Stochastic Disturbance Accommodating Control Using a Kalman Estimator

Jemin George,[*] Puneet Singla,[†] and John L. Crassidis[‡]

University at Buffalo, State University of New York, Amherst, NY, 14260-4400

Disturbance accommodating control theory provides a method for designing feedback controllers which automatically detect and minimize the effect of waveform-structured disturbances. This paper presents a stochastic disturbance accommodating controller which utilizes a Kalman estimator to determine the necessary corrections to the nominal control input and thus minimizes the adverse effects of both model uncertainties and external disturbances on the controlled system. Stochastic stability analysis conducted on the controlled system reveals a lower-bound requirement on the estimator parameters to ensure the stability of the closed-loop system when the nominal control action on the true plant is unstable. Validity of the stability analysis is verified by implementing the proposed technique on a two degree-of-freedom helicopter.

Nomenclature

$(\Omega, \mathcal{F}, \mathbb{P})$	Complete probability space
$E[\cdot]$	Expectation operator
(A, B)	True system state matrix and control distribution matrix, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$
$\mathbf{X}(t,\omega) \in \mathbb{R}^n$	Stochastic state vector, $t \in [t_0 \;\; t_f]$ and $\omega \in \Omega$, for fixed t, $\mathbf{X}(t)$ is a random variable
$\mathbf{Y}(t,\omega) \in \mathbb{R}^m$	Stochastic output vector
$\mathbf{u}(t) \in \mathbb{R}^r$	Input vector
$\mathbf{W}(t,\omega) \in \mathbb{R}^n$	External stochastic disturbance vector
$\mathbf{V}(t,\omega) \in \mathbb{R}^m$	Measurement noise, assumed to be Gaussian white noise, $f_V(\mathbf{v}) \sim \mathcal{N}\big(\mathbf{0}, R\delta(\tau)\big)$
(A_m, B_m)	Assumed state matrix and assumed control distribution matrix, $A_m \in \mathbb{R}^{n\times n}$ and $B_m \in \mathbb{R}^{n\times r}$
$\mathbf{X}_m(t), \mathbf{Y}_m(t)$	State vector and output vector corresponding to the assumed system
C	Known output matrix, $C \in \mathbb{R}^{m\times n}$
$\boldsymbol{\mathcal{D}}(t,\omega) \in \mathbb{R}^n$	True lumped disturbance term
$\boldsymbol{\mathcal{D}}_m(t,\omega) \in \mathbb{R}^n$	Assumed lumped disturbance term
$\mathcal{L}_1(\cdot)$, $\mathcal{L}_2(\cdot)$	Linear mappings: $\mathbb{R}^n \to \mathbb{R}^n$
$\boldsymbol{\mathcal{V}}(t,\omega) \in \mathbb{R}^n$	Zero mean Gaussian white noise process, $f_{\boldsymbol{\mathcal{V}}}(\mathsf{v}) \sim \mathcal{N}\big(\mathbf{0}, \mathcal{Q}\delta(\tau)\big)$
$\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)$	Nominal system states and control input, $\bar{\mathbf{x}}(t) \in \mathbb{R}^n, \bar{\mathbf{u}}(t) \in \mathbb{R}^r$
$\mathbf{Z}(t) \in \mathbb{R}^{2n}$	$\triangleq [\mathbf{X}^T(t) \;\; \boldsymbol{\mathcal{D}}^T(t)]^T$, True augmented state vector
$\mathbf{Z}_m(t) \in \mathbb{R}^{2n}$	$\triangleq [\mathbf{X}_m^T(t) \;\; \boldsymbol{\mathcal{D}}_m^T(t)]^T$, Assumed augmented state vector
(F, D)	True state matrix and control distribution matrix for the augmented system, $F \in \mathbb{R}^{2n\times 2n}$ and $D \in \mathbb{R}^{2n\times r}$
$\mathrm{A}_{\mathcal{D}_m}$	Assumed state matrix for disturbance term, $\mathrm{A}_{\mathcal{D}_m} \in \mathbb{R}^{n\times n}$
(F_m, D_m)	Assumed state matrix and control distribution matrix for the augmented system, $F_m \in \mathbb{R}^{2n\times 2n}$ and $D_m \in \mathbb{R}^{2n\times r}$
$\boldsymbol{\mathcal{W}}(t,\omega) \in \mathbb{R}^n$	Zero mean Gaussian white noise process, $f_{\boldsymbol{\mathcal{W}}}(\mathsf{w}) \sim \mathcal{N}\big(\mathbf{0}, Q\delta(\tau)\big)$
(G, H)	Known disturbance input matrix and output matrix for the augmented system, $G \in \mathbb{R}^{2n\times n}$ and $H \in \mathbb{R}^{m\times 2n}$
$K(t) \in \mathbb{R}^{2n\times m}$	Optimal observer gain or Kalman gain for the assumed system

[*]Graduate Student, Department of Mechanical & Aerospace Engineering, jgeorge3@buffalo.edu, Student Member AIAA.
[†]Assistant Professor, Department of Mechanical & Aerospace Engineering, psingla@eng.buffalo.edu, Member AIAA.
[‡]Professor, Department of Mechanical & Aerospace Engineering, johnc@eng.buffalo.edu, Associate Fellow AIAA.

$P(t) \in \mathbb{R}^{2n\times 2n}$	Assumed estimation error covariance matrix
$K_m(t) \in \mathbb{R}^{r\times n}$	Nominal feedback gain
$I_{n\times m}$	An $n \times m$ identity matrix
$0_{n\times m}$	An $n \times m$ zero matrix
$\hat{}$	Indicate estimated quantities
S	Feedback gain for DAC
$\tilde{\mathbf{Z}}(t)$	$\triangleq \hat{\mathbf{Z}}(t) - \mathbf{Z}(t)$, Estimation error
$\boldsymbol{\mu}_{[\cdot]}$	$\triangleq E[\cdot]$
$\boldsymbol{\mathcal{Z}}(t)$	$\triangleq [\tilde{\mathbf{Z}}^T(t)\ \ \hat{\mathbf{Z}}^T(t)]^T$, Appended vector
$\Upsilon(t)$	State matrix of $\dot{\boldsymbol{\mathcal{Z}}}(t)$
$\boldsymbol{\mathcal{G}}(t)$	$\triangleq [\boldsymbol{\mathcal{V}}^T(t)\ \ \mathbf{V}^T(t)]^T$, Appended noise vector
$\Gamma(t)$	Appended noise distribution matrix
$\Phi(t, t_0)$	Evolution operator generated by $\Upsilon(t)$
$\mathcal{P}(t)$	$\triangleq E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)]$
$\Lambda\delta(\tau)$	$\triangleq E[\boldsymbol{\mathcal{G}}(t)\boldsymbol{\mathcal{G}}^T(t-\tau)]$
$P_{\tilde{\mathbf{Z}}}$, $P_{\hat{\mathbf{Z}}}$, $P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}}$	$\triangleq E[\tilde{\mathbf{Z}}(t)\tilde{\mathbf{Z}}^T(t)]$, $E[\hat{\mathbf{Z}}(t)\hat{\mathbf{Z}}^T(t)]$, and $E[\tilde{\mathbf{Z}}(t)\hat{\mathbf{Z}}^T(t)]$, respectively
$\bar{\hat{\mathbf{Z}}}(t)$, $\bar{\tilde{\mathbf{Z}}}(t)$, $\bar{\boldsymbol{\mathcal{Z}}}(t)$	$\triangleq \hat{\mathbf{Z}}(t)$, $\tilde{\mathbf{Z}}(t)$, and $\boldsymbol{\mathcal{Z}}(t)$ when there is no model uncertainties, i.e., $F = F_m$, $D = D_m$, and $\boldsymbol{\mathcal{V}}(t) = \boldsymbol{\mathcal{W}}(t)$
$\bar{\Upsilon}$, $\bar{\boldsymbol{\mathcal{G}}}$, $\bar{\mathcal{P}}$, $\bar{\Phi}$	$\triangleq \Upsilon(t)$, $\boldsymbol{\mathcal{G}}(t)$, $\mathcal{P}(t)$, and $\Phi(t, t_0)$ when there is no model uncertainties
$\Delta\Upsilon$, Φ_Δ	$\triangleq \Upsilon(t) - \bar{\Upsilon}(t)$, and the evolution operator generated by $\Delta\Upsilon$, respectively
$\inf\{\cdot\}$, $\sup\{\cdot\}$	Infimum, Supremum
$\sigma_{\max}(\cdot)$, $\sigma_{\min}(\cdot)$	Maximum and minimum singular values
$\Vert \cdot \Vert$	Appropriate 2-norm
L, $N(t)$	$\triangleq [G^T\quad 0^T_{2n\times n}]^T$, and $[HP(t)\quad -HP(t)]^T$, respectively
Q^*, R^*, $\mathcal{P}^*$	Appropriate Q & R that would guarantee stability and the corresponding $\mathcal{P}$
$\mathbb{C}^{2,1}$	Family of functions $V(x, t)$ which are twice continuously differentiable in x and once in t.
class-$\mathcal{K}$	Family of continuous non-decreasing functions $\kappa(x) : \mathbb{R}^+ \to \mathbb{R}^+$, $\kappa(0) = 0$, and $\kappa(x) > 0\quad \forall x > 0$
L_1	Family of all functions $\eta(t)$ such that $\int_0^\infty \eta(t)dt < \infty$
$\mathrm{Tr}\{\cdot\}$	Matrix trace

I. Introduction

System uncertainties and noisy measurements can obscure the development of a viable control law. The main objective of a feedback controller design is to develop a compensator that will maintain given design specifications in the presence of realistic ranges of uncertainty. A useful compensator that handles uncertainty is Disturbance-Accommodating Control (DAC), which was first proposed by Johnson in 1971.[1] Though the traditional DAC only considers disturbance functions which exhibit waveform patterns over short intervals of time,[2] a more general formulation of DAC can accommodate the simultaneous presence of both "noise" type disturbances and "waveform structured" disturbances.[3]

The disturbance-accommodating observer approach has shown to be extremely effective for disturbance attenuation;[4–6] however, the performance of the observer can significantly vary for different types of exogenous disturbances, which is due to observer gain sensitivity. This paper presents a robust control approach based on a significant extension of the disturbance accommodating control concept, which compensates for both model parameter uncertainties and external disturbances by estimating a model-error vector (throughout this paper we will use the phrase "disturbance term" to refer to this quantity) in real time that is used as a signal synthesis adaptive correction to the nominal control input to achieve maximum performance. This control approach utilizes a Kalman filter in the feedback loop for simultaneously estimating the system states and the disturbance term from measurements.[7] The estimated states are then used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input to minimize the effect of system uncertainties and the external disturbance. Similar developments of disturbance accommodating controllers using Kalman filter can be found in Refs. 8 and 9. There are several advantages of implementing the Kalman filter in the DAC approach: 1) tuning of the estimator parameters, such as the process-noise covariance matrix, can be done easily unlike the standard DAC techniques in which the adaptation involves the entire feedback gain, 2) the estimated disturbance term is a natural byproduct of state estimation, and 3) the Kalman filter can also be used to filter noisy measurements. A comparative study of the DAC approach to other adaptive techniques, such as the self-tuning regulator and model-reference adaptive control is presented in Ref. 10. Although the disturbance accommodating observer approach using a Kalman filter has been successfully implemented on linear time-invariant (LTI) systems with both noise type and waveform structured disturbances, to the best knowledge of the present authors, a comprehensive stochastic stability analysis has not been conducted before.

A detailed formulation of the stochastic disturbance accommodating controller for multi-input-multi-output (MIMO) systems is given next. Afterwards, a stability analysis is conducted on the proposed control scheme. The stochastic stability analysis indicates a lower-bound requirement on the assumed disturbance term process noise matrix and the measurement noise matrix to guarantee exponential stability in the mean sense when the nominal control action on the true plant would result in an unstable system. The stability analysis also indicates that the controlled stochastic system is almost surely asymptotically stabile if the noise distribution matrix satisfies a given decay rate. The results of the stability analysis are then verified by implementing the proposed control scheme on a two degree-of-freedom helicopter. Finally, conclusions and plans for future work are presented.

