $\bar{J}$, can be rewritten in the quadratic form as

$$
\begin{aligned}
\bar{J} =\ & \frac{1}{2}\,\boldsymbol{\nu}_0^T H_0\left[\hat{\mathbf{x}}(t)\right]\,\boldsymbol{\nu}_0 + \mathbf{g}_0^T\left[\hat{\mathbf{x}}(t), \mathbf{u}(t), \tilde{\mathbf{y}}(t)\right]\,\boldsymbol{\nu}_0 \\
& + q_0\left[\hat{\mathbf{x}}(t), \mathbf{u}(t), \tilde{\mathbf{y}}(t)\right]
\end{aligned}
\tag{21}
$$

where H_0, $\mathbf{g}_0$ and q_0 are functions of $L(k\,h)$.[6] Also, the terminal constraint, $\mathbf{e}(t+T) = 0$, can be formulated as a constraint on $\boldsymbol{\nu}_0$ as follows: Then, the end point constraint is given by

$$
M^T \boldsymbol{\nu}_0 = \mathbf{d} \tag{22}
$$

where M and $\mathbf{d}$ are defined appropriately and the rank of M is m.

Finally, the solution is given by

$$
\begin{aligned}
\boldsymbol{\nu}_0 = -&\left[H_0^{-1} - H_0^{-1} M \left(M^T H_0^{-1} M\right)^{-1}\right. \\
&\left. M^T H_0^{-1}\right] \mathbf{g}_0(t) + \left[H_0^{-1} M \left(M^T H_0^{-1} M\right)^{-1}\right] \mathbf{d}
\end{aligned}
\tag{23}
$$

The first q_w equations give a current model error minimizing the cost function, which leads to a predictive filter structure:

$$
\hat{\mathbf{u}}\left[t;\, \hat{\mathbf{x}}(t),\, \mathbf{u}(t),\, \tilde{\mathbf{y}}(t),\, h\right] = I_{q_w \times N}\ \boldsymbol{\nu}_0 \tag{24}
$$

where $I_{q_w \times N}$ is $q_w \times q_w$ identity matrix with the remaining terms zero.

Since the output is assumed to be constant during the given interval, $\xi \in [t,\, t+T]$, the assumption becomes less accurate as the receding-horizon time T increases and/or the speed of response increases. Therefore, the weights have to be adjusted accordingly such that the portion of measurement for $L(kh)$ is less as k increased. Since the measurement portion of $L(kh)$ is inversely proportional to R_k and W_k, the ideal shape of each weighting matrix is as follows:

$$
R_k = e^{r_p}\, R_{k-1}, \quad \text{for } k = 1, 2, 3 \tag{25a}
$$

$$
W_k = e^{w_p}\, W_{k-1}, \quad \text{for } k = 1, 2 \tag{25b}
$$

where each diagonal term of R_0 is usually set to the covariance of each sensor noise and W_0 is set to a positive real by trial and error, and r_p and w_p are non-negative real values.

ROBUST STABILITY

As stated previously, MECS has an unavoidable time delay effect, shown in Eq. (12). Since the Laplace transform of a time delay is a nonlinear function, i.e., an exponential function, the direct application of linear stability theorem is not possible. To overcome this difficulty, a polynomial approximation for the time delay, the Padé approximation, is used from Refs. [11] and [12]. After the time delay is approximated, the closed loop system is expressed by some polynomials. Finally, the Hermite-Biehler theorem gives the necessary and sufficient conditions for a system to be Hurwitz stable.[7]

Theorem 1 *Hermite-Biehler Theorem*
Consider the following polynomial:

$$d_{cl}(s) = c_n\, s^n + c_{n-1}\, s^{n-1} + \cdots + c_2\, s^2 + c_1\, s + c_0 \quad (26)$$

where $c_n \neq 0$ *and* $d_{cl}(s)$ *can be decomposed as*

$$d_{cl}(s) = p(s) + s\, q(s) \quad (27)$$

where $p(s)$ *contains the even power terms and* $s\, q(s)$ *contains the odd power terms of* $d_{cl}(s)$*. Then* $d_{cl}(s)$ *is Hurwitz stable if and only if* c_n *and* c_{n-1} *are the same sign with all roots of* $p(j\omega)$ *and* $q(j\omega)$ *real, and the nonnegative roots satisfy the following interlacing property:*

$$0 < \omega_{p_1} < \omega_{q_1} < \omega_{p_2} < \omega_{q_2} < \cdots \quad (28)$$

where ω_{p_i} *and* ω_{q_i} *are the roots of* $p(j\omega)$ *and* $q(j\omega)$*, respectively.* ■

The proof can be found in Ref. [7]. In this paper the following scale function is defined to clearly show some widely spread values in the magnitude.

