LIMIT CYCLE OSCILLATION CONTROL OF AEROELASTIC SYSTEM
USING MODEL-ERROR CONTROL SYNTHESIS

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ABSTRACT
Model-error control synthesis is a nonlinear robust control approach that uses an optimal solution to cancel the effects of modeling errors and external disturbances on a system. In this paper the optimal solution for a modified approximate receding-horizon problem is used to determine the model error in an aeroelastic system, which has several uncertain parameters, including a nonlinear spring constant. To verify the control performance the approach is applied to compensate the limit cycle oscillation of the system. Simulation results show the performance of the model-error control synthesis approach.

INTRODUCTION
Model-Error Control Synthesis (MECS) is a signal synthesis adaptive control method.1 Robustness is achieved by applying a correction control, which is determined during the estimation process, to the nominal control vector thereby eliminating the effects of modelling errors at the system output.2 The model-error vector is estimated by using either a one-step ahead prediction approach,1,3 an Approximate Receding-Horizon (ARH) approach,4 or a Modified Approximate Receding-Horizon (MARH) approach.5 Choosing among the one-step ahead prediction approach, the ARH approach, or the MARH approach to determine the model error depends on the particular properties and required robustness in the system to be controlled.

In Ref. [1] MECS with the one-step ahead prediction approach is first applied to suppress the wing rock motion of a slender delta wing, which is described by a highly nonlinear differential equation. Results indicated that this approach provides adequate robustness for this particular system. In Ref. [3] a simple study to test the stability of the closed-loop system is presented using a Padé approximation for the time delay, which showed the relation between the system zeros and the weighting in the cost function. The analysis proved that some systems may not be stabilized using the original model-error estimation algorithm, which lead to the ARH approach in the MECS design to determine the model-error vector in the system.4

The closed-form solution of the ARH approach using Quadratic Programming (QP) is first presented by Lu.6 The model-error vector is determined by the ARH optimal solution.4 Using the ARH approach, the capability of MECS is expanded so that unstable non-minimum phase systems can be stabilized. Furthermore, Ref. [4] shows a method to calculate the stable regions with respect to the weighting and the length of receding-horizon step-time using the Hermite-Biehler theorem.7 After the stable region is found, the weighting and the length of receding-horizon step-time are chosen to minimize the ∞-norm of the sensitivity function.4

The ARH solution for an r-th-order relative degree system shows that the model-error solution is zero before the end of receding-horizon step-time is reached. Some parts of the model-error vector are separated completely from the constraints, so that the optimal solution for those parts are automatically zero. To avoid this situation for all model-error elements of each constraint at the time before the end of receding-horizon step-time, the state prediction is substituted by an r-th-order Taylor series expansion instead of a repeated first-order expansion in the ARH approach. We call this the Modified Approximate Receding-Horizon (MARH) approach, which leads to an even more robust MECS law than with the ARH solution.5

In Ref. [5] the MARH approach is used to the spacecraft attitude control problem for the case where the only available information is attitude-angle measurements, i.e., without angular-velocity measurements.

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In addition the relation between each optimization method and the damping coefficient in the simple mass-spring-damper system is shown in Ref. [8].

In this paper MECS is used to control an aeroelastic system, which has several uncertain parameters, including a nonlinear spring constant. In Ref. [9] to cancel the nonlinear parts and obtain the desired closed-loop dynamics a feedback linearization technique is used. In addition parameter uncertainties in the dynamics are compensated using an adaptive control scheme. Similar to the design steps for the spacecraft attitude maneuver problem in Ref. [5], we adopt the feedback linearization parts of the control in Ref. [9], but instead of the adaptive control parts, MECS will be designed to cancel the model uncertainties.

The organization of this paper is as follows. First, the MARH approach to estimate the model-error vector in a system is summarized. Second, the new in which will be the case for the spacecraft attitude control problem.

In this section the Modified Approximate Receding Horizon (MARH) approach to estimate the model-error vector, \( \hat{u}(t) \), in a system is summarized and the quadratic stability concept for providing nonlinear stability proof is introduced.

**OPTIMAL MODEL ERROR SOLUTION**

The receding-horizon optimization problem is set up as follows:

\[
\min_u J[\hat{x}(t), t, \hat{u}(t)] = \frac{1}{2} \int_t^{t+T} \left[ e^T(\xi) R^{-1}(\xi) e(\xi) + u^T(\xi) W(\xi) \hat{u}(\xi) \right] d\xi
\]

subject to the following:

\[
\dot{\hat{x}}(t) = \hat{f}[\hat{x}(t)] + \hat{B}[\hat{x}(t)] u(t) + \hat{G}[\hat{x}(t)] \hat{u}(t) \quad (4a)
\]

\[
\dot{\hat{y}}(t) = \hat{c}[\hat{x}(t)] \quad (4b)
\]

where \( R^{-1}(\xi) \) and \( W(\xi) \) are positive-definite and symmetric weighting matrices for all \( \xi \in [t, t + T] \), \( \hat{f}[\hat{x}(t)] \in \mathbb{R}^n \) is the assumed model vector, \( \hat{B}[\hat{x}(t)] \in \mathbb{R}^{n \times q_u} \) is the assumed control input distribution matrix, \( \hat{G}[\hat{x}(t)] \in \mathbb{R}^{n \times q_v} \) is the model-error distribution matrix, \( \hat{x}(t) \in X \subseteq \mathbb{R}^n \) is the state estimate vector, \( u(t) \in \Omega_u \subseteq \mathbb{R}^{q_u} \) is the control input, \( \hat{u}(t) \in \Omega_{\hat{u}} \subseteq \mathbb{R}^{q_v} \) is the control input, \( \hat{u}(t) \in \Omega_{\hat{u}} \subseteq \mathbb{R}^{q_v} \) is the control input, \( \hat{u}(t) \in \Omega_{\hat{u}} \subseteq \mathbb{R}^{q_v} \) is the control input, \( \hat{u}(t) \) is the to-be-determined model error, which also includes external disturbances, \( \hat{c}[\hat{x}(t)] \in \mathbb{R}^m \) is the measurement vector (\( m \leq n \) in general), and \( \hat{y}(t) \in \mathbb{R}^m \) is the estimated output vector. Also, we assume that a unique solution for \( \hat{x}(t) \) exists, and \( e(t) = e(t+T) = 0 \) where the residual error is defined by

\[
e(t) = \hat{y}(t) - \hat{y}(t)
\]

where \( \hat{y}(t) \) is the measurement. Note that \( T \) is the receding-horizon optimization-interval, which is not the sampling interval in general.