II. Controller Formulation

A detailed formulation of the DAC for LTI-MIMO systems is presented in this section. Throughout this paper, random vectors are denoted using boldface capital letters and for convenience, the dependency of a stochastic process on ω is not explicitly shown. Consider an n^{th}-order system of the following form:

$$\begin{aligned}\dot{\mathbf{X}}(t) &= A\mathbf{X}(t) + B\mathbf{u}(t) + \mathbf{W}(t), \quad \mathbf{X}(t_0) = \mathbf{x}_0 \\ \mathbf{Y}(t) &= C\mathbf{X}(t) + \mathbf{V}(t)\end{aligned} \tag{1}$$

Here, the true state and control distribution matrices are assumed to be unknown. Also, the system is assumed to be under-actuated, i.e., $r < n$. The external disturbance dynamics is

$$\dot{\mathbf{W}}(t) = \mathcal{L}_1(\mathbf{X}(t), \mathbf{u}(t), \mathbf{W}(t)) + \boldsymbol{\mathcal{V}}(t), \quad \mathbf{W}(t_0) = \mathbf{0} \tag{2}$$

The assumed (known) system model is

$$\begin{aligned}\dot{\mathbf{X}}_m(t) &= A_m\mathbf{X}_m(t) + B_m\mathbf{u}(t), \quad \mathbf{X}_m(t_0) = \mathbf{x}_0 \\ \mathbf{Y}_m(t) &= C\mathbf{X}_m(t) + \mathbf{V}(t)\end{aligned} \tag{3}$$

The external disturbance and the model uncertainties can be lumped into a disturbance term, $\boldsymbol{\mathcal{D}}(t)$, through

$$\boldsymbol{\mathcal{D}}(t) = \Delta A\mathbf{X}(t) + \Delta B\mathbf{u}(t) + \mathbf{W}(t) \tag{4}$$

where $\Delta A = (A - A_m)$ and $\Delta B = (B - B_m)$. Using this disturbance term the true model can be written in terms of the known system matrices as shown by

$$\begin{aligned}\dot{\mathbf{X}}(t) &= A_m\mathbf{X}(t) + B_m\mathbf{u}(t) + \boldsymbol{\mathcal{D}}(t) \\ \mathbf{Y}(t) &= C\mathbf{X}(t) + \mathbf{V}(t)\end{aligned} \tag{5}$$

The control law, $\mathbf{u}(t)$, is selected so that the true system will track the reference model:

$$\dot{\bar{\mathbf{x}}}(t) = A_m\bar{\mathbf{x}}(t) + B_m\bar{\mathbf{u}}(t) \tag{6}$$

The true system tracks the reference model if the following two conditions are satisfied:

$$\mathbf{x}_0 = \bar{\mathbf{x}}(t_0) \tag{7a}$$

$$B_m\mathbf{u}(t) = B_m\bar{\mathbf{u}}(t) - \boldsymbol{\mathcal{D}}(t) \tag{7b}$$

where convergence is understood in the mean-square sense. The disturbance term is not known, but an observer can be implemented in the feedback loop to estimate the disturbance term online. For this purpose, system Eq. (1) is rewritten as the following extended dynamically equivalent system:

$$\begin{aligned}\dot{\mathbf{X}}(t) &= A_m\mathbf{X}(t) + B_m\mathbf{u}(t) + \boldsymbol{\mathcal{D}}(t) \\ \dot{\boldsymbol{\mathcal{D}}}(t) &= \Delta A\dot{\mathbf{X}}(t) + \Delta B\dot{\mathbf{u}}(t) + \mathcal{L}_1(\mathbf{X}(t), \mathbf{u}(t), \mathbf{W}(t)) + \boldsymbol{\mathcal{V}}(t) = \mathcal{L}_2(\mathbf{X}, \boldsymbol{\mathcal{D}}, \mathbf{u}) + \boldsymbol{\mathcal{V}}\end{aligned} \tag{8}$$

The extended system given in Eq. (8) can be written in state space form as

$$\begin{bmatrix} \dot{\mathbf{X}} \\ \dot{\boldsymbol{\mathcal{D}}} \end{bmatrix} = \begin{bmatrix} A_m & I_{(n\times n)} \\ \mathcal{L}_{2\mathbf{X}} & \mathcal{L}_{2\boldsymbol{\mathcal{D}}} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} B_m \\ \mathcal{L}_{2\mathbf{u}} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0}_{(n\times 1)} \\ \boldsymbol{\mathcal{V}} \end{bmatrix} \tag{9}$$

where $\mathcal{L}_{2\mathbf{X}}$, $\mathcal{L}_{2\boldsymbol{\mathcal{D}}}$, and $\mathcal{L}_{2\mathbf{u}}$ are partitions on $\mathcal{L}_2(\cdot)$ that are acting on $\mathbf{X}(t)$, $\boldsymbol{\mathcal{D}}(t)$, and $\mathbf{u}(t)$, respectively. Let $\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \boldsymbol{\mathcal{D}}(t) \end{bmatrix}$, $F = \begin{bmatrix} A_m & I_{(n\times n)} \\ \mathcal{L}_{2\mathbf{X}} & \mathcal{L}_{2\boldsymbol{\mathcal{D}}} \end{bmatrix}$, $D = \begin{bmatrix} B_m \\ \mathcal{L}_{2\mathbf{u}} \end{bmatrix}$, and $G = \begin{bmatrix} 0_{n\times n} \\ I_{n\times n} \end{bmatrix}$. Now the extended system given in Eq. (9) can be written as

$$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) + D\mathbf{u}(t) + G\boldsymbol{\mathcal{V}}(t), \quad \mathbf{Z}(t_0) = [\mathbf{x}_0 \;\; \boldsymbol{\mathcal{D}}_0]^T \tag{10}$$

We do not have precise knowledge about the dynamics of the disturbance term. For simplicity, the disturbance term dynamics is modeled as

$$\dot{\boldsymbol{\mathcal{D}}}_m = \mathrm{A}_{\mathcal{D}_m}\boldsymbol{\mathcal{D}}_m + \boldsymbol{\mathcal{W}}(t), \quad \boldsymbol{\mathcal{D}}_m(t_0) = \mathbf{0} \tag{11}$$

where $\mathrm{A}_{\mathcal{D}_m}$ is Hurwitz. Equation (11) is used solely in the estimator design to estimate the true disturbance term. Now construct the assumed augmented state vector, $\mathbf{Z}_m(t) = \begin{bmatrix} \mathbf{X}_m(t) \\ \boldsymbol{\mathcal{D}}_m(t) \end{bmatrix}$, the assumed model of the system Eq. (9) can be written as

$$\begin{bmatrix} \dot{\mathbf{X}}_m \\ \dot{\boldsymbol{\mathcal{D}}}_m \end{bmatrix} = \begin{bmatrix} A_m & I_{(n\times n)} \\ 0_{(n\times n)} & \mathrm{A}_{\mathcal{D}_m} \end{bmatrix} \begin{bmatrix} \mathbf{X}_m \\ \boldsymbol{\mathcal{D}}_m \end{bmatrix} + \begin{bmatrix} B_m \\ 0_{(n\times r)} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0}_{(n\times 1)} \\ \boldsymbol{\mathcal{W}} \end{bmatrix} \tag{12}$$

The zero elements in the disturbance term dynamics are assumed for the sake of simplicity, the control formulation given here is also valid if non-zero elements are assumed. Equation (12) can be written in terms of the appended state vector, $\mathbf{Z}_m$, as

$$\dot{\mathbf{Z}}_m(t) = F_m\mathbf{Z}_m(t) + D_m\mathbf{u}(t) + G\boldsymbol{\mathcal{W}}(t), \quad \mathbf{Z}_m(t_0) = [\mathbf{x}_0 \;\; \mathbf{0}]^T \tag{13}$$

where $F_m = \begin{bmatrix} A_m & I_{(n\times n)} \\ 0_{(n\times n)} & \mathrm{A}_{\mathcal{D}_m} \end{bmatrix}$ and $D_m = \begin{bmatrix} B_m \\ 0_{(n\times r)} \end{bmatrix}$. Notice that the uncertainty is now only associated with the dynamics of the disturbance term. The assumed output equation can also be written in terms of the appended state vector, $\mathbf{Z}_m$, as

$$\mathbf{Y}_m(t) = \begin{bmatrix} C & 0_{(m\times n)} \end{bmatrix} \mathbf{Z}_m(t) + \mathbf{V}(t) \tag{14}$$

and the measured output equation can be written as

$$\mathbf{Y}(t) = \begin{bmatrix} C & 0_{(m\times n)} \end{bmatrix} \mathbf{Z}(t) + \mathbf{V}(t) \tag{15}$$

Let $H = [C \;\; 0_{m\times n}]$, then $\mathbf{Y} = H\mathbf{Z} + \mathbf{V}$ and $\mathbf{Y}_m = H\mathbf{Z}_m + \mathbf{V}$. Though the disturbance term is unknown, assuming $\boldsymbol{\mathcal{W}}(t)$ and $\mathbf{V}(t)$ possess certain stochastic properties, an optimal estimator such as a Kalman filter can be implemented in the feedback loop to estimate the unmeasured system states and the disturbance term from the noisy measurements. The estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m\mathbf{u}(t) + K(t)[\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)], \quad \hat{\mathbf{Z}}(t_0) = \mathbf{Z}_m(t_0) \tag{16}$$

where $K(t)$ is the Kalman gain and $\hat{\mathbf{Y}} = H\hat{\mathbf{Z}}$. The estimator dynamics can be rewritten as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m\mathbf{u}(t) + K(t)H[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)] + K(t)\mathbf{V}(t) \tag{17}$$

Notice that the estimator uses the assumed system model given in Eq. (13) for the propagation stage and the actual measurements from Eq. (15) for the update stage, i.e., $\hat{\mathbf{Z}}(t) = E[\mathbf{Z}_m(t)|\{\mathbf{Y}_t \ldots \mathbf{Y}_0\}]$. The Kalman gain can be calculated as $K(t) = P(t)H^T R^{-1}$, where $P(t) = E\Big[(\mathbf{Z}_m(t) - \hat{\mathbf{Z}}(t))(\mathbf{Z}_m(t) - \hat{\mathbf{Z}}(t))^T\Big]$ can be obtained by solving the continuous-time matrix differential Riccati equation:[11]

$$\dot{P}(t) = F_m P(t) + P(t){F_m}^T - P(t)H^T R^{-1} H P(t) + GQG^T \tag{18}$$

The total control law, $\mathbf{u}(t)$, consists of a nominal control and necessary corrections to the nominal control to compensate for the disturbance term as shown in Eq. (7b). The nominal control, $\bar{\mathbf{u}}$, is selected so that it guarantees the desired performance of the assumed system. For the system given in Eq. (5), the nominal controller is given as

$$\bar{\mathbf{u}}(t) = -K_m \hat{\mathbf{X}}(t) \tag{19}$$

where $K_m \in \mathbb{R}^{r \times n}$ is the feedback gain. While the nominal controller guarantees the desired performance of the assumed model, the second term, $-\boldsymbol{\mathcal{D}}(t)$, in Eq. (7b) ensures the complete cancelation of the disturbance term which is compensating for the external disturbance and the model uncertainties. Now the control law can be written in terms of the estimated system states and the estimated disturbance term as

$$\mathbf{u}(t) = (B_m^T B_m)^{-1} B_m^T \Big[-B_m K_m \quad -I_{(n \times n)} \Big] \begin{bmatrix} \hat{\mathbf{X}}(t) \\ \hat{\boldsymbol{\mathcal{D}}}(t) \end{bmatrix} = S\hat{\mathbf{Z}}(t) \tag{20}$$

where $S = (B_m^T B_m)^{-1} B_m^T \Big[-B_m K_m \quad -I \Big]$. Notice that $(B_m^T B_m)$ is a nonsingular matrix since B_m is assumed to have linearly independent columns. A summary of the proposed control scheme is given Table. 1.

Table 1. Summary of Overall Control Process

Plant	$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) + D\mathbf{u}(t) + G\boldsymbol{\mathcal{V}}(t)$ $\mathbf{Y}(t) = H\mathbf{Z}(t) + \mathbf{V}(t)$
Initialize	$\hat{\mathbf{Z}}(t_0),\ P(t_0)$
Observer Gain	$\dot{P}(t) = F_m P(t) + P(t){F_m}^T - P(t)H^T R^{-1} H P(t) + GQG^T$ $K(t) = P(t)H^T R^{-1}$
Estimate	$\dot{\hat{\mathbf{Z}}}(t) = F_m \hat{\mathbf{Z}}(t) + D_m \mathbf{u}(t) + K(t)[\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)]$
Control Synthesis	$\mathbf{u}(t) = (B_m^T B_m)^{-1} B_m^T \Big[-B_m K_m \quad -I \Big] \hat{\mathbf{Z}}(t)$

It is important to note that if $Q = 0$, then $\boldsymbol{\mathcal{D}}_m(t) = \boldsymbol{\mathcal{D}}_m(t_0) = \mathbf{0}$ and the total control law becomes just the nominal control. If the nominal control, $\bar{\mathbf{u}}(t)$, on the true plant would result in an unstable system, then selecting a small Q would also result in an unstable system. On the other hand, selecting a large Q value would compel the estimator to completely rely upon the measurement signal and therefore the noise associated with the measurement signal is directly transmitted into the estimates. This could result in noisy control signal which could lead to problems, such as chattering and controller saturation. Also note that as R, the measurement noise covariance, increases, the observer gain decreases and thus the observer fails to update the propagated disturbance term based on measurements. For a highly uncertain system, selecting a small Q or a large R will result in an unstable closed-loop system as shown in Ref. 12. A schematic representation of the proposed controller is given in Fig. 1. In the next section a detailed stability analysis is given, which investigates the dependency of closed-loop system stability on Q and R.

III. Stability Analysis

Notice that $P(t)$ given in Eq. (18) is not the estimation error covariance, a detailed derivation of the true error covariance is considered first. Closed-loop system stability based on the formulation of the true

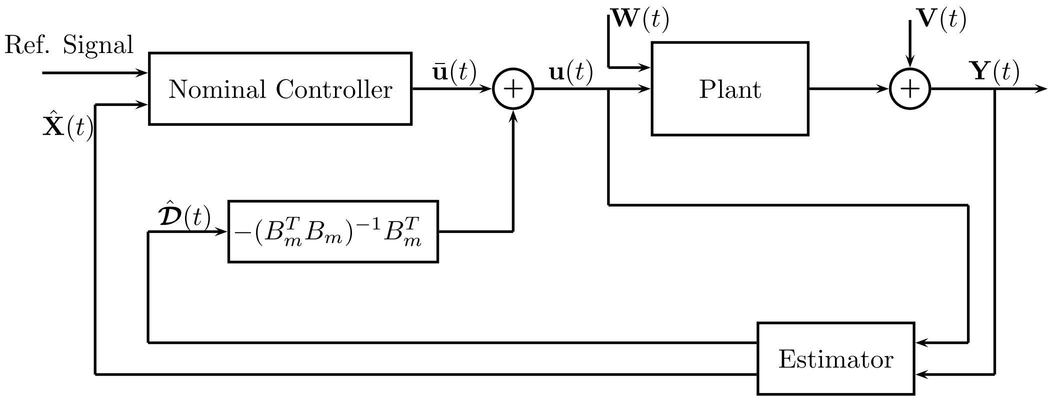

Figure 1. DAC Block Diagram

error covariance is then presented. Finally it is shown that the system stability depends on a lower bound requirement on Q and R^{-1}.