Definition 1 *Scale Function :* $\mathrm{sc}(\cdot)$

$$\mathrm{sc}(x) \equiv \mathrm{sgn}(x)\, \log_{10}(|x| + 1) \quad (29)$$

The scale function is a C^1 *function and is satisfied for all* $x_1 \leq x_2$*,* $sc(x_1) \leq sc(x_2)$*, i.e., it preserves the order. The derivative is given by*

$$\frac{d\,\mathrm{sc}(x)}{dx} = \frac{1}{(|x| + 1)\, \ln(10)}, \quad \textit{for all } x \in \Re \quad (30)$$

where $\ln(\cdot)$ *is the natural logarithm.* ■

From the Hermite-Biehler theorem the following is deduced:

Corollary 1 *Graphical interpretation for a* 6^{th}*-order Polynomial*
Consider the following 6^{th}*-order polynomial:*

$$d_{cl}(s) = c_6\, s^6 + c_5\, s^5 + c_4\, s^4 + c_3\, s^3 + c_2\, s^2 + c_1\, s + c_0 \quad (31)$$

where $c_6 > 0$*, then* $d_{cl}(s)$ *is Hurwitz stable if and only if the stability index,*

$$\varepsilon \equiv \mathrm{sc}(\kappa) \quad (32)$$

is greater than zero, where

$$\begin{aligned}\kappa \equiv \min\big(&\mathrm{I},\ \text{II-a},\ \text{II-b},\ \text{II-c},\\ &\text{III-a},\ \text{III-b},\ \text{III-c},\ \text{III-d}\big)\end{aligned} \quad (33)$$

and

$$\begin{aligned}
&\mathrm{I}:\ \underline{c}_0 > 0 && (34a)\\
&\text{II-a}:\ \mathit{min}\ (c_3\, c_5) > 0,\ \text{II-b}:\ \mathit{min}\ (c_5\, c_6) > 0, &&\\
&\text{II-c}:\ \mathit{min}\ (c_1\, c_5) > 0 && (34b)\\
&\text{III-a}: \bar{c}_6,\ \underline{c}_5,\ \bar{c}_4,\ \bar{c}_3,\ \underline{c}_2,\ \underline{c}_1,\ \bar{c}_0 &&\\
&\text{III-b}: \underline{c}_6,\ \bar{c}_5,\ \underline{c}_4,\ \underline{c}_3,\ \bar{c}_2,\ \bar{c}_1,\ \underline{c}_0 &&\\
&\text{III-c}: \underline{c}_6,\ \bar{c}_5,\ \bar{c}_4,\ \underline{c}_3,\ \underline{c}_2,\ \bar{c}_1,\ \bar{c}_0 &&\\
&\text{III-d}: \bar{c}_6,\ \underline{c}_5,\ \bar{c}_4,\ \bar{c}_3,\ \underline{c}_2,\ \underline{c}_1,\ \underline{c}_0 &&\\
&\textit{are substituted into } \mathrm{III}, \textit{respectively.} && (34c)
\end{aligned}$$

where $\underline{c}_i$ *and* $\bar{c}_i$ *are the lower and the upper bounds of each* c_i*, for* $i = 1, 2, \ldots, 6$*, and*

$$\mathrm{III}:\ -\left(4\, c_1\, c_5\, A^2 + 2\, c_3\, A\, B + B^2\right) > 0 \quad (35)$$

where

$$\begin{aligned}
A &\equiv c_1\, c_5\, c_6 - c_2\, c_5^2 + c_3\, c_4\, c_5 - c_3^2\, c_6 && (36a)\\
B &\equiv 2\, c_0\, c_5^3 - 2\, c_1\, c_4\, c_5^2 + 2\, c_1\, c_3\, c_5\, c_6 && (36b)
\end{aligned}$$

■

Proof:
The polynomial $d_{cl}(s)$ is decomposed as

$$\begin{aligned}
p(s) &= c_6\, s^6 + c_4\, s^4 + c_2\, s^2 + c_0 && (37a)\\
q(s) &= c_5\, s^4 + c_3\, s^2 + c_1 && (37b)
\end{aligned}$$

Substituting $s = j\omega$ gives

$$\begin{aligned}
p(j\omega) &= -c_6\omega^6 + c_4\omega^4 - c_2\omega^2 + c_0 && (38a)\\
q(j\omega) &= c_5\omega^4 - c_3\omega^2 + c_1 && (38b)
\end{aligned}$$

Since the above equations have only even powers, letting $\gamma \equiv \omega^2$ yields

$$\begin{aligned}
p(\gamma) &= -c_6\gamma^3 + c_4\gamma^2 - c_2\gamma + c_0 && (39a)\\
q(\gamma) &= c_5\gamma^2 - c_3\gamma + c_1 && (39b)
\end{aligned}$$

By the Hermite-Biehler theorem, for $d_{cl}(s)$ to be Hurwitz stable, c_6 and c_5 have to be the same sign, and the cubic and the quadratic equations have to have only positive real roots while satisfying the interlacing property. Since general solutions for cubic and quadratic equations exist, the roots can be directly calculated. Using the interlacing property a set of stability inequalities can be formulated, but still may not be easy to solve in general.