For most mechanical systems \( \Omega_u \subseteq \Omega_{\hat{u}} \) i.e., the system is under-actuated or fully actuated at the maximum, so that \( q_u \leq q_v \), where \( q_v \) is the dimension of the dynamics parts. The admissible sets \( X \) and
\(\Omega_u \subset \Omega_a\) are compact and \(X \times \Omega_a\) contains a neighborhood around the origin. One important assumption is \(m \geq q_u\), i.e., the dimension of the measurement vector is at least the dimension of the dynamics. Also, we assume that the rank of \(\hat{G} \{x(t)\}\) is \(q_u\), i.e., full rank. In addition controllability, observability, stable zero dynamics, and well-defined relative degree with respect to \(\hat{u}(t)\) are presumed, and the assumptions about continuity and \(\hat{f}(0) = 0\) hold. Finally, we assume that each element of the model-error vector affects the output.

State-observable measurements are assumed for Eq. (4b) in the following form:

\[
\hat{y}(t) = c[\hat{x}(t)] + v(t)
\]

where \(\hat{y}(t) \in \mathbb{R}^m\) is the measurement vector at time \(t\), and \(v(t) \in \mathbb{R}^m\) is the measurement noise vector, which is assumed to be a zero-mean, stationary, Gaussian noise distributed process with \(E\{v(t)\} = 0\) and \(E\{v(t)v^T(t + \Delta t)\} = R_u \delta(\Delta t)\), where \(E\{\cdot\}\) is expectation, \(R_u \in \mathbb{R}^{m \times m}\) is a positive-definite symmetric covariance matrix, \(\delta(\cdot)\) is Dirac delta function, and \(\Delta t\) is the sampling rate for the discrete-time measurement case.

At each time \(t\), the model-error solution \(\hat{u}\) over a finite horizon \([t, t + T]\) is determined on-line. Define \(h = T/N\) for some integer \(N \geq n/m\), where \(N\) is the number of sub-intervals on \([t, t + T]\). In addition since the future values of \(\hat{y}(t)\) and \(u(t)\) are unknown in general, \(\hat{y}(t)\) and \(u(t)\) are assumed to remain constant over the finite horizon \([t, t + T]\). Now \(\hat{y}(t + kh)\) for each \(k = 1, 2, \ldots, Nh = T\) is approximated. The output prediction at \(t + (k + 1)h\) is given by

\[
\hat{y}[t + (k + 1)h] \approx \hat{y}(t + kh) + z[\hat{x}(t + kh), h] + \Delta(h) S_u[\hat{x}(t + kh)] u(t) + \Delta(h) S_u[\hat{x}(t + kh)] \hat{u}(t + kh)
\]

for \(k = 1, 2, \ldots, N\), where \(\hat{y}(t + kh)\) and \(\hat{x}(t + kh)\) are given by the predictions from the previous stage. This process is repeated up to all \(\hat{x}(t + kh)\) written in terms of \(\hat{x}(t)\). The \(i\)th component of \(z[\hat{x}(t), h]\) is given by

\[
z_i[\hat{x}(t), h] = \sum_{k=1}^{p_i} \frac{h^k}{k!} L^k_i(c_i)
\]

where \(L^k_i(c_i)\) is the \(k\)th Lie derivative, defined by

\[
L^0_i(c_i) = c_i
\]

\[
L^k_i(c_i) = \left[\frac{\partial L^{k-1}_i(c_i)}{\partial \hat{x}}\right]^T \hat{f}, \quad \text{for } k \geq 1
\]

where the gradient is represented by a column vector with elements \((\partial c_i/\partial x)_k = \partial c_i/\partial x_k\). The \(i\)th rows of \(S_u[\hat{x}(t)]\) and \(S_u[\hat{x}(t)]\) are given by

\[
s_{u_i} = \{L_{\hat{b}_1}^i [L_f^{p_{ij} - 1}(c_i)], \ldots, L_{\hat{b}_{q_u}}^i [L_f^{p_{ij} - 1}(c_i)]\}
\]

\[
s_{\hat{u}_i} = \{L_{\hat{g}_1}^i [L_f^{p_{ij} - 1}(c_i)], \ldots, L_{\hat{g}_{q_u}}^i [L_f^{p_{ij} - 1}(c_i)]\}
\]

for \(i = 1, 2, \ldots, m\), where \(\hat{b}_j\) is the \(j\)th column of \(\hat{B} \{\hat{x}(t)\}\), \(\hat{g}_j\) is the \(j\)th column of \(\hat{G} \{\hat{x}(t)\}\), and the Lie derivative in Eq. (10) is defined by

\[
L_{\hat{b}_j}^i [L_f^{p_{ij} - 1}(c_i)] = \left[\frac{\partial L^{p_{ij} - 1}_f(c_i)}{\partial \hat{x}}\right]^T \hat{b}_j
\]

for \(j = 1, 2, \ldots, q_u\), and

\[
L_{\hat{g}_j}^i [L_f^{p_{ij} - 1}(c_i)] = \left[\frac{\partial L^{p_{ij} - 1}_f(c_i)}{\partial \hat{x}}\right]^T \hat{g}_j
\]

for \(j = 1, 2, \ldots, q_u\).