A. Estimation Error Covariance

Substituting the control law, Eq. (20), into the plant dynamics, Eq. (10), the true system can be written as

$$\begin{aligned}\dot{\mathbf{Z}}(t) &= F\mathbf{Z}(t) + DS\hat{\mathbf{Z}}(t) + G\boldsymbol{\mathcal{V}}(t)\\ \mathbf{Y}(t) &= H\mathbf{Z}(t) + \mathbf{V}(t)\end{aligned} \tag{21}$$

The estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m S\hat{\mathbf{Z}}(t) + K(t)H[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)] + K(t)\mathbf{V}(t) \tag{22}$$

From hereon the explicit notation for time varying quantities is omitted when there is no risk of confusion. Let $\tilde{\mathbf{Z}} = \mathbf{Z} - \hat{\mathbf{Z}}$ be the estimation error, then the error dynamics can be written as

$$\begin{aligned}\dot{\tilde{\mathbf{Z}}} = \dot{\mathbf{Z}} - \dot{\hat{\mathbf{Z}}} &= F\mathbf{Z} + DS\hat{\mathbf{Z}} + G\boldsymbol{\mathcal{V}} - F_m\hat{\mathbf{Z}} - D_mS\hat{\mathbf{Z}} - KH[\mathbf{Z} - \hat{\mathbf{Z}}] - K\mathbf{V}\\ &= [F_m - KH + \Delta F]\tilde{\mathbf{Z}} + [\Delta F + \Delta DS]\hat{\mathbf{Z}} + G\boldsymbol{\mathcal{V}} - K\mathbf{V}\end{aligned}$$

where $\triangle F = F - F_m$ and $\triangle D = D - D_m$. Let $\boldsymbol{\mu}_{\tilde{\mathbf{Z}}} = E[\tilde{\mathbf{Z}}]$, and $\boldsymbol{\mu}_{\hat{\mathbf{Z}}} = E[\hat{\mathbf{Z}}]$, i.e.,

$$\begin{aligned}\dot{\boldsymbol{\mu}}_{\tilde{\mathbf{Z}}} = E[\dot{\tilde{\mathbf{Z}}}] &= E\big[(F_m - KH + \Delta F)\tilde{\mathbf{Z}} + (\Delta F + \Delta DS)\hat{\mathbf{Z}} + G\boldsymbol{\mathcal{V}} - K\mathbf{V}\big]\\ &= (F_m - KH + \Delta F)\boldsymbol{\mu}_{\tilde{\mathbf{Z}}} + (\Delta F + \Delta DS)\boldsymbol{\mu}_{\hat{\mathbf{Z}}}\end{aligned}$$

Now $\dot{\boldsymbol{\mu}}_{\hat{\mathbf{Z}}}$ can be written as

$$\dot{\boldsymbol{\mu}}_{\hat{\mathbf{Z}}} = (F_m + D_mS)\boldsymbol{\mu}_{\hat{\mathbf{Z}}} + KH\boldsymbol{\mu}_{\tilde{\mathbf{Z}}}$$

Combining the error dynamics and the estimator dynamics we could write,

$$\begin{bmatrix}\dot{\tilde{\mathbf{Z}}}\\ \dot{\hat{\mathbf{Z}}}\end{bmatrix} = \begin{bmatrix}(F_m - KH + \Delta F) & (\Delta F + \Delta DS)\\ KH & (F_m + D_mS)\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{Z}}\\ \hat{\mathbf{Z}}\end{bmatrix} + \begin{bmatrix}G & -K\\ 0 & K\end{bmatrix}\begin{bmatrix}\boldsymbol{\mathcal{V}}\\ \mathbf{V}\end{bmatrix}$$

or in a more compact form as

$$\dot{\boldsymbol{\mathcal{Z}}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t) \tag{23}$$

where $\boldsymbol{\mathcal{Z}}(t) = \begin{bmatrix} \tilde{\mathbf{Z}}(t) \\ \hat{\mathbf{Z}}(t) \end{bmatrix}$, $\Upsilon(t) = \begin{bmatrix} (F_m - K(t)H + \Delta F) & (\Delta F + \Delta DS) \\ K(t)H & (F_m + D_m S) \end{bmatrix}$, $\Gamma(t) = \begin{bmatrix} G & -K(t) \\ 0 & K(t) \end{bmatrix}$, and $\boldsymbol{\mathcal{G}}(t) = \begin{bmatrix} \boldsymbol{\varphi}(t) \\ \mathbf{v}(t) \end{bmatrix}$. The solution of above equation can be written as

$$\boldsymbol{\mathcal{Z}}(t) = \Phi(t, t_0)\boldsymbol{\mathcal{Z}}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\Gamma(\tau)\boldsymbol{\mathcal{G}}(\tau)d\tau \tag{24}$$

Let $\mathcal{P}(t) \equiv E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)]$, i.e.,

$$\begin{aligned}\mathcal{P}(t) = E\Big[&\Phi(t, t_0)\boldsymbol{\mathcal{Z}}(t_0)\boldsymbol{\mathcal{Z}}^T(t_0)\Phi^T(t, t_0) + \int_{t_0}^{t} \Phi(t, t_0)\boldsymbol{\mathcal{Z}}(t_0)\boldsymbol{\mathcal{G}}^T(\tau)\Gamma^T(\tau)\Phi^T(t, \tau)d\tau + \\ &\int_{t_0}^{t} \Phi(t, \tau)\Gamma(\tau)\boldsymbol{\mathcal{G}}(\tau)\boldsymbol{\mathcal{Z}}^T(t_0)\Phi^T(t, t_0)d\tau + \int_{t_0}^{t}\int_{t_0}^{t} \Phi(t, \tau_1)\Gamma(\tau_1)\boldsymbol{\mathcal{G}}(\tau_1)\boldsymbol{\mathcal{G}}^T(\tau_2)\Gamma^T(\tau_2)\Phi^T(t, \tau_2)d\tau_1 d\tau_2\Big]\end{aligned}$$

Assuming $\boldsymbol{\mathcal{G}}(t)$ and $\boldsymbol{\mathcal{Z}}(t_0)$ are uncorrelated we have $E[\boldsymbol{\mathcal{G}}(t)\boldsymbol{\mathcal{Z}}^T(t_0)] = E[\boldsymbol{\mathcal{Z}}(t_0)\boldsymbol{\mathcal{G}}^T(t)] = 0$. The initial $\mathcal{P}$ is $\mathcal{P}(t_0) = E[\boldsymbol{\mathcal{Z}}(t_0)\boldsymbol{\mathcal{Z}}^T(t_0)]$. Since $\boldsymbol{\mathcal{V}}(t)$ and $\mathbf{V}(t)$ are uncorrelated, the expectation of $\boldsymbol{\mathcal{G}}(\tau_1)\boldsymbol{\mathcal{G}}^T(\tau_2)$ is

$$E[\boldsymbol{\mathcal{G}}(\tau_1)\boldsymbol{\mathcal{G}}^T(\tau_2)] = \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & R \end{bmatrix} \delta(\tau_1 - \tau_2)$$

Let $\Lambda = \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & R \end{bmatrix}$, now $\mathcal{P}(t)$ can be rewritten as

$$\mathcal{P}(t) = \Phi(t, t_0)\mathcal{P}(t_0)\Phi^T(t, t_0) + \int_{t_0}^{t} \Phi(t, \tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\Phi^T(t, \tau)d\tau \tag{25}$$

Taking the time derivative of the above equation results in

$$\begin{aligned}\dot{\mathcal{P}}(t) =& \frac{\partial \Phi(t, t_0)}{\partial t}\mathcal{P}(t_0)\Phi^T(t, t_0) + \Phi(t, t_0)\mathcal{P}(t_0)\frac{\partial \Phi^T(t, t_0)}{\partial t} + \Phi(t, t)\Gamma(t)\Lambda\Gamma^T(t)\Phi^T(t, t) + \\ &\int_{t_0}^{t} \frac{\partial \Phi(t, \tau)}{\partial t}\Gamma(\tau)\Lambda\Gamma^T(\tau)\Phi^T(t, \tau)d\tau + \int_{t_0}^{t} \Phi(t, \tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\frac{\partial \Phi^T(t, \tau)}{\partial t}d\tau\end{aligned}$$

Utilizing the fundamental properties of the evolution operator, the above equation can be rewritten as

$$\begin{aligned}\dot{\mathcal{P}}(t) =& \Upsilon(t)\Phi(t, t_0)\mathcal{P}(t_0)\Phi^T(t, t_0) + \Phi(t, t_0)\mathcal{P}(t_0)\Phi^T(t, t_0)\Upsilon^T(t) + \Gamma(t)\Lambda\Gamma^T(t) + \\ &\Upsilon(t)\int_{t_0}^{t} \Phi(t, \tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\Phi^T(t, \tau)d\tau + \int_{t_0}^{t} \Phi(t, \tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\Phi^T(t, \tau))d\tau\Upsilon^T(t)\end{aligned}$$

Therefore

$$\dot{\mathcal{P}}(t) = \Upsilon(t)\mathcal{P}(t) + \mathcal{P}(t)\Upsilon^T(t) + \Gamma(t)\Lambda\Gamma^T(t) \tag{26}$$

Let

$$\mathcal{P}(t) = \begin{bmatrix} E[\tilde{\mathbf{Z}}\tilde{\mathbf{Z}}^T] & E[\tilde{\mathbf{Z}}\hat{\mathbf{Z}}^T] \\ E[\hat{\mathbf{Z}}\tilde{\mathbf{Z}}^T] & E[\hat{\mathbf{Z}}\hat{\mathbf{Z}}^T] \end{bmatrix} = \begin{bmatrix} P_{\tilde{\mathbf{Z}}} & P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} \\ P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} & P_{\hat{\mathbf{Z}}} \end{bmatrix}$$

Now $\dot{\mathcal{P}}(t)$ can be rewritten as

$$\begin{bmatrix} \dot{P}_{\tilde{\mathbf{Z}}} & \dot{P}_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} \\ \dot{P}_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} & \dot{P}_{\hat{\mathbf{Z}}} \end{bmatrix} = \begin{bmatrix} (F_m - KH + \Delta F) & (\Delta F + \Delta D S) \\ KH & (F_m + D_m S) \end{bmatrix} \begin{bmatrix} P_{\tilde{\mathbf{Z}}} & P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} \\ P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} & P_{\hat{\mathbf{Z}}} \end{bmatrix} + \begin{bmatrix} P_{\tilde{\mathbf{Z}}} & P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} \\ P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} & P_{\hat{\mathbf{Z}}} \end{bmatrix} \begin{bmatrix} (F_m - KH + \Delta F)^T & (KH)^T \\ (\Delta F + \Delta D S)^T & (F_m + D_m S)^T \end{bmatrix} + \begin{bmatrix} (GQG^T + KRK^T) & -KRK^T \\ -KRK^T & KRK^T \end{bmatrix}$$

From the above equation, $\dot{P}_{\tilde{\mathbf{Z}}}$, $\dot{P}_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}}$, and $\dot{P}_{\hat{\mathbf{Z}}}$ can be written as

$$\begin{aligned} \dot{P}_{\tilde{\mathbf{Z}}} =&(F_m - KH + \Delta F)P_{\tilde{\mathbf{Z}}} + (\Delta F + \Delta D S)P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} + P_{\tilde{\mathbf{Z}}}(F_m - KH + \Delta F)^T + \\ & P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}}(\Delta F + \Delta D S)^T + GQG^T + KRK^T \end{aligned} \tag{27}$$

$$\dot{P}_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} =(F_m - KH + \Delta F)P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} + (\Delta F + \Delta D S)P_{\hat{\mathbf{Z}}} + P_{\tilde{\mathbf{Z}}}(KH)^T + P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}}(F_m + D_m S)^T - KRK^T \tag{28}$$

$$\dot{P}_{\hat{\mathbf{Z}}} =(KH)P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}} + (F_m + D_m S)P_{\hat{\mathbf{Z}}} + P_{\tilde{\mathbf{Z}}\hat{\mathbf{Z}}}(KH)^T + P_{\hat{\mathbf{Z}}}(F_m + D_m S)^T + KRK^T \tag{29}$$

Thus the true estimation error covariance is

$$E\Big[(\tilde{\mathbf{Z}}(t) - \boldsymbol{\mu}_{\tilde{\mathbf{Z}}}(t))(\tilde{\mathbf{Z}}(t) - \boldsymbol{\mu}_{\tilde{\mathbf{Z}}}(t))^T\Big] = P_{\tilde{\mathbf{Z}}}(t) - \boldsymbol{\mu}_{\tilde{\mathbf{Z}}}(t)\boldsymbol{\mu}^T_{\tilde{\mathbf{Z}}}(t) \tag{30}$$

Since the model errors are unknown, the above equation cannot be utilized in the filter implementation.