Let us consider the graphical property of the given problem. First, without loss of generality, let c_6 be positive. So one of the necessary conditions for $p(\gamma)$ to have only real roots is

$$\text{I}: \; p(0) = c_0 > 0 \tag{40}$$

The following two roots of $q(\gamma)$ have to be positive:

$$\text{II}: \; \gamma_{1,2} = \frac{c_3 \pm \sqrt{c_3^2 - 4\,c_1\,c_5}}{2\,c_5} > 0$$
$$\text{iff} \quad c_3^2 > 4\,c_1\,c_5, \; c_1 > 0, \; c_3 > 0, \; \text{and } c_5 > 0 \tag{41}$$

and finally, to satisfy the interlacing property, the values of $p(\gamma)$ at $\gamma_{1,2}$ must be

$$\text{III}: \; p(\gamma_1) < 0 \;\text{ and }\; p(\gamma_2) > 0 \tag{42}$$

Since $q(\gamma_1) = 0$ or $q(\gamma_2) = 0$ and $\gamma_1 < \gamma_2$, then

$$p(\gamma_1) = \frac{1}{c_5^2}\left(c_1\,c_5\,c_6 - c_2\,c_5^2 + c_3\,c_4\,c_5 - c_3^2\,c_6\right)\gamma_1 + \frac{1}{c_5^2}\left(c_0\,c_5^2 - c_1\,c_4\,c_5 + c_1\,c_3\,c_6\right) \tag{43}$$

With some algebraic manipulations and the definitions of A and B in Eq. (36), we have

$$c_3\,A + B < A\sqrt{c_3^2 - 4\,c_1\,c_5} \tag{44}$$

and for $\gamma = \gamma_2$:

$$c_3\,A + B > -A\sqrt{c_3^2 - 4\,c_1\,c_5} \tag{45}$$

Hence,

$$\left|\,c_3\,A + B\right| < \left|A\sqrt{c_3^2 - 4\,c_1\,c_5}\right| \tag{46}$$

Note that if the above inequality is satisfied, then $c_3^2 > 4\,c_1\,c_5$ is always true. Hence, $c_3^2 > 4\,c_1\,c_5$ is not required to be checked for stability. Squaring both sides of the above inequality gives

$$4\,c_1\,c_5\,A^2 + 2\,c_3\,A\,B + B^2 < 0 \tag{47}$$

Finally, the inequalities to be satisfied are summarized as follows:

$$\text{I}: \; c_0 > 0 \tag{48a}$$
$$\text{II}: \; c_1 > 0, \; c_3 > 0 \text{ and } c_5 > 0 \tag{48b}$$
$$\text{III}: \; -\left(4, c_1\,c_5\,A^2 + 2\,c_3\,A\,B + B^2\right) > 0 \tag{48c}$$

with the assumption $c_6 > 0$. Now, bound each of the coefficients as follows:

$$\underline{c}_i \le c_i \le \bar{c}_i, \;\text{ for } i = 1, 2, \ldots, 6 \tag{49}$$

Then, the first two conditions are easy to check, i.e.,

$$\text{I}: \; \underline{c}_0 > 0 \tag{50a}$$
$$\text{II-a}: \; \min\;(c_3\,c_5) > 0 \tag{50b}$$
$$\text{II-b}: \; \min\;(c_5\,c_6) > 0 \tag{50c}$$
$$\text{II-c}: \; \min\;(c_1\,c_5) > 0 \tag{50d}$$

However, the third condition, III, is difficult to solve analytically. It can be formulated as a minimization problem, but the problem is highly nonlinear and the solution cannot be guaranteed to be a global minimum. To overcome this difficulty, the extreme polynomials defined in Ref. [13] are used, i.e., if III is satisfied for the following extreme polynomials, it is satisfied for the rest of uncertainty space. The coefficients for the four extreme conditions are given by

$$\text{III-a}: \bar{c}_6, \; \underline{c}_5, \; \bar{c}_4, \; \bar{c}_3, \; \underline{c}_2, \; \underline{c}_1, \; \bar{c}_0 \tag{51a}$$
$$\text{III-b}: \underline{c}_6, \; \bar{c}_5, \; \underline{c}_4, \; \underline{c}_3, \; \bar{c}_2, \; \bar{c}_1, \; \underline{c}_0 \tag{51b}$$
$$\text{III-c}: \underline{c}_6, \; \bar{c}_5, \; \bar{c}_4, \; \underline{c}_3, \; \underline{c}_2, \; \bar{c}_1, \; \bar{c}_0 \tag{51c}$$
$$\text{III-d}: \bar{c}_6, \; \underline{c}_5, \; \bar{c}_4, \; \bar{c}_3, \; \underline{c}_2, \; \underline{c}_1, \; \underline{c}_0 \tag{51d}$$

In fact if the minimum of the above conditions is greater than zero, Hurwitz stability is guaranteed by the Hermite-Biehler theorem. Define κ as follows:

$$\kappa \equiv \min\left(\text{I}, \; \text{II-a}, \; \text{II-b}, \; \text{II-c}, \; \text{III-a}, \; \text{III-b}, \; \text{III-c}, \; \text{III-d}\right) \tag{52}$$

and the stability index, ε, is defined by

$$\varepsilon \equiv \text{sc}(\kappa) \tag{53}$$

Therefore, if $\varepsilon > 0$, then $d_{cl}(s)$ is Hurwitz stable, otherwise it is not stable.