Define the following:

\[
L(k h) \equiv e^T(t + k h) R_k^{-1} e(t + k h)
\]

\[
+ \hat{u}^T(t + k h) W_k \hat{u}(t + k h)
\]

The cost function to be minimized is approximated using a trapezoidal formula or Simpson’s rule as follows:

\[
\bar{J} = \frac{h}{2} \sum_{k=1}^{N} \left\{ \frac{1}{2} L [(k - 1) h] + \frac{1}{2} L (k h) \right\}
\]

when \(N\) is odd,

\[
\bar{J} = \frac{h}{6} \sum_{k=0}^{(N/2) - 1} \left\{ L (2k h) + 4L (2k + 1) h \right\}
\]

\[
\bar{J} = \frac{h}{6} \sum_{k=0}^{(N/2) - 1} \left\{ L (2h) + 4L (2k + 1) h \right\}
\]

With the following definition:

\[
\nu_0 \equiv \left\{ \hat{u}^T(t), \hat{u}^T(t + h), \ldots, \hat{u}^T(t + (N - 1)h) \right\}^T
\]

The approximate cost, \(\bar{J}\), can be rewritten in quadratic form as

\[
\bar{J} = \frac{1}{2} \nu_0^T H_0 \nu_0 + g_0^T (\hat{x}, u, \hat{y}) \nu_0 + q_0(\hat{x}, u, \hat{y})
\]

where \(H_0, g_0\) and \(q_0\) are functions of \(L(k h)\). Also, the terminal constraint, \(e(t + T) = 0\), can be formulated as a constraint on \(\nu_0\) as follows:

\[
M^T \nu_0 = d(t)
\]
Finally, the solution of the QP problem is given by
\[ \nu_0 = - \left[ H_0^{-1} - H_0^{-1} M \left( M^T H_0^{-1} M \right)^{-1} M^T H_0^{-1} \right] g_0(t) \]
\[ + \left[ H_0^{-1} M \left( M^T H_0^{-1} M \right)^{-1} \right] d(t) \]  
(19)
where the rank of \( M \) is \( m \). The first \( q_w \) equations give a current model error minimizing the cost function, which leads to a predictive filter structure:
\[ \dot{u}_i(t); \dot{x}(t), u(t), \dot{y}(t), h = I_{q_w \times N} \nu_0 \]  
(20)
where \( I_{q_w \times N} \) is a \( \min(q_w, N) \times \min(q_w, N) \) identity matrix with zeros for the remaining elements.

**QUADRATIC STABILITY**

To provide a stability proof for nonlinear systems, the following are summarized from Ref. [10]:

**Definition 1** Quadratically Stable

Consider the following system with nonlinear uncertainty \( \Delta f[x(t)] \):
\[ \dot{x}(t) = Ax(t) + \Delta f[x(t)] \]  
(21)
where \( x(t) \in \mathbb{R}^n \) and the nonlinear uncertainty \( \Delta f[x(t)] = E_f \delta [x(t)] \) is a \( C^0 \) function, and \( \delta [x(t)] \) is an element of the following set:
\[ \Omega = \{ \delta [x(t)] \mid \| \delta [x(t)] \|_\infty \leq \| N_f x(t) \|_\infty, \ \forall x(t) \} \]  
(22)
where \( E_f \) and \( N_f \) are some constant matrices. The system, Eq. (21), is said to be quadratically stable if there exists a positive-definite symmetric matrix \( P_q > 0 \) such that
\[ \{ Ax(t) + \Delta f[x(t)] \}^T P_q x(t) \]
\[ + x^T(t) P_q \{ Ax(t) + \Delta f[x(t)] \} < 0 \]  
(23)
for all nonzero \( x(t) \in \mathbb{R}^n \) and all admissible nonlinear uncertainty, \( \Delta f[x(t)] \).

**Definition 2** Quadratic Cost Matrix, \( P_q \)

A positive definite matrix \( P_q > 0 \) is said to be a quadratic cost matrix for Eq. (21) and the following cost function:
\[ J_q = \int_0^\infty x^T(t) Q_q x(t) \, dt \]  
(24)
where \( Q_q \geq 0 \), if
\[ \{ Ax(t) + \Delta f[x(t)] \}^T P_q x(t) \]
\[ + x^T(t) P_q \{ Ax(t) + \Delta f[x(t)] \} < -x^T(t) Q_q x(t) \]  
(25)
for all nonzero \( x(t) \in \mathbb{R}^n \) and all admissible nonlinear uncertainty, \( \Delta f[x(t)] \).

**Theorem 1** The Cost Function Bound

If \( P_q > 0 \) is a quadratic cost matrix of Eq. (21), then the cost function is bounded by
\[ J_q \leq x^T(0) P_q x(0) \]  
(26)
and if the system is quadratically stable, then there exists a quadratic cost matrix. ■

**Lemma 1** \( H_\infty \) Norm Bound Condition

For the system, Eq. (21), there exists a quadratic cost matrix, \( P_q > 0 \), if and only if the following conditions hold:
1. \( A \) is a stable matrix.
2. The following \( H_\infty \) norm bound is satisfied for some \( \epsilon > 0 \):
\[ \left\| \begin{bmatrix} N_f & \sqrt{\epsilon Q_q} \end{bmatrix} (s I_n - A)^{-1} E_f \right\|_\infty < 1 \]  
(27)
Then, for such \( \epsilon \), the Riccati equation
\[ A^T P_q + P_q A + \epsilon Q_f E_f^T P_q + \frac{1}{\epsilon} N_f^T N_f = -Q_q \]  
(28)
has a solution. ■

**AEROELASTIC SYSTEM**

In this section the MECS design will be applied to the control of limit cycle oscillations of the aeroelastic system shown in Fig. 2. The aeroelastic system has two degrees of freedom, i.e., pitch angle, \( \alpha \), and plunge displacement, \( \tilde{h} \), which are measured by US Digital E2 digital optical encoders.9