B. Closed-Loop Stability and Transient Response Bound for Systems with No Uncertainties

A detailed analysis of the closed-loop system's asymptotic stability in the mean when there are no uncertainties is now given. As shown here, a transient bound on the system response mean can be obtained in terms of the time varying correlation matrix. Most of the definitions and formulations given in this section are similar to the ones given in Ref. 13 for deterministic systems.

Consider a case where there is no model error, i.e., $F = F_m$, $D = D_m$, and $\boldsymbol{\mathcal{V}}(t) = \boldsymbol{\mathcal{W}}(t)$. If there is no model error, then the estimator is unbiased, i.e., $E[\bar{\tilde{\mathbf{Z}}}] = \boldsymbol{\mu}_{\bar{\tilde{\mathbf{Z}}}} = \mathbf{0}$. Now we could write

$$\begin{bmatrix} \dot{\bar{\tilde{\mathbf{Z}}}} \\ \dot{\bar{\hat{\mathbf{Z}}}} \end{bmatrix} = \begin{bmatrix} (F_m - KH) & 0 \\ KH & (F_m + D_m S) \end{bmatrix} \begin{bmatrix} \bar{\tilde{\mathbf{Z}}} \\ \bar{\hat{\mathbf{Z}}} \end{bmatrix} + \begin{bmatrix} G & -K \\ 0 & K \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{W}} \\ \mathbf{V} \end{bmatrix}$$

or in a more compact form as

$$\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t) \tag{31}$$

Before discussing the stability analysis, a few definitions regarding the stability of stochastic processes are presented.

Definition 1. *Given $M \geq 1$ and $\beta \in \mathbb{R}$, the system $\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t)$ is said to be (M, β)-stable in the mean if*

$$\| \bar{\Phi}(t, t_0)\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t_0) \| \leq M e^{\beta(t-t_0)} \| \boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t_0) \| \tag{32}$$

where $\bar{\Phi}(t, t_0)$ is the evolution operator generated by $\bar{\Upsilon}(t)$ and $\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t) = E[\bar{\boldsymbol{\mathcal{Z}}}(t)]$.

Since most applications involve the case where $\beta \leq 0$, (M, β)-stability guarantees both a specific decay rate of the mean (given by β) and a specific bound on the transient behavior of the mean (given by M).

Definition 2. *If a stochastic system, $\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t)$, is (M, β)-stable in the mean, then the transient bound of the system mean response for the exponential rate β is defined to be*

$$M_\beta = \inf\Big\{ M \in \mathbb{R}; \forall t \geq t_0 : \| \bar{\Phi}(t, t_0) \| \leq M e^{\beta(t-t_0)} \Big\} \tag{33}$$

The optimal transient bound $M_\beta = 1$ can be achieved by choosing a sufficiently large β, i.e.,

$$\beta(t-t_0) \geq \int_{t_0}^{t} \parallel \bar{\Upsilon}(\tau) \parallel d\tau \quad \Longrightarrow \quad \parallel \bar{\Phi}(t,t_0) \parallel \leq e^{\int_{t_0}^{t} \|\bar{\Upsilon}(\tau)\| d\tau} \leq e^{\beta(t-t_0)}\,, \quad t \geq t_0$$

Therefore it is of interest to know the smallest $\beta \in \mathbb{R}$ such that $\parallel \bar{\Phi}(t,t_0) \parallel \leq e^{\beta(t-t_0)}$, $t \geq t_0$. Given the system, $\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t)$, which is (M,β)-stable in the mean, the transient bound M_β of the system mean can be readily obtained based on the premises of the following theorem.

Theorem 1. *Suppose the system* $\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t)$ *is (M,β)-stable in the mean, then there exists a continuously differentiable positive definite matrix function $\bar{\mathcal{P}}(t)$ ($\bar{\mathcal{P}} = E[\bar{\boldsymbol{\mathcal{Z}}}\bar{\boldsymbol{\mathcal{Z}}}^T]$) satisfying the matrix Lyapunov differential equation*

$$\dot{\bar{\mathcal{P}}}(t) = \bar{\Upsilon}(t)\bar{\mathcal{P}}(t) + \bar{\mathcal{P}}(t)\bar{\Upsilon}^T(t) + \Gamma(t)\bar{\Lambda}\Gamma^T(t) \tag{34}$$

such that

$$M_\beta^2 \leq \sup_{t \geq t_0} \sigma_{\max}(\bar{\mathcal{P}}(t))/\sigma_{\min}(\bar{\mathcal{P}}(t_0)) \tag{35}$$

Proof. Solution to Eq. (34) can be written as

$$\bar{\mathcal{P}}(t) = \bar{\Phi}(t,t_0)\bar{\mathcal{P}}(t_0)\bar{\Phi}^T(t,t_0) + \int_{t_0}^{t} \bar{\Phi}(t,\tau)\Gamma(\tau)\bar{\Lambda}\Gamma^T(\tau)\bar{\Phi}^T(t,\tau)d\tau$$

Notice $\bar{\mathcal{P}}(t) \geq \bar{\Phi}(t,t_0)\bar{\mathcal{P}}(t_0)\bar{\Phi}^T(t,t_0) \geq \sigma_{\min}(\bar{\mathcal{P}}(t_0))\bar{\Phi}(t,t_0)\bar{\Phi}^T(t,t_0)$, i.e.,

$$\sigma_{\max}(\bar{\mathcal{P}}(t)) \geq \parallel \bar{\Phi}(t,t_0)\bar{\mathcal{P}}(t_0)\bar{\Phi}^T(t,t_0) \parallel \geq \sigma_{\min}(\bar{\mathcal{P}}(t_0)) \parallel \bar{\Phi}(t,t_0) \parallel^2$$

Now Eq. (35) follows from

$$\sigma_{\max}(\bar{\mathcal{P}}(t))/\sigma_{\min}(\bar{\mathcal{P}}(t_0)) \geq \parallel \bar{\Phi}(t,t_0) \parallel^2$$

□

C. Closed-Loop Stability and Transient Response Bound for Uncertain Systems

Consider a scenario where model error is present, i.e.,

$$\begin{bmatrix} \dot{\tilde{\mathbf{Z}}} \\ \dot{\hat{\mathbf{Z}}} \end{bmatrix} = \begin{bmatrix} (F_m - KH + \Delta F) & (\Delta F + \Delta D S) \\ KH & (F_m + D_m S) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Z}} \\ \hat{\mathbf{Z}} \end{bmatrix} + \begin{bmatrix} G & -K \\ 0 & K \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{V}} \\ \mathbf{V} \end{bmatrix}$$

or in a more compact form as

$$\dot{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t) \tag{36}$$

where

$$\Delta\Upsilon(t) = \begin{bmatrix} (\Delta F) & (\Delta F + \Delta D S) \\ 0 & 0 \end{bmatrix}$$

In the previous section we analyzed the stability of the unperturbed system $\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t)$. Here we will analyze the stability of the perturbed system, $\dot{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t)$.

The correlation matrix $\mathcal{P}(t) = E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)]$ satisfies the following matrix Lyapunov differential equation

$$\dot{\mathcal{P}}(t) = (\bar{\Upsilon}(t) + \Delta\Upsilon(t))\mathcal{P}(t) + \mathcal{P}(t)(\bar{\Upsilon}(t) + \Delta\Upsilon(t))^T + \Gamma(t)\Lambda\Gamma^T(t) \tag{37}$$

Note that $\Gamma(t)\Lambda\Gamma^T(t)$ can be factored as shown below:

$$\Gamma(t)\Lambda\Gamma^T(t) = \begin{bmatrix} (G\mathcal{Q}G^T + KRK^T) & -KRK^T \\ -KRK^T & KRK^T \end{bmatrix} = \begin{bmatrix} (G\mathcal{Q}G^T) & -0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (KRK^T) & -KRK^T \\ -KRK^T & KRK^T \end{bmatrix}$$
$$= \begin{bmatrix} G \\ 0 \end{bmatrix} \mathcal{Q} \begin{bmatrix} G^T & 0 \end{bmatrix} + \begin{bmatrix} PH^T \\ -PH^T \end{bmatrix} R^{-1} \begin{bmatrix} HP & -HP \end{bmatrix} = LQL^T + N(t)R^{-1}N^T(t)$$

Theorem 2. *The uncertain system, $\dot{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t)$, is (M, β)-stable in the mean if*

$$\| \Delta\Upsilon(t)\bar{\mathcal{P}}(t) \|^2 \leq \sigma_{\min}(Q)\sigma_{\min}(R^{-1}) \| L \|^2 \| N(t) \|^2 \tag{38}$$

where $\bar{\mathcal{P}}(t)$ satisfying

$$\dot{\bar{\mathcal{P}}}(t) = \bar{\Upsilon}(t)\bar{\mathcal{P}}(t) + \bar{\mathcal{P}}(t)\bar{\Upsilon}^T(t) + LQL^T + N(t)R^{-1}N(t)^T \tag{39}$$

Proof. In order to show the asymptotic stability of the mean we consider the following equation:

$$\dot{\boldsymbol{\mu}}_{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) + \Delta\Upsilon(t)\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)$$

Construct the following Lyapunov candidate function:

$$V[\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)] = \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T(t)\bar{\mathcal{P}}^{-1}(t)\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) \tag{40}$$

The matrix $\bar{\mathcal{P}}(t)$ is required to be a positive definite matrix, therefore $\bar{\mathcal{P}}^{-1}(t)$ exists and $V[\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)] > 0$ for all $\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) \neq \mathbf{0}$. Since $\bar{\mathcal{P}}(t)\bar{\mathcal{P}}^{-1}(t) = I$, the time derivative of $\bar{\mathcal{P}}(t)\bar{\mathcal{P}}^{-1}(t)$ is 0:

$$\frac{d}{dt}\Big[\bar{\mathcal{P}}(t)\bar{\mathcal{P}}^{-1}(t)\Big] = \dot{\bar{\mathcal{P}}}(t)\bar{\mathcal{P}}^{-1}(t) + \bar{\mathcal{P}}(t)\dot{\bar{\mathcal{P}}}^{-1}(t) = 0$$

Solving the above equation for $\dot{\bar{\mathcal{P}}}^{-1}(t)$ and substituting Eq. (39) gives

$$\begin{aligned} \dot{\bar{\mathcal{P}}}^{-1}(t) &= -\bar{\mathcal{P}}^{-1}(t)\dot{\bar{\mathcal{P}}}(t)\bar{\mathcal{P}}^{-1}(t) \\ &= -\bar{\mathcal{P}}^{-1}(t)\bar{\Upsilon}(t) - \bar{\Upsilon}^T(t)\bar{\mathcal{P}}^{-1}(t) - \bar{\mathcal{P}}^{-1}(t)LQL^T\bar{\mathcal{P}}^{-1}(t) - \bar{\mathcal{P}}^{-1}(t)N(t)R^{-1}N(t)^T\bar{\mathcal{P}}^{-1}(t) \end{aligned} \tag{41}$$

Now the time derivative of Eq. (40) can be written as

$$\begin{aligned} \dot{V}[\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)] =& \dot{\boldsymbol{\mu}}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\dot{\bar{\mathcal{P}}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}\dot{\boldsymbol{\mu}}_{\boldsymbol{\mathcal{Z}}} \\ =& [\bar{\Upsilon}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}]^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}\bar{\Upsilon}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\Upsilon}^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \\ & \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}LQL^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}NR^{-1}N^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}[\bar{\Upsilon}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}] \\ =& \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\Delta\Upsilon^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}\Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}LQL^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\mathcal{P}}^{-1}NR^{-1}N^T\bar{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} \\ =& \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\Big\{\Delta\Upsilon^T\bar{\mathcal{P}}^{-1} + \bar{\mathcal{P}}^{-1}\Delta\Upsilon - \bar{\mathcal{P}}^{-1}LQL^T\bar{\mathcal{P}}^{-1} - \bar{\mathcal{P}}^{-1}NR^{-1}N^T\bar{\mathcal{P}}^{-1}\Big\}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} \end{aligned}$$

We have asymptotic stability in the mean if

$$\Big\{-\bar{\mathcal{P}}\Delta\Upsilon^T - \Delta\Upsilon\bar{\mathcal{P}} + LQL^T + NR^{-1}N^T\Big\} > 0$$

Note

$$\begin{aligned} -\Delta\Upsilon\bar{\mathcal{P}} - \bar{\mathcal{P}}\Delta\Upsilon^T + LQL^T + NR^{-1}N^T =& LQL^T - \bar{\mathcal{P}}\Delta\Upsilon^T(NR^{-1}N^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}} + \\ & \Big[\bar{\mathcal{P}}\Delta\Upsilon^T(NR^{-1}N^T)^{-1} - I\Big](NR^{-1}N^T)\Big[\bar{\mathcal{P}}\Delta\Upsilon^T(NR^{-1}N^T)^{-1} - I\Big]^T \end{aligned}$$

Now we need to show

$$LQL^T \geq \bar{\mathcal{P}}\Delta\Upsilon^T(NR^{-1}N^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}}$$

Note the following:

$$
\begin{aligned}
&\sigma_{\min}(Q)LL^T \leq LQL^T \\
&\sigma_{\min}(R^{-1})NN^T \leq NR^{-1}N^T \Rightarrow \bar{\mathcal{P}}\Delta\Upsilon^T(\sigma_{\min}(R^{-1})NN^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}} \geq \bar{\mathcal{P}}\Delta\Upsilon^T(NR^{-1}N^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}}
\end{aligned}
$$