Q.E.D.

Corollary 1 only gives the minimum requirements for $d_{cl}(s)$ to be Hurwitz stable. Therefore, for a system to be satisfied given design specifications, such as settling time, maximum overshoot, phase margin, gain margin, etc., the performance of the closed-loop response has to be examined.

Although the determined model error is based on a certain optimization technique, the solution does not give any explicit information about how to choose the optimal weighting and optimal length of optimization interval. To qualify the model error with respect to the weighting and the optimization length, another performance measure is required. H_∞ and H_2 norms of the weighted sensitivity or the weighted complementary sensitivity functions are adopted for the performance measure.

As illustrated in Fig. 3, the steps involved in the optimization problem are as follows: First, find a stable region with the given uncertainty bounds using Corollary 1; second, inside the stable region for the nominal system calculate the H_∞ or H_2 norm and some performance specifications, e.g., settling time, maximum overshoot, etc.; finally, choose the optimal weighting and length of the optimization interval such that the norms are minimized. This method is adopted from Ref. [14].

In Ref. [14] the order of controller is fixed, such as a PI or PID Controller, and after the stable region is

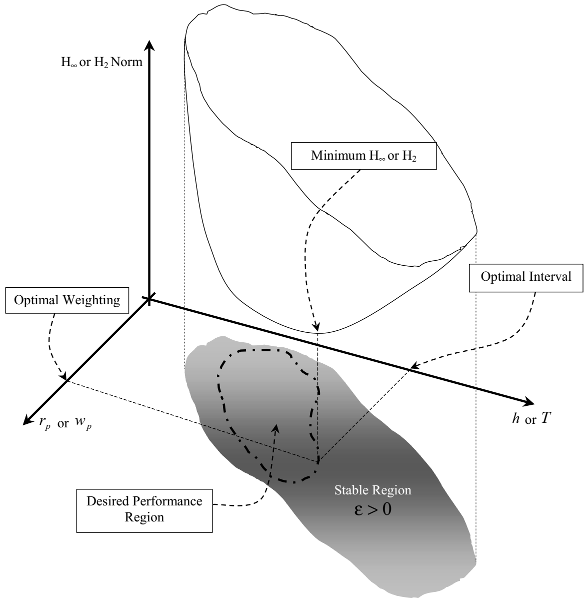

Fig. 3 Optimal Length of the Interval & Weighting

found, inside the stable region each gain is selected with performance specifications such that the norm is minimized. Also, in Ref. [14] the extended version of the Hermite-Biehler theorem is derived for application to higher-order characteristic equations. Finally, the optimization problem is given by

$$\min ||W_S(j\omega)S(j\omega)||_\infty \text{ or } \min ||W_T(j\omega)T(j\omega)||_\infty \quad \text{subject to } \varepsilon > 0 \tag{54}$$

where $W_S(j\omega)$ and $W_T(j\omega)$ are the weighting functions.

MASS-SPRING-DAMPER SYSTEM

Consider the following Single-Input-Single-Output (SISO) time invariant second-order linear system, i.e., a simple mass-spring-damper system:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = u(t) + \hat{u}(t) \tag{55}$$

where $\zeta = 0.5$ and $\omega_n = 1$, which are the nominal values and the whole model-error effect is dumped into $\hat{u}(t)$. The following PI nominal controller is chosen to minimize the sensitivity function of Eq. (55):

$$\bar{u}(t) = \frac{2.5s + 0.1}{s}\,[\hat{y}(t)] \tag{56}$$

ARH APPROACH

For $N = 3$, using a trapezoidal formula we have the model error for the system, i.e. Eq. (55), as follows:

$$\hat{u}(t) = a_1\,\hat{x}(t) + a_2\,\dot{\hat{x}}(t) + a_3\,u(t) + a_4\,\tilde{y}(t) \tag{58}$$

where

$$a_1 \equiv 2\,\left[(\,h-2\,)\left(\,h^3 - 3\,h^2 + 1\,\right)\,w_1\,r_2 + h^4\,\left(\,h^2 - 1\,\right)\right]/a_d \tag{59a}$$

$$a_2 \equiv 2\,h\,\left[-3\,(\,h-1\,)(\,h-2\,)\,w_1\,r_2 + h^4\,(\,h-2\,)\right]/a_d \tag{59b}$$

$$a_3 \equiv -2\,h^2\,\left[(\,h-2\,)(\,h-3\,)\,w_1\,r_2 + h^4\right]/a_d \tag{59c}$$

$$a_4 \equiv -2\,\left[(h-2\,)\,w_1\,r_2 - h^4\right]/a_d \tag{59d}$$

and

$$a_d \equiv 2\,h^2\,\left[(\,h-2\,)^2\,w_1\,r_2 + h^4 + \frac{w_0\,r_2}{2}\right] \tag{60}$$

where a_d is never equal to zero for $h > 0$ and positive weighting because H_0 and M^T are always full rank. Note that a_2 is always zero at $h = 2$ sec regardless of the values of the weighting. Also, a_1, a_3, and a_4 are not functions of $(w_1\,r_2)$ at $h = 2$, but only a function of $(w_0\,r_2)$ in a_d. Hence, a_1, a_3, and a_4 converge to zero as r_p is increased with $h = 2$, i.e., less correction of the model error. As a result, the maximum degree of freedom to adjust each coefficient is three, i.e., h, $(w_0\,r_2)$, and $(w_1\,r_2)$; however, when $h = 2$, one of the degrees of freedom, $(w_1\,r_2)$, vanishes.