**PROBLEM FORMULATION**

The equations of motion for the system shown in Fig. 2 are given by
\[ \begin{bmatrix} m_T & m_W x_o b \\ m_W x_o b & I_a \end{bmatrix} \begin{bmatrix} \tilde{h} \\ \tilde{\alpha} \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_o \end{bmatrix} \begin{bmatrix} \tilde{h} \\ \tilde{\alpha} \end{bmatrix} = \begin{bmatrix} -L \\ M \end{bmatrix} \]  
(29)
where \( m_T \) is the total mass, \( m_W \) is the mass of the wing only, and \( I_a \) is the moment of inertia about the elastic axis. The terms \( a \) and \( x_\alpha \) are non-dimensionalized elastic axis and center of mass locations by the length of midchord, \( b \), respectively. The terms \( c_h \) and \( c_\alpha \) are plunge and pitch structural damping coefficients, and the structural stiffness for plunge and pitch motions are \( k_h \) and \( k_\alpha \), respectively.

The term \( k_\alpha(\alpha) \) is a nonlinear function of pitch angle as follows:

\[
k_\alpha(\alpha) = \sum_{i=0}^{\infty} k_{\alpha i} \alpha^i
\]

where the \( k_{\alpha i} \)'s are constants. For numerical simulation purposes, the following 4th-order approximation is used as the real value of \( k_\alpha(\alpha) \):

\[
k_\alpha(\alpha) = 6.833 + 9.997\alpha + 667.685\alpha^2 + 26.569\alpha^3 - 5087.931\alpha^4 \text{ [N·m/rad]} \quad (31)
\]

As shown in Ref. [9], the above approximation well matches the experimental results within ±11.49° of \( \alpha \). In addition the following quasi-steady aerodynamic model for the lift, \( L \), and the moment, \( M \) are used:

\[
L = \rho U^2 c_{\alpha \alpha} \left[ a + \frac{h}{U} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right] + \rho U^2 c_{\alpha \beta} \beta \quad (32a)
\]

\[
M = \rho U^2 c_{\alpha \alpha} \left[ a + \frac{h}{U} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right] + \rho U^2 c_{\alpha \beta} \beta \quad (32b)
\]

where \( \rho \) is air density, \( U \) is the freestream velocity, \( c_{\alpha \alpha} \) and \( c_{\alpha \beta} \) are the aerodynamic lift and moment coefficients, respectively, and \( \beta \) is the flap deflection.

Define the following:

\[
\phi_1 = \alpha, \quad \phi_2 = \dot{\alpha}, \quad \phi_3 = h, \quad \phi_4 = -g_3 \dot{\alpha} + g_4 h \quad (33)
\]

The state-space form of Eq. (29) is given by

\[
\dot{\phi}(t) = f(U, \alpha, k_\alpha) + B(\alpha) \beta(t)
\]

\[
y(t) = \begin{bmatrix} \phi(t) \\ \dot{h}(t) \end{bmatrix} = C \phi(t)
\]

where

\[
\phi(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) & \phi_3(t) & \phi_4(t) \end{bmatrix}^T
\]

\[
f(U, \alpha, k_\alpha) = \begin{bmatrix} f_1(\phi(t)) \\ f_2(\phi(t)) \\ A_{32} \phi_2(t) + A_{34} \phi_4(t) \\ f_2(\phi(t)) \phi_1(t) + A_{42} \phi_2(t) + A_{43} \phi_3(t) + A_{44} \phi_4(t) \end{bmatrix}
\]

\[
B(\alpha) = \begin{bmatrix} 0 & g_4 U^2 & 0 & 0 \end{bmatrix}^T
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

For simplicity, it is assumed that the uncertainty in the dynamics occurs from the velocity, \( U \), the location of elastic axis, \( \alpha \), which has the significant role in the stability, and the nonlinear torsional spring constant, \( k_\alpha \). The measurement is given by

\[
\dot{y}(t) = C \phi(t) + v(t)
\]

where \( v(t) \in \mathbb{R}^2, E\{v(t)\} = 0 \) and

\[
E \{v(t) v^T(t + \Delta t)\} = \begin{bmatrix} r_{\alpha} & 0 \\ 0 & r_h \end{bmatrix} \delta(\Delta t)
\]

The model-error representation of the state-space form is as follows:

\[
\dot{\hat{\phi}}(t) = \hat{f}(\hat{U}, \hat{\alpha}, \hat{k_\alpha}) + \hat{B}(\hat{U}, \hat{\alpha}) \beta(t) + \hat{G} \hat{u}(t)
\]

where \( \hat{U} \) and \( \hat{\alpha} \) are the nominal values of \( U \) and \( \alpha \), respectively. The nominal value of \( k_\alpha(\alpha) \) is given by

\[
\hat{k_\alpha} = 6.833 \text{ [N·m/rad]} \quad (40)
\]

i.e., only the linear part in the resulting moment by the torsional spring with respect to \( \alpha \) is assumed:

\[
\hat{G} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and the model-error, \( \hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) & \hat{u}_2(t) & \hat{u}_3(t) \end{bmatrix}^T \), is to be determined.