Thus now we need to show

$$
LQL^T \geq \sigma_{\min}(Q)LL^T \geq \bar{\mathcal{P}}\Delta\Upsilon^T(\sigma_{\min}(R^{-1})NN^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}} \geq \bar{\mathcal{P}}\Delta\Upsilon^T(NR^{-1}N^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}}
$$

$$
\begin{gathered}
\sigma_{\min}(Q)\sigma_{\min}(R^{-1}) \parallel LL^T \parallel \geq \parallel \bar{\mathcal{P}}\Delta\Upsilon^T(NN^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}} \parallel \\
\parallel \bar{\mathcal{P}}\Delta\Upsilon^T \parallel\parallel (NN^T) \parallel^{-1}\parallel \Delta\Upsilon\bar{\mathcal{P}} \parallel \geq \parallel \bar{\mathcal{P}}\Delta\Upsilon^T(NN^T)^{-1}\Delta\Upsilon\bar{\mathcal{P}} \parallel
\end{gathered}
$$

Hence we have

$$
\sigma_{\min}(Q)\sigma_{\min}(R^{-1}) \parallel L \parallel^2 \parallel N \parallel^2 \geq \parallel \Delta\Upsilon\bar{\mathcal{P}} \parallel^2 \tag{42}
$$

□

Therefore (M, β)-stability in the mean is guaranteed if the inequality Eq. (38) is satisfied. Let Q^* and R^* is chosen so that the above inequality is satisfied. Now substituting Q^* and R^* into Eq. (37) we have

$$
\dot{\mathcal{P}}^*(t) = (\bar{\Upsilon}(t) + \Delta\Upsilon(t))\mathcal{P}^*(t) + \mathcal{P}^*(t)(\bar{\Upsilon}(t) + \Delta\Upsilon(t))^T + LQ^*L^T + N(t){R^*}^{-1}N^T(t) \tag{43}
$$

The solution of the above equation is

$$
\begin{aligned}
\mathcal{P}^*(t) =& [\bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0)]\mathcal{P}^*(t_0)[\bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0)]^T + \\
&\int_{t_0}^{t} [\bar{\Phi}(t,\tau) + \Phi_\Delta(t,\tau)]\{LQ^*L^T + N(\tau){R^*}^{-1}N^T(\tau)\}[\bar{\Phi}(t,\tau) + \Phi_\Delta(t,\tau)]^T d\tau
\end{aligned} \tag{44}
$$

Corollary 1. *If the system given in Eq. (36) is (M, β)-stable in the mean, then there exists a continuously differentiable positive definite symmetric matrix function $\mathcal{P}^*(t)$ given by Eq. (44) such that*

$$
M_\beta^2 \leq \sup_{t \geq t_0} \sigma_{\max}(\mathcal{P}^*(t))/\sigma_{\min}(\mathcal{P}^*(t_0)) \tag{45}
$$

where M_β represents the transient bound of the perturbed system's mean response.

Proof. If $\mathcal{P}^*(t)$ satisfies Eq. (44), then

$$
\mathcal{P}^*(t) \geq [\bar{\Phi}(t,t_0)+\Phi_\Delta(t,t_0)]\mathcal{P}^*(t_0)[\bar{\Phi}(t,t_0)+\Phi_\Delta(t,t_0)]^T \geq \sigma_{\min}(\mathcal{P}^*(t_0))[\bar{\Phi}(t,t_0)+\Phi_\Delta(t,t_0)][\bar{\Phi}(t,t_0)+\Phi_\Delta(t,t_0)]^T
$$

i.e.,

$$
\sigma_{\max}(\mathcal{P}^*(t)) \geq \parallel [\bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0)]\mathcal{P}^*(t_0)[\bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0)]^T \parallel \geq \sigma_{\min}(\mathcal{P}^*(t_0)) \parallel [\bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0)] \parallel^2
$$

Now we have

$$
\sigma_{\max}(\mathcal{P}^*(t))/\sigma_{\min}(\mathcal{P}^*(t_0)) \geq \parallel \bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0) \parallel^2
$$

Therefore the transient bound, M_β^2, of the perturbed system can be obtained from

$$
M_\beta^2 \leq \sup_{t \geq t_0} \sigma_{\max}(\mathcal{P}^*(t))/\sigma_{\min}(\mathcal{P}^*(t_0))
$$

□

D. Mean Square Stability

Previously we analyzed stability in the mean. Here, it is shown that the (M, β)-stability in the mean implies mean square stability. More details on mean square stability can be found in Refs. 14 and 15.

Definition 3. *A stochastic system of the following form* $\dot{\boldsymbol{\mathcal{Z}}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t)$ *is mean square stable if*

$$\lim_{t\to\infty} E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)] < \mathrm{C} \tag{46}$$

where C *is a constant square matrix whose elements are finite.*

Note that

$$\frac{d}{dt}E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)] = \dot{\mathcal{P}}(t) = \Upsilon(t)\mathcal{P}(t) + \mathcal{P}(t)\Upsilon^T(t) + \Gamma(t)\Lambda\Gamma^T(t)$$

and the solution to the above equation can be written as

$$\mathcal{P}(t) = \int_{-\infty}^{t} \Phi(t,\tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\Phi^T(t,\tau)d\tau$$

The exponentially stable in the mean implies the system matrix, $\Upsilon(t) = \bar{\Upsilon}(t) + \Delta\Upsilon(t)$, generates a stable evolution operator, therefore $\mathcal{P}(t)$ has a bounded solution.[16]

E. Almost Sure Asymptotic Stability

Solution to the stochastic system given in Eq. (36) cannot be based on the ordinary mean square calculus because the integral involved in the solution depends on $\boldsymbol{\mathcal{G}}(t)$, which is of unbounded variation. For the treatment of this class of problems, the stochastic differential equation can be rewritten in Itô form as[17]

$$d\boldsymbol{\mathcal{Z}}(t) = [\bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t)]dt + \Gamma(t)\Lambda^{1/2}d\mathcal{B}(t)$$

or simply as

$$d\boldsymbol{\mathcal{Z}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t)dt + \Gamma(t)\Lambda^{1/2}d\mathcal{B}(t) \tag{47}$$

where $d\mathcal{B}(t)$ is an increment of Brownian motion process with zero-mean, Gaussian distribution and covariance

$$E[d\mathcal{B}(t)d\mathcal{B}^T(t)] = Idt \tag{48}$$

The solution $\boldsymbol{\mathcal{Z}}(t)$ of Eq. (47) is a semimartingale process that is also Markov.[18]

Definition 4. *The linear stochastic system given in Eq.* (47) *is asymptotically stable with probability* 1*, or almost surely asymptotically stable, if*

$$\mathbb{P}\big(\boldsymbol{\mathcal{Z}}(t) \to \boldsymbol{0} \quad \text{as} \quad t\to\infty\big) = 1 \tag{49}$$

Given below is the well-known classical result on the global asymptotic stability for stochastic systems:[14,19]

Theorem 3. *Assume that there are functions* $V(\boldsymbol{z},t) \in \mathbb{C}^{2,1}$ *and* $\kappa_1, \kappa_2, \kappa_3 \in$ class-$\mathcal{K}$ *such that*

$$\kappa_1(\| \boldsymbol{z} \|) \le V(\boldsymbol{z},t) \le \kappa_2(\| \boldsymbol{z} \|) \tag{50a}$$

$$\mathfrak{L}V(\boldsymbol{z},t) \le -\kappa_3(\| \boldsymbol{z} \|) \tag{50b}$$

for all $(\boldsymbol{z},t) \in R^{4n} \times R_+$*, where* $\boldsymbol{z}$ *indicate a sample path of* $\boldsymbol{\mathcal{Z}}(t,\omega)$*, i.e.,* $\boldsymbol{z}(t) = \boldsymbol{\mathcal{Z}}(t,\omega_\imath)\mid_{\omega_\imath\in\Omega}$*. Then, for every initial valuse* $\boldsymbol{\mathcal{Z}}_0$*, the solution of Eq.* (47) *has the property that*

$$\boldsymbol{\mathcal{Z}}(t) \to \boldsymbol{0} \quad \text{almost surely as } t\to\infty \tag{51}$$

The operator $\mathfrak{L}\{\cdot\}$ *acting on* $V(\boldsymbol{z},t)$ *is given by*

$$\mathfrak{L}V(\boldsymbol{z},t) = \lim_{dt\to 0}\frac{1}{dt}E\big[dV(\boldsymbol{\mathcal{Z}}(t),t)|\boldsymbol{\mathcal{Z}}(t) = \boldsymbol{z}\big] \tag{52}$$

where $dV(\boldsymbol{\mathcal{Z}}(t),t)$ *can be calculated using the Itô Formula.*

If is often very difficult to show the negative definiteness of $\mathfrak{L}V(\boldsymbol{z},t)$. One way to get around this problem is to replace the condition given in Eq. (50b) with two weaker conditions.

Theorem 4. *Assume that there are functions $V(\boldsymbol{z},t) \in \mathbb{C}^{2,1}$, $\kappa_1, \kappa_2, \kappa_3 \in$ class-$\mathcal{K}$, and $\eta(t) \in L_1$ such that*

$$\kappa_1(\| \boldsymbol{z} \|) \leq V(\boldsymbol{z},t) \leq \kappa_2(\| \boldsymbol{z} \|) \tag{53a}$$

$$\mathfrak{L}V(\boldsymbol{z},t) \leq \eta(t) \qquad \text{and} \tag{53b}$$

$$\mathfrak{L}V(\boldsymbol{z},t) \leq \eta(t) + \| V_{\boldsymbol{z}}^T(\boldsymbol{z},t)\Gamma(t) \|^2 - \kappa_3(\| \boldsymbol{z} \|) \tag{53c}$$

where $V_{\boldsymbol{z}} = \frac{\partial V}{\partial \boldsymbol{z}}$. Then the conclusion of Theorem 3 still holds.

More detailed derivation and proof of this theorem can be found in Ref. 20. Notice that if the inequality Eq. (38) is satisfied, then there exists a $\mathcal{P}(t)$ which satisfies the following equation:

$$\dot{\mathcal{P}}(t) = \Upsilon(t)\mathcal{P}(t) + \mathcal{P}(t)\Upsilon^T(t) + \Gamma(t)\Lambda\Gamma^T(t)$$

Consider the function $V(\boldsymbol{z},\mathcal{P}^{-1}) = \frac{\boldsymbol{z}^T(t)\mathcal{P}^{-1}(t)\boldsymbol{z}(t)}{\mathcal{M}_{\text{sup}}}$, where $\mathcal{M}_{\text{sup}} = \sup_{\infty > \tau \geq t_0} \text{Tr}\Big\{\mathcal{P}^{-1}(\tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\Big\}$. Now $dV(\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t))$ can be written as

$$dV(\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t)) = V(\boldsymbol{\mathcal{Z}}(t) + d\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t) + d\mathcal{P}^{-1}(t)) - V(\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t))$$

Using a Taylor series up to second order, we have

$$V(\boldsymbol{\mathcal{Z}}(t) + d\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t) + d\mathcal{P}^{-1}(t)) \approx V(\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t)) + \text{Tr}\Big\{d\boldsymbol{\mathcal{Z}}\Big(\frac{\partial V}{\partial \boldsymbol{\mathcal{Z}}}\Big)^T\Big\} + \text{Tr}\Big\{d\mathcal{P}^{-1}\Big(\frac{\partial V}{\partial \mathcal{P}^{-1}}\Big)\Big\} + \frac{1}{2}\text{Tr}\Big\{d\boldsymbol{\mathcal{Z}}d\boldsymbol{\mathcal{Z}}^T\Big(\frac{\partial^2 V}{\partial \boldsymbol{\mathcal{Z}}\partial \boldsymbol{\mathcal{Z}}^T}\Big)\Big\}$$

here $d\boldsymbol{\mathcal{Z}}$ is given in Eq.(47) and $d\mathcal{P}^{-1}$ is

$$d\mathcal{P}^{-1} = \Big\{-\mathcal{P}^{-1}\Upsilon - \Upsilon^T\mathcal{P}^{-1} - \mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\bar{\mathcal{P}}^{-1}\Big\}dt$$

The partials are

$$\frac{\partial V}{\partial \boldsymbol{\mathcal{Z}}} = \frac{2}{\mathcal{M}_{\text{sup}}}\mathcal{P}^{-1}\boldsymbol{\mathcal{Z}}, \quad \frac{\partial V}{\partial \mathcal{P}^{-1}} = \frac{1}{\mathcal{M}_{\text{sup}}}\boldsymbol{\mathcal{Z}}\boldsymbol{\mathcal{Z}}^T, \quad \text{and} \quad \frac{\partial^2 V}{\partial \boldsymbol{\mathcal{Z}}\partial \boldsymbol{\mathcal{Z}}^T} = \frac{2}{\mathcal{M}_{\text{sup}}}\mathcal{P}^{-1}$$