The ARH approach with $N = 3$, a_2 in Eq. (58) is always zero at $h = 1/(\zeta\,\omega_n)$. If the length of the optimization subinterval, h, is equal to the time constant, $1/(\zeta\,\omega_n)$, the determined model error is independent of the velocity, $\dot{\hat{x}}(t)$. This is caused by the fact that the only non-zero off-diagonal term of H_0 is given by

$$h_{0_{12}} = -\frac{h^5}{r_3}\times(h\,\zeta\,\omega_n - 1) \tag{61}$$

where $h_{0_{12}}$ is the first row and the second column element of H_0, and $h_{0_{12}}$ is zero at $h = 1/(\zeta\,\omega_n)$, and so the first element of $M^T = \{\,m_1,\,m_2,\,m_3\,\}$, which is given by $2\,h\,(1 - \zeta\,\omega_n\,h)$, is zero. As a result the coefficient for $\dot{\hat{x}}(t)$ of the first element of $\mathbf{g}_0$ is zero and the first element of the matrix multiplied by $d(t)$, i.e., $H_0^{-1}\,M\,\left(M^T\,H_0^{-1}\,M\right)^{-1}$, is zero. Therefore, the solution is given by

$$\boldsymbol{\nu}_0 = \begin{bmatrix}\hat{u}(t)\\ \hat{u}(t+h)\\ \hat{u}(t+2h)\end{bmatrix} = \begin{bmatrix}h_{0_{11}}^{-1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & h_{0_{33}}^{-1}\end{bmatrix}\mathbf{g}_0(t) + \begin{bmatrix}0\\ \zeta\,\omega_n\\ 0\end{bmatrix} d(t) = \begin{bmatrix}h_{0_{11}}^{-1}\,g_{0_1}(t)\\ \zeta\,\omega_n\,d(t)\\ h_{0_{33}}^{-1}\,g_{0_3}(t)\end{bmatrix} \tag{62}$$

So, $a_2 = 0$ and a_1, a_3, and a_4, at $h = 1/(\zeta\,\omega_n)$ are only a function of $(w_0\,r_2)$ as follows: $a_1 = 2\left(1 - \zeta^2\right)\omega_n^2/a_d$, $a_3 = -2/a_d$, $a_4 = 2\,\zeta^2\,\omega_n^2\,/a_d$.

MARH APPROACH

For $N = 2$, the coefficients of the MARH approach for the mass-spring-damper system are given in Eq. (57):

$$a_1 = \left\{ \omega_n^6 \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{6}{\omega_n^2} \right) \left(h^4 + \frac{4\,\zeta}{\omega_n}\,h^3 - \frac{8}{\omega_n^2}\,h^2 + \frac{4}{\omega_n^4} \right) w_1\,r_1\,r_2 + h^4\,(\,h-2\,)\;r_2 \right.$$
$$\left. - \frac{h^5}{16}\left[\left(\omega_n^2 - 1 \right)\,h + 4\,\zeta\,\omega_n - 2 \right] \left(h^4 + 2\,h^3 - 8\,h^2 + 4 \right) \right\} / \left(h^2\,a_d \right) \tag{57a}$$
$$a_2 = \left\{ 2\,\omega_n^4\,\left(\zeta\,\omega_n\,h - 1 \right) \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{6}{\omega_n^2} \right) \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{4}{\omega_n^2} \right) w_1\,r_1\,r_2 \right.$$
$$\left. + h^4\,(\,h-2\,)\;r_2 - \frac{h^5}{16}\left[\left(\omega_n^2 - 1 \right)\,h + 4\,\zeta\,\omega_n - 2 \right] (h-2)\,\left(h^2 + 2\,h - 4 \right) \right\} / \left(h\,a_d \right) \tag{57b}$$
$$a_3 = \left\{ -\omega_n^4 \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{6}{\omega_n^2} \right) \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{8}{\omega_n^2} \right) w_1\,r_1\,r_2 \right.$$
$$\left. - h^4\,r_2 + \frac{h^5}{16}\left[\left(\omega_n^2 - 1 \right)\,h + 4\,\zeta\,\omega_n - 2 \right] (h-2)\,(h+4) \right\} / a_d \tag{57c}$$
$$a_4 = \left\{ -4\,\omega_n^2 \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{6}{\omega_n^2} \right) w_1\,r_1\,r_2 + 2\,h^4\,r_2 + \frac{h^5}{4}\left[\left(\omega_n^2 - 1 \right)\,h + 4\,\zeta\,\omega_n - 2 \right] \right\} / \left(h^2\,a_d \right) \tag{57d}$$
$$a_d = \omega_n^4 \left(h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{6}{\omega_n^2} \right)^2 w_1\,r_1\,r_2 + \left(h^4 + w_0\,r_1 \right) r_2 \tag{57e}$$