**NOMINAL CONTROLLER DESIGN**

We summarize the nominal controller design using feedback linearization in Ref. [9]. The system is given by

\[
\dot{\hat{\phi}}(t) = \hat{f}(\hat{U}, \hat{\alpha}, \hat{k_\alpha}) + \hat{B}(\hat{U}, \hat{\alpha}) \beta(t) + \hat{G} \hat{u}(t)
\]

where

\[
\dot{\hat{\phi}}(t) = \hat{f}(\hat{U}, \hat{\alpha}, \hat{k_\alpha}) + \hat{B}(\hat{U}, \hat{\alpha}) \beta(t) + \hat{G} \hat{u}(t)
\]
The partial feedback linearization control input is given by

\[
\beta(t) = \frac{-F[\phi(t)] + \frac{m_T}{d} \dot{\phi}_1(t) + \nu(t) - \nu(t - \tau)}{\dot{g}_4 U^2}
\]

(43)

where

\[ F[\phi(t)] = -\dot{k}_4 \ddot{\phi}_1(t) - \left[ \dot{c}_4 + \frac{\ddot{g}_3}{g_4} \right] \ddot{\phi}_2(t) \]

(44a)

\[
-\dot{k}_3 \ddot{\phi}_3(t) - \frac{\ddot{c}_3}{g_4} \ddot{\phi}_4(t)
\]

(44b)

Also, \( \nu(t - \tau) \) is the to-be-determined model-error correction and \( \dot{\mathbf{f}}(\dot{U}, a, \dot{k}_a) \), i.e., a function value with the nominal values. Then, the closed-loop dynamics is given by

\[
\begin{bmatrix}
\dot{\phi}_1(t) \\
\dot{\phi}_2(t) 
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dot{k}_1 & -\dot{k}_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1(t) \\
\dot{\phi}_2(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{\nu}(t - \tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{u}_1(t)
\]

(45)

In Ref. [9] an adaptive control part is designed so that \( \dot{u}_1(t) \) approaches zero as time increases. Similar to the spacecraft attitude maneuver we will determine the model-error to minimize a cost function so that the model-error effects will vanish through the correction flap deflection, namely, \(-\dot{\nu}(t - \tau)/(\dot{g}_4 U^2)\).

Let us consider the resulting zero dynamics, which are given by

\[
\begin{bmatrix}
\dot{\phi}_3(t) \\
\dot{\phi}_4(t) 
\end{bmatrix} = \begin{bmatrix} 0 & A_{34}(U, a) \\ A_{43}(U, a) & A_{44}(U, a) \end{bmatrix} \begin{bmatrix} \dot{\phi}_3(t) \\
\dot{\phi}_4(t) \end{bmatrix} + \begin{bmatrix} \dot{u}_2(t) \\ \dot{u}_3(t) \end{bmatrix}
\]

(46)

The above zero dynamics are Hurwitz stable in the range of \(-1 \leq a \leq 1\) and \(0 < U \leq 30\) [m/sec] as shown in Ref. [9]. Therefore, the main concern to cancel the model-error effects are given by Eq. (45) from a control point of view.

**MODEL ERROR ESTIMATION**

The model-error for the state estimator and control input correction will be determined. The nominal values are given by Table 1, which are the experiment setup by the Aeroelastic Group in Department of Aerospace Engineering, Texas A&M University at College Station, TX (the experimental data and the physical parameters are used with the permission by Dr. T. Strganac, who is the director of the Aeroelastic Group). The span of this wing model is 0.6 m. First, the Predictive filter will be designed and verified by the experimental data. Second, to cancel the model-error effects in Eq. (45) through the flap angle deflection, the optimal correction will be determined. Finally, the closed-loop response will be shown by simulation.

**Table 1  Aeroelastic System Parameters**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_T )</td>
<td>12.3870 [kg]</td>
</tr>
<tr>
<td>( m_W )</td>
<td>2.0490 [kg]</td>
</tr>
<tr>
<td>( b ) (without flap)</td>
<td>0.1064 [m]</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1.225 [kg/m³]</td>
</tr>
<tr>
<td>( a )</td>
<td>( \bar{a} = -0.6 )</td>
</tr>
<tr>
<td>( x_a )</td>
<td>( [0.0873 - (b + a \cdot b)] / b ) [m]</td>
</tr>
<tr>
<td>( I_a )</td>
<td>( m_W x_a^2 b^2 + 0.0517 ) [kg m²]</td>
</tr>
<tr>
<td>( k_h )</td>
<td>2844.4 [N/m]</td>
</tr>
<tr>
<td>( c_{t_a} )</td>
<td>6.28</td>
</tr>
<tr>
<td>( c_{t_f} )</td>
<td>3.58</td>
</tr>
<tr>
<td>( c_{c_{t_a}} )</td>
<td>( (0.5 + a) c_{t_a} )</td>
</tr>
<tr>
<td>( c_{c_{t_f}} )</td>
<td>-1.94</td>
</tr>
<tr>
<td>( c_h )</td>
<td>27.43 [kg/sec]</td>
</tr>
<tr>
<td>( c_{\alpha} )</td>
<td>0.036 [kg m²/sec]</td>
</tr>
</tbody>
</table>

**MODEL ERROR FOR STATE ESTIMATOR**

With the nominal values given in Table 1, the state-space form of the model is obtained as follows:

\[
\dot{\mathbf{\dot{\phi}}}(t) = \hat{A}\mathbf{\dot{\phi}}(t) + \hat{B}\beta(t) + \hat{G}_{ss}\mathbf{\dot{u}}_s(t)
\]

(47)

where each matrix is given in the Appendix A.

Since we have a measurement of \( \mathbf{b} \), the model-error, \( \mathbf{\dot{u}}_2(t) \), can be compensated by using a simple estimator such as a Luenberger observer or a Kalman filter, where \( \mathbf{\dot{u}}_3(t) \) is negligible.