Now we have

$$\begin{aligned}
dV(\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t)) &\approx \frac{2}{\mathcal{M}_{\text{sup}}}\boldsymbol{\mathcal{Z}}^T\mathcal{P}^{-1}d\boldsymbol{\mathcal{Z}} + \frac{1}{\mathcal{M}_{\text{sup}}}\boldsymbol{\mathcal{Z}}^T d\mathcal{P}^{-1}\boldsymbol{\mathcal{Z}} + \frac{1}{\mathcal{M}_{\text{sup}}}\text{Tr}\Big\{\mathcal{P}^{-1}d\boldsymbol{\mathcal{Z}}d\boldsymbol{\mathcal{Z}}^T\Big\} \\
\approx \frac{2}{\mathcal{M}_{\text{sup}}}\boldsymbol{\mathcal{Z}}^T\mathcal{P}^{-1}\Upsilon\boldsymbol{\mathcal{Z}}dt &+ \frac{2}{\mathcal{M}_{\text{sup}}}\boldsymbol{\mathcal{Z}}^T\mathcal{P}^{-1}\Gamma\Lambda^{1/2}d\mathcal{B} - \frac{1}{\mathcal{M}_{\text{sup}}}\boldsymbol{\mathcal{Z}}^T\Big\{\mathcal{P}^{-1}\Upsilon + \Upsilon^T\mathcal{P}^{-1} + \mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\mathcal{P}^{-1}\Big\}\boldsymbol{\mathcal{Z}}dt + \\
&\frac{1}{\mathcal{M}_{\text{sup}}}\text{Tr}\Big\{\mathcal{P}^{-1}\Big(\Upsilon\boldsymbol{\mathcal{Z}}\boldsymbol{\mathcal{Z}}^T\Upsilon^T dt^2 + \Gamma\Lambda^{1/2}d\mathcal{B}\boldsymbol{\mathcal{Z}}^T\Upsilon^T dt + \Upsilon\boldsymbol{\mathcal{Z}}d\mathcal{B}^T\Lambda^{1/2}\Gamma^T dt + \Gamma\Lambda^{1/2}d\mathcal{B}d\mathcal{B}^T\Lambda^{1/2}\Gamma^T\Big)\Big\}
\end{aligned}$$

Now taking the conditional expectation, we obtain

$$\begin{aligned}
E\big[dV(\boldsymbol{\mathcal{Z}}(t),\mathcal{P}^{-1}(t))|\boldsymbol{\mathcal{Z}}(t) = \boldsymbol{z}\big] = &\frac{2}{\mathcal{M}_{\text{sup}}}\boldsymbol{z}^T\mathcal{P}^{-1}\Upsilon\boldsymbol{z}dt + \frac{1}{\mathcal{M}_{\text{sup}}}\text{Tr}\Big\{\mathcal{P}^{-1}\Big(\Upsilon\boldsymbol{z}\boldsymbol{z}^T\Upsilon^T dt^2 + \Gamma\Lambda\Gamma^T dt\Big)\Big\} - \\
&\frac{1}{\mathcal{M}_{\text{sup}}}\boldsymbol{z}^T\Big\{\mathcal{P}^{-1}\Upsilon + \Upsilon^T\mathcal{P}^{-1} + \mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\mathcal{P}^{-1}\Big\}\boldsymbol{z}dt
\end{aligned}$$

Now we can calculate

$$\begin{aligned}
\mathfrak{L}V(\boldsymbol{z},\mathcal{P}^{-1}) = &\frac{2}{\mathcal{M}_{\text{sup}}}\boldsymbol{z}^T\mathcal{P}^{-1}\Upsilon\boldsymbol{z} - \frac{1}{\mathcal{M}_{\text{sup}}}\boldsymbol{z}^T\Big\{\mathcal{P}^{-1}\Upsilon + \Upsilon^T\mathcal{P}^{-1} + \mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\mathcal{P}^{-1}\Big\}\boldsymbol{z} + \frac{1}{\mathcal{M}_{\text{sup}}}\text{Tr}\Big\{\mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\Big\} \\
= &-\frac{1}{\mathcal{M}_{\text{sup}}}\boldsymbol{z}^T\mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\mathcal{P}^{-1}\boldsymbol{z} + \frac{1}{\mathcal{M}_{\text{sup}}}\text{Tr}\Big\{\mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\Big\}
\end{aligned}$$

Notice $\mathcal{M}_{\text{sup}} \geq \text{Tr}\Big\{\mathcal{P}^{-1}(t)\Gamma(t)\Lambda\Gamma^T(t)\Big\}, \quad \forall t \geq t_0$, i.e.,

$$\mathfrak{L}V(\boldsymbol{z}, \mathcal{P}^{-1}) \leq 1 - \frac{\sigma_{\min}\{\mathcal{P}^{-1}\Gamma\Lambda\Gamma^T\mathcal{P}^{-1}\}}{\mathcal{M}_{\text{sup}}} \parallel \boldsymbol{z} \parallel^2$$

Let $k(t) = \frac{\sigma_{\min}\{\mathcal{P}^{-1}(t)\Gamma(t)\Lambda\Gamma^T(t)\mathcal{P}^{-1}(t)\}}{\mathcal{M}_{\text{sup}}}$, thus $\mathfrak{L}V(\boldsymbol{z}, \mathcal{P}^{-1}) \leq 1 - k(t) \parallel \boldsymbol{z}(t) \parallel^2$.

Notice that

$$\lim_{t\to\infty} \parallel \boldsymbol{z}(t) \parallel^2 \leq \frac{1}{k(t)} \Longrightarrow 1 - k \parallel \boldsymbol{z} \parallel^2 \nleq \eta(t)$$

Thus we do not have almost sure asymptotic stability for the stochastic system given in Eq. (47). In fact, given a $\Upsilon(t)$ that generates an asymptotically stable evolution for the linear system in Eq. (47), the necessary and sufficent condition for the almost sure asymptotic stability is

$$\lim_{t\to\infty} \parallel \Gamma(t) \parallel^2 \log(t) = 0 \tag{54}$$

Detailed proof of this argument can be found in Ref. 21. Equation (54) constitutes the sufficent condition for the almost sure asymptotic stability of a linear stochastic system gievn (M, β)-stability in the mean.

IV. Results

A detailed investigation of the above Lyapunov stability analysis through numerical simulations is given in this section. For simulation purposes, we consider a two degree of freedom helicopter that pivots about the pitch axis by angle θ and about the yaw axis by angle ψ. As shown in Fig. 2, the pitch is defined positive when the nose of the helicopter goes up and the yaw is defined positive for a counterclockwise rotation. Also in Fig. 2, there is a thrust force F_p acting on the pitch axis that is normal to the plane of the front propeller and a thrust force F_y acting on the yaw axis that is normal to the rear propeller. Therefore a pitch torque is being applied at a distance r_p from the pitch axis and a yaw torque is applied at a distance r_y from the yaw axis. The gravitational force, F_g, generates a torque at the helicopter center of mass that pulls down on the helicopter nose. As shown in Fig. 2, the center of mass is a distance of l_{cm} from the pitch axis along the helicopter body length.

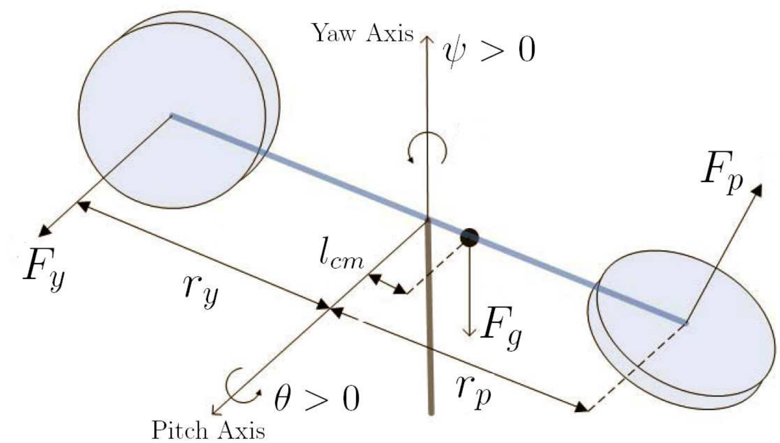

Figure 2. Two Degree of Freedom Helicopter

The helicopter equations of motion can be written as

$$(J_{eq,p} + m_{heli}l_{cm}^2)\ddot{\theta} = K_{pp}V_{m,p} + K_{py}V_{m,y} - B_p\dot{\theta} - m_{heli}gl_{cm}\cos(\theta) - m_{heli}l_{cm}^2\cos(\theta)\sin(\theta)\dot{\psi}^2 \tag{55a}$$

$$(J_{eq,y} + m_{heli}l_{cm}^2\cos(\theta)^2)\ddot{\psi} = K_{yy}V_{m,y} + K_{yp}V_{m,p} - B_y\dot{\psi} + 2m_{heli}l_{cm}^2\cos(\theta)\sin(\theta)\dot{\psi}\dot{\theta} \tag{55b}$$

After linearizing about $\theta_0 = \psi_0 = \dot{\theta}_0 = \dot{\psi}_0 = 0$, the helicopter equations of motion can be written as

$$(J_{eq,p} + m_{heli} l_{cm}^2)\ddot{\theta} = K_{pp} V_{m,p} + K_{py} V_{m,y} - B_p \dot{\theta} - m_{heli} g l_{cm} \quad (56a)$$

$$(J_{eq,y} + m_{heli} l_{cm}^2)\ddot{\psi} = K_{yy} V_{m,y} + K_{yp} V_{m,p} - B_y \dot{\psi} \quad (56b)$$

A detailed description of system parameters and assumed values are given in Table 2. Note that the negative viscous damping about the yaw axis is purposefully selected to ensure that the nominal control on the true plant is unstable.

Table 2. Two Degree-of-Freedom Helicopter Model Parameters

System Parameter	Description	Assumed Values	True Values	Unit
B_p	Equivalent viscous damping about pitch axis	0.8	1	N/V
B_y	Equivalent viscous damping about yaw axis	0.318	-0.3021	N/V
$J_{eq,p}$	Total moment of inertia about yaw pivot	0.0384	0.0288	$Kg{\cdot}m^2$
$J_{eq,y}$	Total moment of inertia about pitch pivot	0.0432	0.0496	$Kg{\cdot}m^2$
K_{pp}	Trust torque constant acting on pitch axis from pitch motor/propeller	0.204	0.2552	$N{\cdot}m/V$
K_{py}	Trust torque constant acting on pitch axis from yaw motor/propeller	0.0068	0.0051	$N{\cdot}m/V$
K_{yp}	Trust torque constant acting on yaw axis from pitch motor/propeller	0.0219	0.0252	$N{\cdot}m/V$
K_{yy}	Trust torque constant acting on yaw axis from yaw motor/propeller	0.072	0.0684	$N{\cdot}m/V$
m_{heli}	Total mass of the helicopter	1.3872	1.3872	Kg
l_{cm}	Location of center-of-mass along helicopter body	0.186	0.176	m

The control input to the system are the input voltages of the pitch and yaw motors, $V_{m,p}$ and $V_{m,y}$, respectively. Let $\mathbf{u} = [u_1 \quad u_2]^T = [V_{m,p} \quad V_{m,y}]^T$. Now the linearized equations can be rewritten as

$$\ddot{\theta} = a_1 \dot{\theta} + b_1 u_1 + b_2 u_2 - m_{heli} g l_{cm} \quad (57a)$$

$$\ddot{\psi} = a_2 \dot{\psi} + b_3 u_1 + b_4 u_2 \quad (57b)$$

where

$$a_1 = \frac{-B_p}{(J_{eq,p} + m_{heli} l_{cm}^2)} \qquad a_2 = \frac{-B_y}{(J_{eq,y} + m_{heli} l_{cm}^2)}$$

$$b_1 = \frac{K_{pp}}{(J_{eq,p} + m_{heli} l_{cm}^2)} \qquad b_2 = \frac{K_{py}}{(J_{eq,p} + m_{heli} l_{cm}^2)}$$

$$b_3 = \frac{K_{yp}}{(J_{eq,y} + m_{heli} l_{cm}^2)} \qquad b_4 = \frac{K_{yy}}{(J_{eq,y} + m_{heli} l_{cm}^2)}$$

Let $\mathbf{x} = [\theta \quad \psi \quad \dot{\theta} \quad \dot{\psi}]^T$. Now the state-space representation of the above system is

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + \mathbf{w} \quad (58)$$

where $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ -m_{heli} g l_{cm} \\ 0 \end{bmatrix}$. The state-space representation of the assumed system model is

$$\dot{\mathbf{x}}_m = A_m \mathbf{x}_m + B_m \mathbf{u} \quad (59)$$

where $A_m = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{1_m} & 0 \\ 0 & 0 & 0 & a_{2_m} \end{bmatrix}$ and $B_m = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_{1_m} & b_{2_m} \\ b_{3_m} & b_{4_m} \end{bmatrix}$. The measured output and the assumed output equations are given as

$$\mathbf{Y} = C\mathbf{x} + \mathbf{V} \tag{60}$$

$$\mathbf{Y}_m = C\mathbf{x}_m + \mathbf{V} \tag{61}$$

where $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Notice that the disturbance term, $\mathbf{d} = [0 \quad 0 \quad d_{\dot{\theta}} \quad d_{\dot{\psi}}]^T$, can be written as

$$d_{\dot{\theta}} = (a_1 - a_{1_m})\dot{\theta} + (b_1 - b_{1_m})u_1 + (b_2 - b_{2_m})u_2 - m_{heli}gl_{cm} = \triangle a_1\dot{\theta} + \triangle b_1 u_1 + \triangle b_2 u_2 - m_{heli}gl_{cm} \tag{62a}$$

$$d_{\dot{\psi}} = (a_2 - a_{2_m})\dot{\psi} + (b_3 - b_{3_m})u_1 + (b_4 - b_{4_m})u_2 = \triangle a_2\dot{\psi} + \triangle b_3 u_1 + \triangle b_4 u_2 \tag{62b}$$