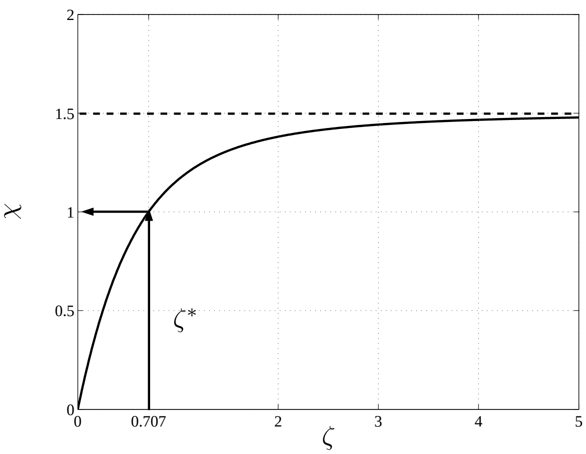

Fig. 4 Damping Ratio vs. χ

Similar to ARH case, a_d is never equal to zero for $h > 0$ and positive weighting. Also, a_2 is always zero at $h = 2$ sec regardless of the values of the weighting, but the dependence of a_4 to $(w_1\,r_1)$ does not vanish. The singular h to reduce the degree of freedom is the roots of the following:

$$h^2 + 2\,h - 6 = 0 \tag{63}$$

and the positive root is $h = \sqrt{7} - 1$ sec.

With the assumptions $\omega_n > 0$ and $\zeta > 0$, for the system, Eq. (55), the singular h is the positive roots of the following:

$$h^2 + \frac{4\,\zeta}{\omega_n}\,h - \frac{6}{\omega_n^2} = 0 \tag{64}$$

The positive singular h, denoted in h_s, is given by

$$h_s = \frac{1}{\zeta\,\omega_n}\,\chi \tag{65}$$

where

$$\chi = \zeta\,\left(\sqrt{4\,\zeta^2 + 6} - 2\,\zeta \right) \tag{66}$$

If $h = h_s$, one of the degrees of freedom to adjust the a_i's vanishes. The value of χ is equal to 1 at the following, as shown in Fig. 4:

$$\zeta^* = \frac{1}{\sqrt{2}} \approx 0.707 \tag{67}$$

Then, when the damping ratio is ζ^*, h_s is equal to the one for the ARH approach, i.e., $1/(\zeta\omega_n)$. If the damping ratio is greater than ζ^*, h_s is greater than the one of the ARH approach and vice versa. Also, the upper bound of χ is

$$\lim_{\zeta\to\infty} \chi = \lim_{\zeta\to\infty} \left\{ \zeta\,\left(\sqrt{4\,\zeta^2 + 6} - 2\,\zeta \right) \right\} = \frac{3}{2} \tag{68}$$

then, $\chi \in [0,\,3/2)$.

The stable regions are shown in Table 1 and Fig. 5 for three different cases of the nominal values of ζ for the ARH approach with $N = 3$ and the MARH approach with $N = 2$. For $\zeta > \zeta^*$, h_s for the ARH approach is less than the one for the MARH approach and vice versa. When the nominal value of the damping coefficient is less than $1/\sqrt{2}$, the MARH approach is preferable since it gives smaller h, which increases the model-error correction. When the nominal value of ζ is greater than $1/\sqrt{2}$, the ARH approach is preferable. In addition if the nominal value of the damping coefficient is ζ^*, the choice of h and r_p in the $h < h_s$ region is maximized. This is another interpretation of the optimal damping coefficient, 0.707. If the system is designed to have the optimal damping coefficient, it gives many benefits to the control design procedure. The choice of the optimization method depends on the

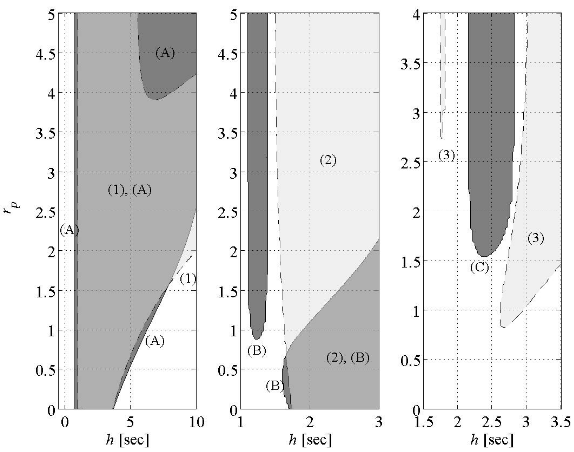

Fig. 5 $\varepsilon > 0$ **Regions for Different** ζ **of ARH (Dark) and MARH (Bright)**