Using the MARH approach with the following values:

\[
N = 2, \quad h = 0.006 \text{ [sec]}
\]

(48a)

\[
r_{0h} = r_{0a} = 1 \times 10^{-6}, \quad r_p = 1
\]

(48b)

\[
w_{0h} = w_{0a} = 0.1, \quad w_p = 1
\]

(48c)

the model-error is determined by

\[
\mathbf{\dot{u}}_1(t) = -18008.7 \mathbf{\dot{\phi}}_1(t) - 217.5 \mathbf{\dot{\phi}}_2 - 810.2 \mathbf{\dot{\phi}}_3 + 14.7 \mathbf{\dot{\phi}}_4 + 136.2 \beta(t) + 18218.8 \alpha(t) + 227.3 \mathbf{\dot{\mathbf{h}}}(t)
\]

(49a)

\[
\mathbf{\dot{u}}_3(t) = -1038.5 \mathbf{\dot{\phi}}_1(t) + 15.0 \mathbf{\dot{\phi}}_2 + 28595.7 \mathbf{\dot{\phi}}_3 - 845.1 \mathbf{\dot{\phi}}_4 - 0.5825 \beta(t) + 1035.8 \alpha(t) - 28759.3 \mathbf{\dot{\mathbf{h}}}(t)
\]

(49b)

where Eqs. (49a) and (49b) are substituted into Eq. (47) with \( \dot{U} = 16 \) m/sec. Since the measurement sampling time is 0.002 sec, the discrete form of the estimator with zero-order hold is given by

\[
\mathbf{\dot{\phi}}(k + 1) = \hat{A}_d \mathbf{\dot{\phi}}(k) + \hat{B}_d \beta(k) + \hat{G}_{ys} \mathbf{\dot{y}}(k)
\]

(50)

where each matrix is given in the Appendix A.
The above estimator is tested for 24 sets of the experimental data shown in Table 2. To compare the estimates we have to estimate \( \dot{\alpha}(t) \) and \( \dot{\hat{b}}(t) \) using the whole measurement information including the acceleration measurements, i.e., \( \dot{\hat{\alpha}}(t) \) and \( \dot{\hat{\hat{b}}}(t) \). Toward this end the following models are used:

\[
\begin{align*}
\begin{bmatrix} \dot{\hat{\alpha}}(t) \\ \dot{\hat{\hat{b}}}(t) \\ \hat{b}_\alpha(t) \\ \hat{b}_b(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \hat{\alpha}(t) \\ \dot{\alpha}(t) \\ \hat{\hat{b}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} w_{\alpha}(t) \\
\begin{bmatrix} \dot{\hat{\alpha}}(t) \\ \dot{\hat{\hat{b}}}(t) \\ \hat{b}_\alpha(t) \\ \hat{b}_b(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\alpha}(t) \\ \hat{\hat{b}}(t) \\ \hat{\hat{b}}(t) \\ \hat{b}_b(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_{\alpha}(t)
\end{align*}
\] (51a)  

where \( b_\alpha \) and \( b_b \) are the biases in the acceleration measurements, which are given by random walk models through \( w_{\alpha} \) and \( w_{\alpha} \), which are zero-mean Gaussian white noise, respectively. The measurement equation is given by

\[
\begin{align*}
\dot{\hat{\alpha}}(t) &= \dot{\alpha}(t) + b_\alpha(t) + v_{\alpha}, \\
\dot{\hat{\hat{b}}}(t) &= \dot{\hat{b}}(t) + b_b(t) + v_{\alpha} \\
\end{align*}
\] (52a)  

Fig. 3 Absolute Values of \( \hat{\alpha}(t) \) and \( \hat{\hat{b}}(t) \) Estimation Error Mean

Fig. 4 Absolute Values of \( \hat{\dot{\alpha}}(t) \) and \( \hat{\dot{\hat{b}}}(t) \) Estimation Error Mean

\[
\begin{align*}
\hat{\dot{\alpha}}(t) &= \dot{\alpha}(t) + b_\alpha(t) + v_{\alpha}, \\
\hat{\dot{\hat{b}}}(t) &= \dot{\hat{b}}(t) + b_b(t) + v_{\alpha} \\
\end{align*}
\] (52b)

After transforming to discrete form with the sampling rate, 0.002 sec, and using the Rauch-Tung-Striebel (RTS) smoother, we can obtain accurate velocity estimates. The RTS smoother provides optimal estimates through an optimal combination of the forward and the backward Kalman filter.\(^{12,13}\) Based on the velocity estimates from the RTS smoother the performance of the Predictive filter will be evaluated.

The absolute values of the mean of the errors between the estimate by the Predictive filter and the RTS smoother are shown in Figs. 3 and 4. As shown in the figures, the mean values of the pitch angle and the plunge displacement estimation errors are less than 0.23 arcsec and 0.19\( \mu \)m. The absolute values of the mean for the \( \dot{\alpha}(t) \) and \( \dot{\hat{\dot{b}}}(t) \) are 0.37\( ^\circ \)/sec and 0.77\( \times 10^{-3} \)/(m-scec), respectively. As a result the Predictive filter, Eq. (50), estimates \( \dot{\alpha}(t) \) and \( \dot{\hat{\dot{b}}}(t) \) with small estimation errors in the sense of the estimates by RTS smoother for all the cases in Table 2. The time histories of \( \dot{\alpha}(t) \) for experiment #19 and \( \dot{\hat{\dot{b}}}(t) \) for experiment #13, which are the largest mean error cases by the Predictive filter are shown in Fig. 5. As shown in the figure, the Predictive filter yields accurate performance results, compared the RTS smoother results.

Since the midchord length, \( b \), with flap is given by 0.135 m, each matrix used in the Predictive filter is as
Predictive filter for the closed-loop control simulation. From now on, the above matrices will be used in the optimization method will be better than ARH approach.

From Eq. (45) with \( \bar{\zeta} = 4 \) and \( \bar{\omega} = 1.2 \), the damping ratio of the model is \( \zeta = 0.3 < 0.707 \) and \( \chi = \zeta \left( \sqrt{4\zeta^2 + 6} - 2\zeta \right) \approx 0.566 < 1 \) (more details about \( \chi \) can be found in Ref. [8]). Therefore, the MARH optimization method will be better than ARH approach. In addition the model-error correction of MARH with \( N = 2 \) is given in the Appendix B with the following definition:

\[
\omega_n = \sqrt{\bar{k}_1}, \quad \zeta = \frac{\bar{k}_2}{2\sqrt{\bar{k}_1}}
\]

Then, the model-error correction input is given by

\[
\hat{\nu}(t) = a_1 \hat{\phi}(t) + a_2 \dot{\hat{\phi}}(t) - a_3 \nu(t - \tau) + a_4 \check{\alpha}(t)
\]

where each of the \( a_i \)'s will be determined in the next section.