The first two zero elements in the disturbance term indicate the perfect knowledge of the system kinematics. The disturbance term in vector notation can be written as

$$\mathbf{d} = \triangle A\mathbf{x} + \triangle B\mathbf{u} + \mathbf{w} \tag{63}$$

where $\triangle A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \triangle a_1 & 0 \\ 0 & 0 & 0 & \triangle a_2 \end{bmatrix}$ and $\triangle B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \triangle b_1 & \triangle b_2 \\ \triangle b_3 & \triangle b_4 \end{bmatrix}$. Using the disturbance term the true model can be written in terms of the assumed parameters as shown below:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{1_m} & 0 \\ 0 & 0 & 0 & a_{2_m} \end{bmatrix} \begin{bmatrix} \theta \\ \psi \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_{1_m} & b_{2_m} \\ b_{3_m} & b_{4_m} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ d_{\dot{\theta}} \\ d_{\dot{\psi}} \end{bmatrix} \tag{64}$$

or in vector notation:

$$\dot{\mathbf{x}} = A_m\mathbf{x} + B_m\mathbf{u} + \mathbf{d} \tag{65}$$

The disturbance term dynamics is modeled as

$$\dot{d}_{\dot{\theta}_m} = -d_{\dot{\theta}_m} + \mathcal{W}_1(t) \tag{66a}$$

$$\dot{d}_{\dot{\psi}_m} = -3d_{\dot{\psi}_m} + \mathcal{W}_2(t) \tag{66b}$$

Since the model uncertainty is only associated with the dynamics, only the nonzero elements of the disturbance term need to be appended to the system states. Let the extended assumed state vector, $\mathbf{Z}_m = [\mathbf{x}_m^T \quad d_{\dot{\theta}_m} \quad d_{\dot{\psi}_m}]^T$. Now the assumed extended state-space equation can be written as

$$\dot{\mathbf{Z}}_m = F_m\mathbf{Z}_m + D_m\mathbf{u} + G\boldsymbol{\mathcal{W}} \tag{67}$$

where $F_m = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_{1_m} & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{2_m} & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$, $D_m = \begin{bmatrix} B_m \\ 0_{2\times 2} \end{bmatrix}$, $G = \begin{bmatrix} 0_{4\times 2} \\ I_{2\times 2} \end{bmatrix}$, and $\boldsymbol{\mathcal{W}} = \begin{bmatrix} \mathcal{W}_1(t) \\ \mathcal{W}_2(t) \end{bmatrix}$. The assumed output equation can be written in terms of the appended state vector, $\mathbf{z}_m$, as

$$\mathbf{Y}_m = H\mathbf{Z}_m + \mathbf{V} \tag{68}$$

where $H = [C \quad 0_{2\times 2}]$. A Kalman filter is implemented in the feedback loop to estimate the system rates and the disturbance term. The filter dynamics is

$$\dot{\hat{\mathbf{Z}}} = F_m\hat{\mathbf{Z}} + D_m\mathbf{u} + KH[\mathbf{Z} - \hat{\mathbf{Z}}] + K\mathbf{Z} \tag{69}$$

The reference model that is of interest is

$$\dot{\bar{\mathbf{x}}} = A_m\bar{\mathbf{x}} + B_m\bar{\mathbf{u}} \tag{70}$$

where the nominal controller is a linear quadratic regulator which minimizes the cost function

$$J = \frac{1}{2}\int_0^\infty \big((\mathbf{x}(t) - \mathbf{x}_d)^T \mathcal{Q}_{\mathbf{x}}(\mathbf{x}(t) - \mathbf{x}_d) + \mathbf{u}^T(t)\mathcal{R}_{\mathbf{u}}\mathbf{u}(t)\big)dt \tag{71}$$

where $\mathbf{x}_d^T = [\theta_d \quad \psi_d \quad 0 \quad 0]$, θ_d and ψ_d are some desired final values of θ and ψ, respectively, and $\mathcal{Q}_{\mathbf{x}}$ and $\mathcal{R}_{\mathbf{u}}$ are two symmetric positive definite matrices. The nominal control that minimizes the above cost function is

$$\bar{\mathbf{u}}(t) = -K_m(\mathbf{x}(t) - \mathbf{x}_d) \tag{72}$$

where K_m is the feedback gain that minimizes the cost Eq. (71). Now the total control law can be written in terms of the estimated states and the estimated disturbance term as

$$\mathbf{u} = (B_m^T B_m)^{-1} B_m^T \Big[- B_m K_m \quad - I_{2\times 2}\Big] \begin{bmatrix} \hat{\mathbf{x}} - \mathbf{x}_d \\ \hat{d}_{\dot{\theta}} \\ \hat{d}_{\dot{\psi}} \end{bmatrix} = S\hat{\mathbf{Z}} + K_m\mathbf{x}_d \tag{73}$$

Since Eq. (65) does not contain any noise-like external disturbances, after substituting the above control law into Eq. (63), the true disturbance-term dynamics can be written as

$$\dot{\mathbf{d}} = (\triangle AA + \triangle BSKC)\mathbf{x} + (\triangle AB + \triangle BSD_m)(S\hat{\mathbf{Z}} + K_m\mathbf{x}_d) + \triangle BS(F_m - KH)\hat{\mathbf{Z}} + \triangle BSK\mathbf{v} \tag{74}$$

Equation (74) indicates that selecting a large Q or small R would amplify the measurement noise effect on the disturbance term dynamics. This is clearly shown in the simulation results given next.

Table 3. Nominal Controller/Estimator Matrices

LQR Weighting Matrices	Covariance Matrices
$\mathcal{R}_{\mathbf{u}} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$	$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$ $R = \begin{bmatrix} 1\times 10^{-5} & 0 \\ 0 & 1\times 10^{-5} \end{bmatrix}$
$\mathcal{Q}_{\mathbf{x}} = \begin{bmatrix} 2000 & 0 & 0 & 0 \\ 0 & 2000 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$	$P(t_0) = \begin{bmatrix} 1\times 10^{-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\times 10^{-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Table 3 shows the nominal controller and estimator matrices. Since the measurement noise covariance, R, can be obtained from sensor calibration, the process noise matrix, Q, is treated as a tuning parameter. Based on the weighting matrices given in Table 3, the feedback gain is calculated to be

$$K_m = \begin{bmatrix} 14.0529 & 1.5865 & 2.1762 & 0.3790 \\ -1.5865 & 14.0529 & -0.1712 & 3.6387 \end{bmatrix}$$

For simulation purposes the initial states are selected to be $[\theta_0 \quad \psi_0 \quad \dot{\theta}_0 \quad \dot{\psi}_0]^T = [-45^o \quad 0 \quad 0 \quad 0]^T$ and the desired states θ_d and ψ_d are selected to be 45^o and 30^o, respectively.

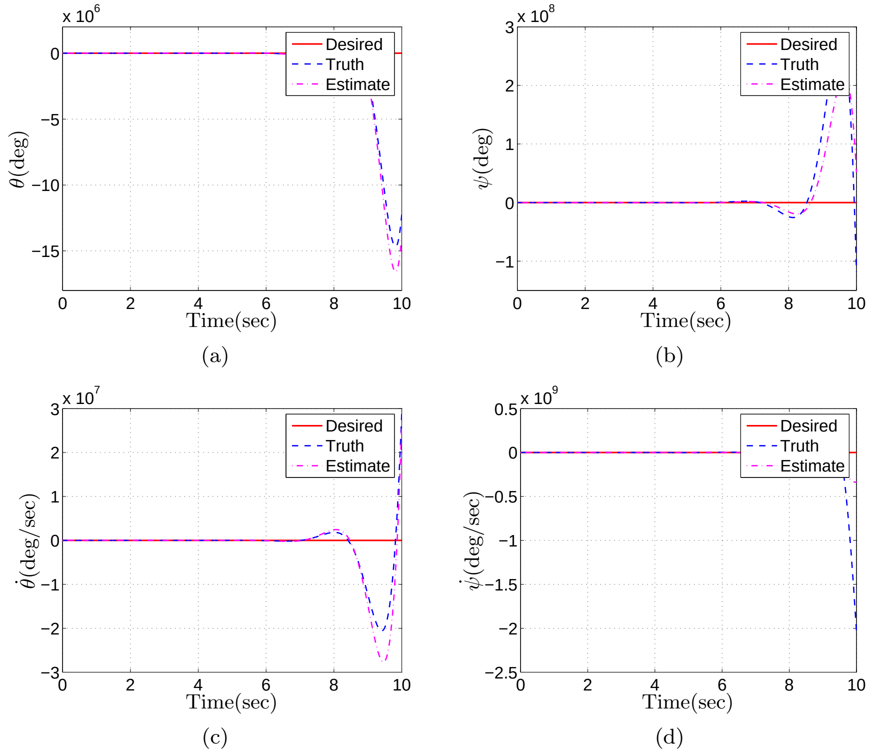

Figure 3. Unstable System Response: $q_1 = q_2 = 0.10$

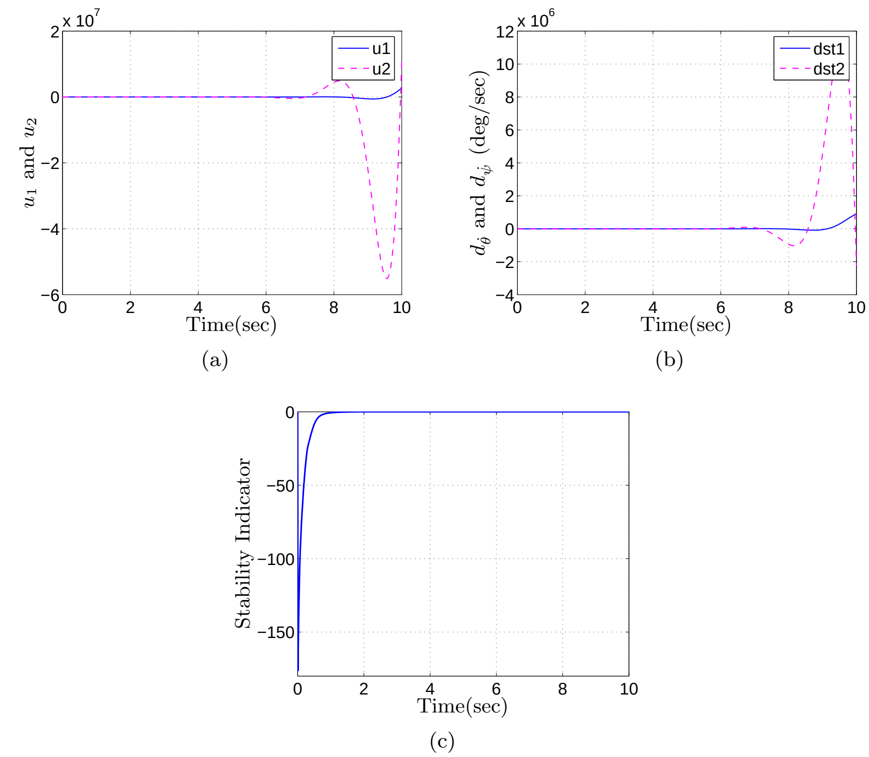

Figure 4. Control Input, Estimated Disturbance Term, and Stability Indicator: $q_1 = q_2 = 0.10$

Figure 3 shows the unstable system response obtained for the first simulation. The desired response given in Fig. 3 is the system response to nominal control when there is no model error and external disturbance. Figure 4 shows the system control input, estimated disturbance term and stability indicator obtained for the first simulation. The stability indicator is calculated as

$$\text{Stability Indicator} = \sigma_{\min}(Q)\sigma_{\min}(R^{-1}) \parallel L \parallel^2 \parallel N(t) \parallel^2 - \parallel \Delta\Upsilon(t)\bar{\mathcal{P}}(t) \parallel^2$$

Notice that the negative values in stability indicator reveal that the inequality Eq. (38) is violated for the selected Q matrix.

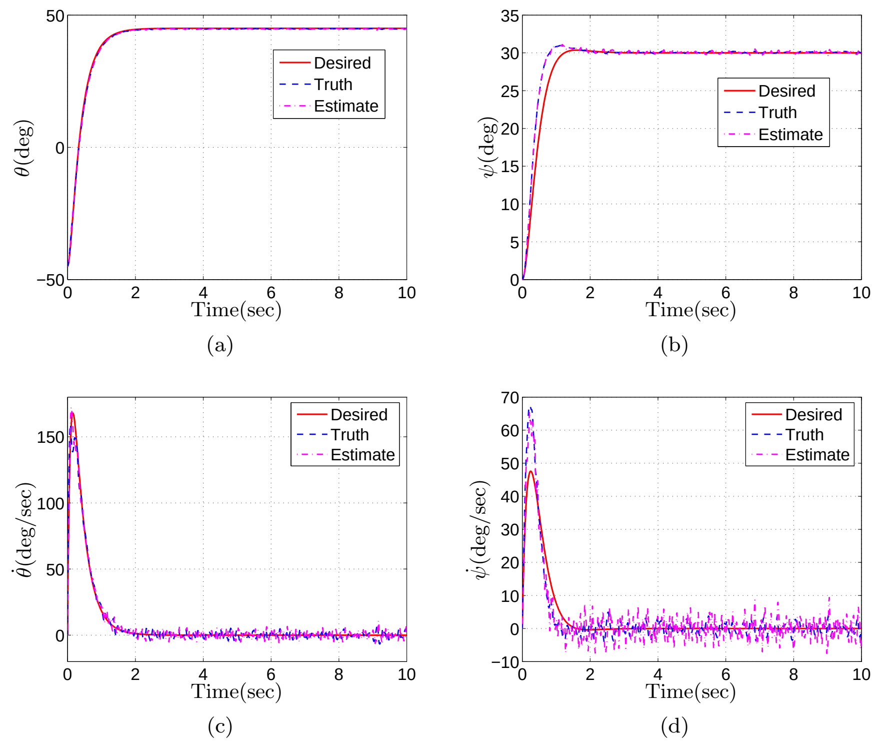

Figure 5. Stable System Response: $q_1 = q_2 = 1 \times 10^4$

A second set of simulations are conducted using $Q = \begin{bmatrix} 1 \times 10^4 & 0 \\ 0 & 1 \times 10^4 \end{bmatrix}$. The system response obtained for the second simulation is given in Fig. 5. The system is stable when Q increased because a large Q satisfies the inequality Eq. (38) as shown in Fig. 6(c). Figure 6 shows the system control input, estimated disturbance term and stability indicator obtained for the second simulation. Notice that the estimated system rates, estimated disturbance term and the control input are highly noisy because of the large Q selected.

