Adaptive Disturbance Accommodating Controller for Uncertain Stochastic Systems

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***Abstract*—This paper presents a Kalman filter-based adaptive disturbance-accommodating stochastic control scheme for linear uncertain systems to minimize the adverse effects of both model uncertainties and external disturbances. A rigorous stochastic stability analysis reveals a lower bound requirement on system process noise covariance to ensure the stability of the controlled system when the nominal control action on the true plant is unstable. Finally, an adaptive law is synthesized for the selection of stabilizing system process noise covariance. Simulation results are presented where the proposed control scheme is implemented on a two degree-of-freedom helicopter.**

I. Introduction

Uncertainty in dynamic systems may take numerous forms, but among them, the most significant are noise/disturbance uncertainty and model/parameter uncertainty. External disturbances and system uncertainties can obscure the development of a viable control law. The main objective of Disturbance Accommodating Controller (DAC) is to make necessary corrections to the nominal control input to accommodate for external disturbances and system uncertainties [1]–[4]. The disturbance accommodating observer approach has shown to be extremely effective for disturbance attenuation [5]–[7]; however, the performance of the observer can significantly vary for different types of exogenous disturbances, which is due to observer gain sensitivity.

This paper presents a robust control approach based on a significant extension of the conventional observer-based disturbance-accommodating control concept, which compensates for both unknown model parameter uncertainties and external disturbances by estimating a model-error vector (throughout this paper we will use the phrase “disturbance term” to refer to this quantity) in real time. The estimated model-error vector is further used as a signal synthesis adaptive correction to the nominal control input to achieve maximum performance. This control approach utilizes a Kalman filter in the feedback loop for simultaneously estimating the system states and the disturbance term from measurements [8]–[10]. The estimated states are then used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input to minimize the effects of both unknown system uncertainties and the external disturbance.

It is a well-known fact that the closed-loop performance and stability of the Kalman filter-based DAC approach depends on the selection of process noise covariance associated with the unknown disturbance term [11], [12]. Although the Kalman filter-based DAC approach has been successfully utilized for practical applications, there has not been any rigorous stochastic stability analysis to reveal the interdependency between the estimator process noise covariance and controlled system stability. The two main contributions of this paper are 1) a stochastic stability analysis, and 2) an adaptive law for updating the selected process noise covariance online. The stochastic stability analysis indicates a lower-bound requirement on the assumed disturbance term process noise covariance matrix and the measurement noise covariance matrix to guarantee exponential stability in the mean sense when the nominal control action on the true plant would result in an unstable system. Based on the stochastic Lyapunov analysis, an adaptive law is developed for updating the selected process noise covariance online so that the controlled system is stable.

The structure of paper is as follows: A detailed formulation of the stochastic disturbance accommodating controller for MIMO systems followed by a stochastic stability analysis is given next. Afterwards, an adaptive scheme is developed for the selection of stabilizing disturbance term process noise covariance. Simulation results are then presented where the proposed control scheme is implemented on a two degree-of-freedom helicopter.

II. Controller Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space. Consider an n^{th}-order linear time-invariant stochastic system of the following form:

$$\begin{aligned} \dot{\mathbf{X}}(t) &= A\mathbf{X}(t) + B\mathbf{u}(t) + \mathbf{W}(t), \quad \mathbf{X}(t_0) = \mathbf{x}_0 \\ \mathbf{Y}(t) &= C\mathbf{X}(t) + \mathbf{V}(t) \end{aligned} \tag{1}$$

Here, the stochastic state vector, $\mathbf{X}(t) \triangleq \mathbf{X}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^n$, is an n-dimensional random variable for fixed t. Throughout this paper, random vectors are denoted using boldface capital letters and for convenience, the dependency of a stochastic process on ω is not explicitly shown. The true state and control distribution matrices, $(A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r})$, are assumed to be unknown. Also, the system is assumed to be under-actuated, i.e., $r < n$. The stochastic measurement vector is given as $\mathbf{Y}(t) \triangleq \mathbf{Y}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^m$ and the known output matrix is $C \in \mathbb{R}^{m\times n}$. The measurement noise, $\mathbf{V}(t) \triangleq \mathbf{V}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^m$, is assumed to be zero mean Gaussian white noise, i.e., $f_V(\mathbf{v}) \sim \mathcal{N}\big(\mathbf{0}, R\delta(\tau)\big)$. Finally, the stochastic external disturbance $\mathbf{W}(t) \triangleq \mathbf{W}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^n$ is assumed to satisfy the matching condition and it is modeled as a linear time-invariant system driven by a Gaussian white noise process, i.e.,

$$\dot{\mathbf{W}}(t) = \mathcal{L}_1(\mathbf{X}(t), \mathbf{u}(t), \mathbf{W}(t)) + \boldsymbol{\mathcal{V}}(t), \quad \mathbf{W}(t_0) = \mathbf{0} \tag{2}$$