**OPTIMAL DESIGN**

From the Hermite-Biehler theorem the following is deduced:

**Corollary 1** Graphical interpretation for a 6th-order Polynomial

Consider the following 6th-order polynomial:

\[
d_d(s) = c_6 s^6 + c_5 s^5 + c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0
\]

where \( c_6 > 0 \), then \( d_d(s) \) is Hurwitz stable if and only if the stability index,

\[
\varepsilon \equiv \text{sc}(\kappa)
\]

is greater than zero, where

\[
\kappa \equiv \min (I, II-a, II-b, III-a, III-b, III-c, III-d)
\]

and

\[
I : \quad \zeta_0 > 0
\]

\[
II-a : \quad \min (c_3 c_5) > 0, \quad II-b : \quad \min (c_5 c_6) > 0,
\]

\[
II-c : \quad \min (c_1 c_5) > 0
\]

\[
III-a : \quad \zeta_6, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0
\]

\[
III-b : \quad \zeta_6, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0
\]

\[
III-c : \quad \zeta_6, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0
\]

\[
III-d : \quad \zeta_6, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0
\]

are substituted into III, respectively.

where \( \zeta_i \) and \( \bar{\zeta}_i \) are the lower and the upper bounds of each \( c_i \), for \( i = 1, 2, \ldots, 6 \), and

\[
III : \quad - \left( 4 c_1 c_5 A^2 + 2 c_3 A B + B^2 \right) > 0
\]

where

\[
A \equiv c_1 c_5 c_6 - c_2 c_5^2 + c_3 c_4 c_5 - c_4^2 c_6
\]

\[
B \equiv 2 c_0 c_2^2 - 2 c_1 c_4 c_5^2 + 2 c_1 c_3 c_5 c_6
\]

The proof and detail about Corollary 1 can be found in Refs. [4] and [5].

The optimal weighting and length of receding-horizon step-time are determined using the same procedure as the spacecraft attitude maneuver problem in Ref. [5]. The design goal is to determine \( w_p \) and/or \( r_p \) and \( h \) that minimizes the \( \infty \)-norm of sensitivity function for the system inside the stable region, which is found using Corollary 1.
The model-error with Padé approximation for the time delay is given by

$$\hat{\nu}(t) = \frac{d(s)}{d(s) + a_3 \, n(s)} \left\{ \left[ a_1 + a_4, \quad a_2, \quad 0, \quad 0 \right] \hat{\phi}(t) + a_4 \, v_n(t) \right\}$$  

(65)

where each matrix is given by

$$A_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4.0 & -1.2 & 0 & 0 \\ 0 & 0.0054 & 0 & -1.33 \\ 8.63 & -0.23 & 185.6 & -3.58 \end{bmatrix}$$  

(67a)

$$B_e = \{0 \, 1 \, 0 \, 0\}^T$$  

(67b)

$$G_e = \begin{bmatrix} 0_{1 \times 3} \\ I_{3 \times 3} \end{bmatrix}$$  

(67c)

In addition each matrix for Eq. (27) is given by

$$E_f = \begin{bmatrix} 0_{1 \times 3} \\ I_{3 \times 3} \end{bmatrix}$$  

(68a)

$$N_f = \begin{bmatrix} 0.2 \times I_{4 \times 4} & 0_{4 \times 4} \end{bmatrix}$$  

(68b)

$$Q_f = \begin{bmatrix} I_{4 \times 4} & 0_{4 \times 4} \\ 0_{4 \times 4} & 0_{4 \times 4} \end{bmatrix}$$  

(68c)

The $\infty$-norm of Eq. (27) with respect to $\epsilon$ is shown in Fig. 7. The maximum value of $\epsilon$ satisfying the norm less than 1 is 0.0492.

$$\|\text{Eq. (27)}\|_{\infty}$$  

Fig. 7  $\infty$-norm Condition for the Quadratic Stability with Respect to $\epsilon$

**SIMULATION**

The simulation parameters are given in Table 1 with $b = 0.135$ m, $a = -0.6847$ and $U = 13$ m/sec. The nonlinear spring constant, $k_0$, is given by Eq. (31). The nominal values of the three uncertainty parameters are given by: $\hat{\alpha} = -0.6$, $\hat{U} = 16$ m/sec, and $\hat{k}_0 = 6.833$. The initial condition for the plunge displacement, $h$, is equal to 0.2 m. To show the capability of the design controller to compensate limit cycle oscillations, the controller is activated at 1 sec after the open-loop response falls into a limit cycle oscillation.

The results are shown in Figs. 8 and 9. As shown in the figures, without MECS the nominal controller fails to stabilize the response. However, with MECS the limit cycle oscillation is stabilized. Also, as shown in the flap deflection history, Fig. 9, the flap deflection is inside the constraints.
Fig. 8 Time Histories of $\alpha(t)$ and $h(t)$ for Each Case

Fig. 9 Time History of $\beta(t)$ for Each Case

**CONCLUSION**

The modelling errors in the aeroelastic system were determined using a modified approximate receding-horizon expression with a Taylor series expansion at each instant of time. This approach was used in the model-error control synthesis design to provide robustness with respect to the bounded modelling errors. A Predictive filter was designed to estimate the angular velocity and the linear velocity, which were subsequently used in the overall controller, and the estimated velocities were compared to the ones from RTS smoother. Simulation results indicated that a nominal controller combined with the model-error control synthesis approach produced the ability to compensate the limit cycle oscillation. In addition the closed-loop system is globally quadratically stable for a norm bounded nonlinear uncertainty.