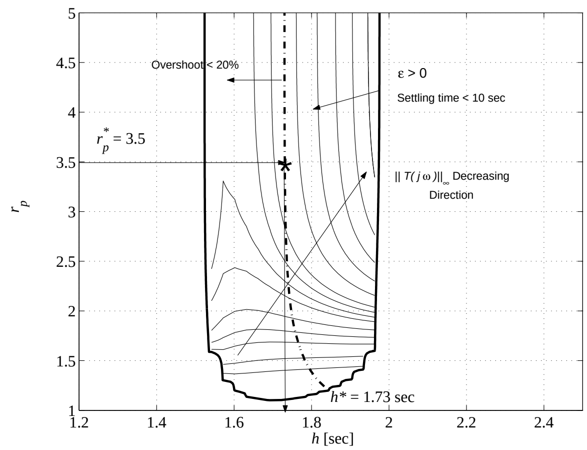

Fig. 6 **Optimal** r **and** h_p **for ARH Approach with** $N = 3$

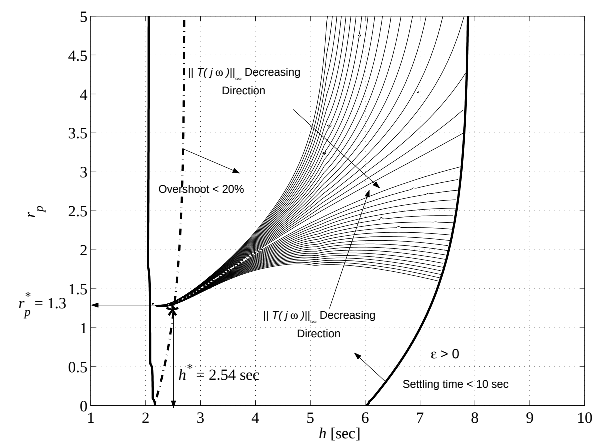

Fig. 7 **Optimal** r **and** h_p **for MARH Approach with** $N = 2$

nominal value of χ as summarized in Table 2. However, if there is a constraint on the control input, the choice will be different because a small h might saturate the actuator. In addition since $\sup(\chi) = 3/2$, the case for $\chi > 1$ has less effect than the $\chi < 1$ case.

Table 1 $\varepsilon > 0$ **Region Index of Fig. 5**

Optimization	Stable	ζ	h_s [sec]
ARH (3)	(A)	$2\,\zeta^*$	0.707
MARH (2)	(1)	$2\,\zeta^*$	0.9132
ARH (3)	(B)	$\zeta^* = 1/\sqrt{2}$	1.414
MARH (2)	(2)	$\zeta^* = 1/\sqrt{2}$	1.414
ARH (3)	(C)	$\zeta^*/2$	2.828
MARH (2)	(3)	$\zeta^*/2$	1.842

Table 2 **Choice of Optimization Method by** χ

ζ	h_s	χ	Method?
$\zeta > \zeta^*$	$h_s < 1/(\zeta^*\,\omega_n)$	$\chi > 1$	ARH
$\zeta < \zeta^*$	$h_s > 1/(\zeta^*\,\omega_n)$	$\chi < 1$	MARH

Remark 1 *Case of* $\chi > 1$
If $\chi > 1$*, the ARH approach is not always necessary and the MARH approach requires less N than the ARH approach. In terms of the damping ratio, if the nominal damping ratio is greater than* $1/\sqrt{2}$*, the choice of the optimization method is not critical. Conversely, when* $\chi < 1$*, the choice of the optimization method is critical. This is another advantage of the MARH approach over the ARH approach.* ■

Remark 2 *Physical Interpretation of* h_s
Since the assumed system model includes the bandwidth assumption of the open-loop response, a model-error correction with a time constant faster than the time constant of the assumed model is difficult to estimate. If the assumed model is chosen properly or close enough to the real plant, a model-error time constant faster than $1/(\zeta\,\omega_n)$ *will not be seen by the system itself, i.e., the faster frequency model-error does not need to be corrected. Also, physically, if the damping coefficient is greater than* $1/\sqrt{2}$*, the system is easier to stabilize, i.e., it's easy to cancel the effect of the model error because* h_s *is smaller than the one of the* $\zeta < 1/\sqrt{2}$ *case and* h *can be chosen smaller.* ■

OPTIMAL DESIGN

The optimal h and the weights will be chosen for each approach to minimize the H_∞ norm of the complementary sensitivity function within the following bounds: settling time for the unit impulse $w(t)$ is less than 10 sec and maximum overshoot for the same $w(t)$ is less than 20%.

- ARH Approach

As shown in Fig. 6, the smallest r_p to minimize $\|T(j\omega)\|_\infty$ in the region, which satisfies the design specifications, is 3.5 and the corresponding h is 1.73

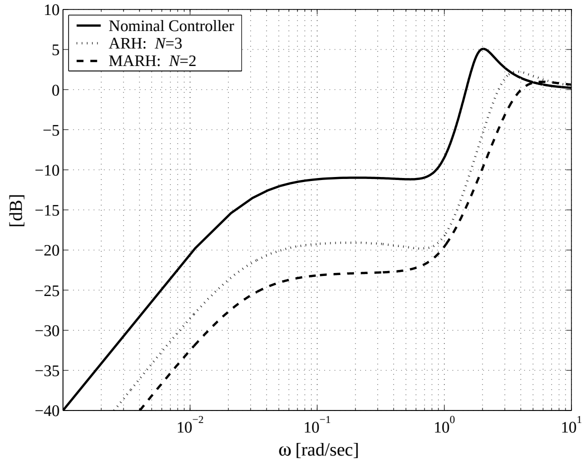

Fig. 8 $\|S(j\omega)\|_\infty$ **for Each Approach**

sec. The designed model-error correction is given by

$$\hat{u}(t) \approx 0.444\,\hat{x}_1(t) + 0.586\,\hat{x}_2(t) - 0.604\,u(t) + 0.160\,\tilde{y}(t) \tag{69}$$