A third and final simulation is conducted after tuning Q to $Q = \begin{bmatrix} 10 & 0 \\ 0 & 200 \end{bmatrix}$. The system response obtained for the third simulation is given in Fig. 7. Figure 8 shows the system control input and estimated disturbance term. Notice that the estimated system rates, estimated disturbance term and the control input are relatively less noisier after tuning Q.

The simulation results given here explicitly reveal the direct dependency of the proposed control scheme on the disturbance term process noise matrix, Q. Since the nominal control action on the true plant is unstable, selecting a very low Q value resulted in an unstable system. Conversely, selecting a large Q stabilized the system but resulted in a highly noisy control input. The third simulation indicates that there is an optimal Q value that would minimize the noise in the control input and guarantee stability. Though the closed-loop stability depends on Q and R, here we only consider the variations in Q only since the measurement noise covariance can easily be determined from sensor calibration while the process noise covariance is more or less a tuning parameter.

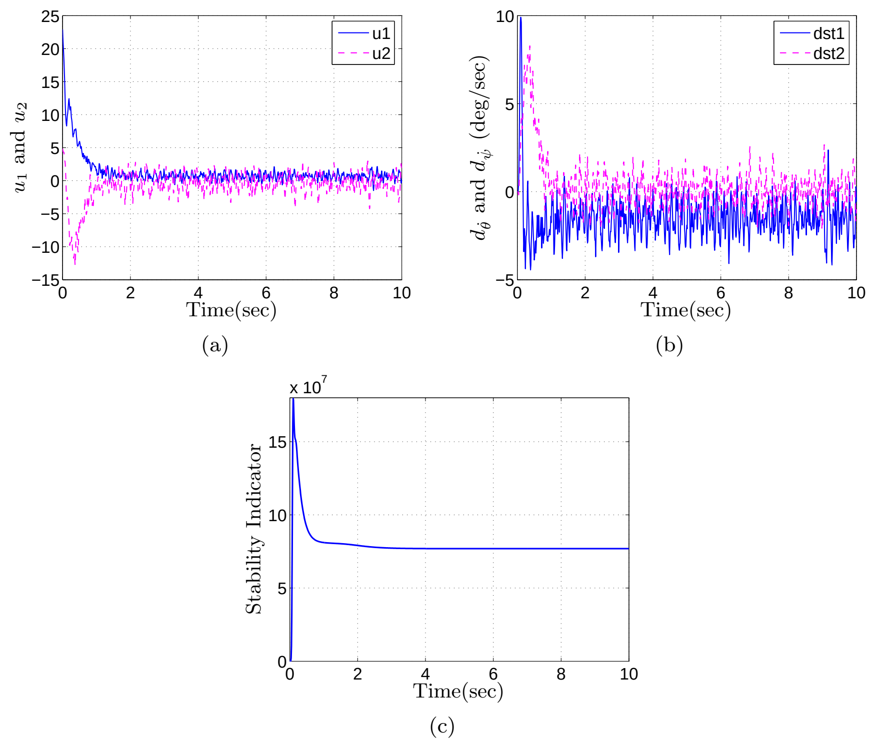

Figure 6. Control Input, Estimated Disturbance Term, and Stability Indicator: $q_1 = q_2 = 1 \times 10^4$

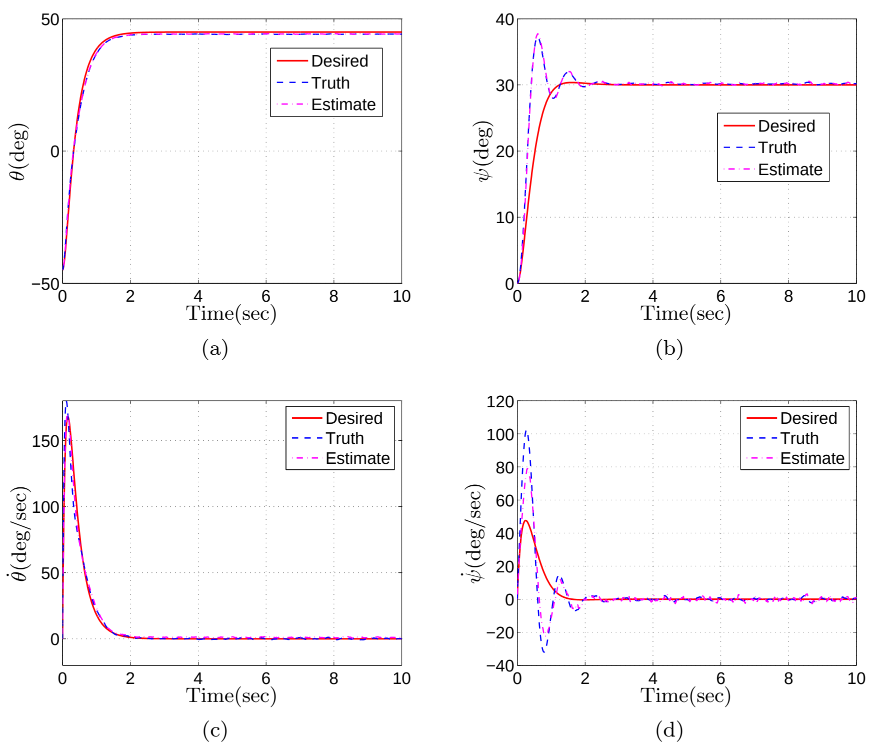

Figure 7. Stable System Response: $q_1 = 10, q_2 = 200$

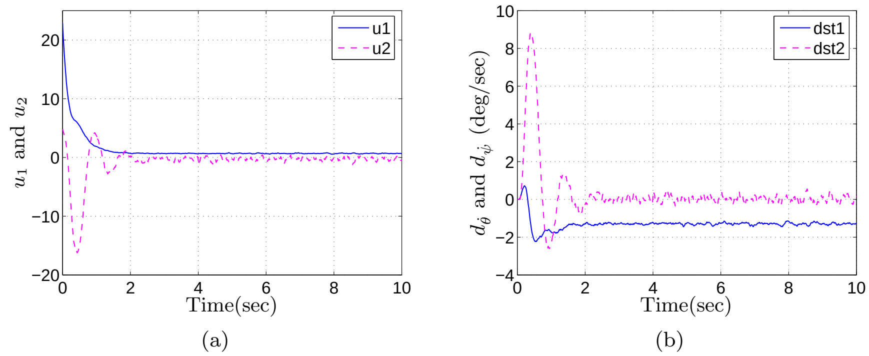

Figure 8. Control Input and Estimated Disturbance Term: $q_1 = 10, q_2 = 200$

V. Conclusions

This paper presents the formulation of a stochastic disturbance accommodating control with observer approach for linear time-invariant multi-input-multi-output systems which automatically detects and minimizes the adverse effects of both model uncertainties and external disturbances on a controlled system. Assuming all system uncertainties and external disturbances can be lumped in a disturbance term, this control approach utilizes a Kalman filter in the feedback loop for simultaneously estimating the system states and the disturbance term from measurements. The estimated states are then used to develop a nominal control law while the estimated disturbance-term is used to make necessary corrections to the nominal control input to minimize the effect of system uncertainties and the external disturbance.

The stochastic stability analysis conducted on the controlled system reveals a lower-bound requirement on the estimator matrices, Q and R^{-1}, to ensure stability in the mean or the mean-square stability of the closed-loop system. If the nominal control on the true plant would result in an unstable system, then selecting a small Q would also result in an unstable system. On the other hand, selecting a large Q value would compel the estimator to completely rely upon the measurement signal and therefore the noise associated with the measurement signal is directly transmitted to the estimates. This could result in noisy control signal which could lead to problems, such as chattering and controller saturation. Also note that as R, the measurement noise covariance, increases, the observer gain decreases and thus the observer fails to update the propagated disturbance term based on the measurements. Thus for a highly uncertain systems, selecting a small Q or a large R will result in an unstable closed-loop system. The stochastic Lyapunov style analysis indicates that the controlled stochastic system is almost surely asymptotically stable if the noise distribution matrix, $\Gamma(t)$, satisfies a specific decay rate. Since the measurement noise covariance can be obtained from sensor calibration, the process noise matrix Q is treated as a tuning parameter. The simulation results reveal that if the selected Q is too low, then the system is unstable and if the selected Q is too large, then the resulted control input is highly noisy. Simulation results also indicate that there is an optimal parameter that would guarantee stability with minimal control input noise. Future research plans include developing an adaptive law for Q that would guarantee asymptotic stability in the mean based on the stochastic Lyapunov analysis, and also extending the current approach to nonlinear systems where the disturbance term also accommodate for system nonlinearities.

Acknowledgments

This material is based upon work supported in part by U.S. Air Force Research Laboratory, Space Vehicles Directorate. The authors would like to thank Khanh D. Pham for his contributions.

References

[1]Johnson, C., "Accommodation of External Disturbances in Linear Regulator and Servomechanism Problems," *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, December 1971, pp. 635–644.

[2]Johnson, C. and Kelly, W., "Theory of Disturbance-Utilizing Control: Some Recent Developments," *Proceedings of IEEE*

Southeast Conference, Huntsville, AL, 1981, pp. 614–620.

[3]Johnson, C., "Disturbance-Utilizing Controllers for Noisy Measurements and Disturbances," *International Journal of Control*, Vol. 39, No. 5, 1984, pp. 859–862.

[4]Biglari, H. and Mobasher, A., "Design of Adaptive Disturbance Accommodating Controller with Matlab and Simulink," *NAECON, Proceedings of the IEEE National Aerospace and Electronics Conference (NAECON*, Dayton, OH, 2000, pp. 208–211.

[5]Joseph A. Profeta III, W. G. and Mickle, M. H., "Disturbance Estimation and Compesation in Linear Systems," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 26, No. 2, March 1990, pp. 225–231.

[6]Kim, J.-H. and Oh, J.-H., "Disturbance Estimation using Sliding Mode for Discrete Kalman Filter," *Proceedings of the* 37^{th} *IEEE Conference on Dicision and Control*, Tampa, FL, 1998, pp. 1918–1919.

[7]Sorrells, J. E., "Design of Disturbance Accommodating Controllers for Stochastic Environments," *Proceedings of the* 1982 *American Control Conference*, Arlington, VA, 1982, pp. 2603–2608.

[8]Sorrells, J., "Comparison of contemporary adaptive control design techniques," *System Theory, 1989. Proceedings., Twenty-First Southeastern Symposium on*, March 1989, pp. 636–640.

[9]Davari, A. and Chandramohan, R., "Design of linear adaptive controller for nonlinear systems," *System Theory, 2003. Proceedings of the 35th Southeastern Symposium on*, March 2003, pp. 222–226.

[10]Sorrells, J., "Adaptive Control Law Design for Model Uncertainty Compensation," Tech. rep., Dynetics, Inc, Huntsville, AL, 1989.

[11]Crassidis, J. L. and Junkins, J. L., *Optimal Estimation of Dynamic System*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[12]George, J. and Crassidis, J. L., "Sensitivity Analysis of Disturbance Accommodating Control with Kalman Filter Estimation," *Proceedings of the AIAA Guidance, Navigation and Control Conference and Exhibit*, Hilton Head, SC, 2007.

[13]Hinrichsen, D. and Pritchard, A. J., *Mathematical Systems Theory I; Modeling, State Space Analysis, Stability, and Robustness*, Vol. 48 of *Text in Applied Mathematics*, Springer, Heidelberg, Germany, 2005.

[14]Kushner, H. J., *Stochastic Stability and Control*, Academic Press, New York, NY, 1967.

[15]Soong, T. T., *Random Differential Equations in Science and Engineering*, Academic Press, New York, NY, 1973.

[16]Abou-Kandil, H., Freiling, G., Ionescu, V., and Jank, G., *Matrix Riccati Equations in Control and Systems Theory*, Systems & Control: Foundations & Applications, Birkhäuser, Basel, Switzerland, 1st ed., 2003.

[17]Soong, T. T. and Grigoriu, M., *Random Vibration of Mechanical and Structural Systems*, Prentice Hall, Englewood Cliffs, NJ, 1993.

[18]Grigoriu, M., *Stochastic Calculus*, Birkhäuser, Boston, MA, 2002.

[19]Arnold, L., *Stochastic Differential Equations: Theory and Applications*, Wiley, New York, NY, 1972.

[20]Mao, X., "Some Contributions to Stochastic Asymptotic Stability and Boundedness via Multiple Lyapunov Functions," *Journal of Mathematical Analysis and Applications*, Vol. 260, No. 2, August 2001, pp. 325–340.

[21]Appleby, J. A. D., "Almost Sure Stability of Linear Itô-Volterra Equations with Damped Stochastic Perturbations," *Electronic Communications in Probability*, Vol. 7, August 2002, pp. 223–234.