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**REFERENCES**


APPENDIX A

The definitions of the parameters in Eq. (34) are given by

\[
\begin{align*}
P_U[\phi_1(t)] &= k_4 U^2 + q[\phi_1(t)] \\
Q_U[\phi_1(t)] &= k_2 U^2 + p[\phi_1(t)] \\
A_{32} &= g_3/g_4 \\
A_{34} &= 1/g_4 \\
A_{42} &= -c_1 g_3 - c_2 g_4 + c_3 g_3^2/g_4 + c_4 g_3 \\
A_{43} &= k_3 g_4 - k_1 g_4 \\
A_{44} &= c_3 g_3/g_4 - c_1 \\
p[\phi_1] &= -m_W x_a b k_a(\phi_1)/d \\
q[\phi_1] &= m_T k_a(\phi_1)/d \\
d &= m_T I - m_W^2 x_a^2 b^2
\end{align*}
\]

where

\[
\begin{align*}
k_1 &= I_a k_h/d \\
k_2 &= (I_a \rho b c_1 + m_W x_a \rho b^3 c_{ma}) / d \\
k_3 &= -m_W x_a b k_h/d \\
k_4 &= -(m_W x_a \rho b^2 c_1 + m_T \rho b^3 c_{ma}) / d \\
c_1 &= [I_a (c_h + \rho U b c_1a) + m_W x_a \rho U b^3 c_{ma}] / d \\
c_2 &= \left\{ \left[I_a \rho U b^2 c_1 + m_W x_a \rho U b^3 c_{ma}\right] \left(\frac{1}{2} - a\right) \\
&\quad -m_W x_a b c_1a\right\} / d \\
c_3 &= -m_W x_a b \left[c_h + \rho U b c_1a\right] - m_T \rho U b^2 c_{ma} / d \\
c_4 &= \left\{ m_T \rho b^3 c_{ma} + m_W x_a \rho U b^3 c_{ma}\right\} / d \\
g_3 &= -(I_a \rho b c_1 c_3 + m_W x_a b^3 c_{mb}) / d \\
g_4 &= \left(m_W x_a \rho b^2 c_1 c_3 + m_T \rho b^2 c_{mb}\right) / d
\end{align*}
\]

Each matrix in Eq. (47) is given by

\[
\hat{\mathbf{A}} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-121.0 - 0.156 U^2 & -0.299 - 0.032 U & 0 & 0.078 \\
-4.26 + 0.021 U^2 & -0.339 + 0.004 U & 0 & 109.1 - 2.59 - 0.053 U \\
0 & 447.0 & 10.6 + 0.383 U & -2.46 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\hat{\mathbf{B}} = \begin{bmatrix}
0 \\
-0.406 U^2 \\
0 \\
0
\end{bmatrix}, \quad \mathbf{G}_s = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
\hat{\mathbf{u}}_s(t) = \begin{bmatrix}
\hat{u}_1(t) \\
\hat{u}_2(t) \\
\hat{u}_3(t)
\end{bmatrix}^T
\]

APPENDIX B

For the following simple mass-spring-damper system

\[
\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = u(t) + \ddot{u}(t)
\]

For \(N = 2\), the coefficients of the MARH approach are as follows:

\[
\ddot{u}(t) = a_1 x(t) + a_2 \dot{x}(t) + a_3 u(t) + a_4 \ddot{u}(t)
\]

where

\[
a_1 = \begin{bmatrix}
\omega_n^6 & h^2 + 4\zeta \omega_n h - \frac{6}{\omega_n^2} \\
-\frac{8}{\omega_n^2} h^2 + \frac{4}{\omega_n^4} & w_1 \dot{r}_1 \dot{r}_2 + h^4 (h - 2) r_2 \\
\frac{h^5}{16} & [(\omega_n^2 - 1) h + 4 \zeta \omega_n - 2] \\
(h^4 + 2 h^3 - 8 h^2 + 4) / (h^2 a_d)
\end{bmatrix}
\]

\[
a_2 = \begin{bmatrix}
2 \omega_n^4 & (\zeta \omega_n h - 1) \left(h^2 + \frac{4 \zeta \omega_n h - \frac{6}{\omega_n^2}}{}\right) \\
\left(h^2 + \frac{4 \zeta \omega_n h - \frac{4}{\omega_n^2}}{}\right) w_1 \dot{r}_1 \dot{r}_2 + h^4 (h - 2) r_2 \\
-(\omega_n^2 - 1) h + 4 \zeta \omega_n - 2 \\
(h - 2) (h^2 + 2 h - 4) / (h a_d)
\end{bmatrix}
\]

\[
a_3 = \begin{bmatrix}
-\omega_n^4 & h^2 + \frac{4 \zeta \omega_n h - \frac{6}{\omega_n^2}}{} \\
\left(h^2 + \frac{4 \zeta \omega_n h - \frac{8}{\omega_n^2}}{}\right) w_1 \dot{r}_1 \dot{r}_2 \\
-h^4 r_2 + \frac{h^5}{16} [(\omega_n^2 - 1) h + 4 \zeta \omega_n - 2] \\
(h - 2) (h + 4) / a_d
\end{bmatrix}
\]

\[
a_4 = \begin{bmatrix}
-4 \omega_n^2 \left(h^2 + \frac{4 \zeta \omega_n h - \frac{6}{\omega_n^2}}{}\right) w_1 \dot{r}_1 \dot{r}_2 + 2 h^4 r_2 \\
+\frac{h^5}{4} [(\omega_n^2 - 1) h + 4 \zeta \omega_n - 2] / (h^2 a_d)
\end{bmatrix}
\]

and

\[
a_d = \omega_n^4 \left(h^2 + \frac{4 \zeta \omega_n h - \frac{6}{\omega_n^2}}{}ight)^2 w_1 \dot{r}_1 \dot{r}_2 \\
+ (h^4 + w_0 r_1) r_2
\]