- MARH Approach

As shown in Fig. 7, the optimal values of h and r_p to minimize $\|T(j\omega)\|_\infty$ in the region are uniquely determined as $h^* = 2.54$ and $r_p^* = 1.3$. Then, the designed model-error correction is given by

$$\hat{u}(t) \approx 0.717\,\hat{x}_1(t) + 0.262\,\hat{x}_2(t) - 0.719\,u(t) + 0.002\,\tilde{y}(t) \tag{70}$$

The sensitivity function plots for each case are shown in Fig. 8. In this case since the nominal values are $\hat{m} = 1$, $\hat{c} = 1$, and $\hat{k} = 1$, the nominal value of ζ is $1/2$, which is less than $1/\sqrt{2}$. The value of χ is calculated from Eq. (66) as follows:

$$\chi = \frac{1}{2}\left(\sqrt{4 \times \frac{1}{2^2} + 6} - 2 \times \frac{1}{2}\right) \approx 0.823 < 1 \tag{71}$$

Therefore, by Table 2 the MARH approach is preferable over the ARH approach. As shown in Fig. 8, the minimum ∞-norm is achieved by the MARH approach and the norm is the smaller than the others at most frequencies.

CONCLUSION

Model-error control synthesis has been shown to provide robustness with respect to modelling errors and external disturbances The model error in the system is determined from either an approximate receding-horizon approach or a modified approximate receding-horizon approach. An analysis of the overall control design using these two approaches has been carried out for a simple mass-spring-damper system. The relation between the damping ratio and the choice of optimization approach has been shown. It has been found that the optimal choice of the optimization approach depends on a simple scalar parameter. This parameter is a function of the damping coefficient and gives an optimal value when the damping is optimal, i.e. $1/\sqrt{2}$.

REFERENCES

[1] Crassidis, J. L., "Robust Control of Nonlinear Systems Using Model-Error Control Synthesis," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 4, July-August 1999, pp. 595–601.

[2] Crassidis, J. L. and Markley, F. L., "Predictive Filtering for Nonlinear Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, May-June 1997, pp. 566–572.

[3] Kim, J. and Crassidis, J. L., "Linear Stability Analysis of Model-Error Control Synthesis," *AIAA GN&C Conference & Exhibit*, Denver, CO, 2000, Aug. 14-16, AIAA-2000-3963.

[4] Kim, J. and Crassidis, J. L., "Model-Error Control Synthesis Using Approximate Receding-Horizon Control Laws," *AIAA GN&C Conference & Exhibit*, Montreal, QB, Canada, 2001, Aug. 6-9, AIAA-2001-4220.

[5] Kim, J. and Crassidis, J. L., "Robust Spacecraft Attitude Control Using Model-Error Control Synthesis," *AIAA GN&C Conference & Exhibit*, Monterey, CA, 2002, Aug. 5-8, AIAA-2002-4576.

[6] Lu, P., "Approximate Nonlinear Receding-Horizon Control Laws in Closed Form," *International Journal of Control*, Vol. 71, No. 1, 1998, pp. 19–34.

[7] Gantmacher, F. R., *The Theory of Matrices*, Vol. II, Chelsea Publishing Company, 1959.

[8] Kim, J. and Crassidis, J. L., "Limit Cycle Oscillation Control of Aeroelastic System Using Model-Error Control Synthesis," *AIAA GN&C Conference & Exhibit*, Austin, TX, 2003, Aug. 11-14, AIAA-2003-5506.

[9] Shanmugan, K. S. and Breipohl, A. M., *Random Signals: Detection, Estimation and Data Analysis*, John Wiley & Sons, 1988.

[10] Gelb, A., *Applied Optimal Estimation*, The MIT Press, 1974.

[11] Franklin, G. F., Powell, J. D., and Emani-Naeini, A., *Feedback Control of Dynamic Systems*, Addison-Wesley Publishing Company Inc., 3rd ed., 1994.

[12] Wang, Z. and Hu, H., "Robust Stability Test for Dynamic Systems with Short Delay by Using Padé Approximation," *Nonlinear Dynamics*, Vol. 18, No. 3, 1999, pp. 275–287.

[13] Argoun, M. B., "On the Stability of Low-Order Perturbed Polynomials," *IEEE Transactions on Automatic Control*, Vol. 35, No. 2, February 1999, pp. 180–182.

[14] Ho, M. T., Datta, A., and Bhattacharyya, S. P., "A Linear Programming Characterization of All Stabilizing PID Controllers," *Proceedings of the American Control Conference*, Albuquerque, NM, Jun. 1997, pp. 3922–3928.