where $\mathcal{L}_1(\cdot)$ is an unknown linear operator and $\boldsymbol{\mathcal{V}}(t) \triangleq \boldsymbol{\mathcal{V}}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^n$, is assumed to be zero mean Gaussian white noise process, i.e., $f_{\boldsymbol{\mathcal{V}}}(\mathsf{v}) \sim \mathcal{N}\big(\mathbf{0}, \mathcal{Q}\delta(\tau)\big)$. The assumed (known) system model is

$$\begin{aligned} \dot{\mathbf{X}}_m(t) &= A_m \mathbf{X}_m(t) + B_m \mathbf{u}(t), \quad \mathbf{X}_m(t_0) = \mathbf{x}_0 \\ \mathbf{Y}_m(t) &= C\mathbf{X}_m(t) + \mathbf{V}(t) \end{aligned} \tag{3}$$

The external disturbance and the model uncertainties can be lumped into a disturbance term, $\boldsymbol{\mathcal{D}}(t)$, through

$$\boldsymbol{\mathcal{D}}(t) = \Delta A \mathbf{X}(t) + \Delta B \mathbf{u}(t) + \mathbf{W}(t) \tag{4}$$

where $\Delta A = (A - A_m)$ and $\Delta B = (B - B_m)$. Using this disturbance-term the true model can be written in terms of the known system matrices as follows:

$$\dot{\mathbf{X}}(t) = A_m \mathbf{X}(t) + B_m \mathbf{u}(t) + \boldsymbol{\mathcal{D}}(t), \; \mathbf{Y}(t) = C\mathbf{X}(t) + \mathbf{V}(t) \tag{5}$$

The control law, $\mathbf{u}(t)$, is selected so that the true system will track the reference model $\dot{\bar{\mathbf{x}}}(t) = A_m \bar{\mathbf{x}}(t) + B_m \bar{\mathbf{u}}(t)$. The true system tracks the reference model if the following two conditions are satisfied:

$$\mathbf{x}_0 = \bar{\mathbf{x}}(t_0), \; B_m \mathbf{u}(t) = B_m \bar{\mathbf{u}}(t) - \boldsymbol{\mathcal{D}}(t) \tag{6}$$

where convergence is understood in the mean square sense. The disturbance term is not known, but an observer can be implemented in the feedback-loop to estimate the disturbance term online. For this purpose, the system in (1) is rewritten as the following extended dynamically equivalent system:

$$\begin{bmatrix} \dot{\mathbf{X}} \\ \dot{\boldsymbol{\mathcal{D}}} \end{bmatrix} = \begin{bmatrix} A_m & I_{(n\times n)} \\ \mathcal{L}_{2\mathbf{X}} & \mathcal{L}_{2\boldsymbol{\mathcal{D}}} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} B_m \\ \mathcal{L}_{2\mathbf{u}} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{V}} \end{bmatrix} \tag{7}$$

where $\mathcal{L}_{2\mathbf{X}}$, $\mathcal{L}_{2\boldsymbol{\mathcal{D}}}$, and $\mathcal{L}_{2\mathbf{u}}$ are partitions on $\mathcal{L}_2(\cdot)$, a realization of unknown $\mathcal{L}_1(\cdot)$. Let $\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \boldsymbol{\mathcal{D}}(t) \end{bmatrix}$, $F = \begin{bmatrix} A_m & I_{(n\times n)} \\ \mathcal{L}_{2\mathbf{X}} & \mathcal{L}_{2\boldsymbol{\mathcal{D}}} \end{bmatrix}$, $D = \begin{bmatrix} B_m \\ \mathcal{L}_{2\mathbf{u}} \end{bmatrix}$, and $G = \begin{bmatrix} 0_{n\times n} \\ I_{n\times n} \end{bmatrix}$. Now the extended system given in (7) can be written as

$$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) + D\mathbf{u}(t) + G\boldsymbol{\mathcal{V}}(t), \quad \mathbf{Z}(t_0) = \mathbf{Z}_0 \tag{8}$$

where $\mathbf{Z}_0 = [\mathbf{x}_0 \;\; \boldsymbol{\mathcal{D}}_0]^T$. It should be noted that we do not have precise knowledge about the dynamics of the disturbance term. For simplicity, the disturbance term is modeled as

$$\dot{\boldsymbol{\mathcal{D}}}_m(t) = \mathrm{A}_{\mathcal{D}_m} \boldsymbol{\mathcal{D}}_m(t) + \boldsymbol{\mathcal{W}}(t), \quad \boldsymbol{\mathcal{D}}_m(t_0) = \mathbf{0} \tag{9}$$

where $\mathrm{A}_{\mathcal{D}_m}$ is Hurwitz and $\boldsymbol{\mathcal{W}}(t) \triangleq \boldsymbol{\mathcal{W}}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^n$ is zero mean Gaussian white noise process, i.e., $f_{\boldsymbol{\mathcal{W}}}(\mathsf{w}) \sim \mathcal{N}\big(\mathbf{0}, Q\delta(\tau)\big)$. Equation (9) is used solely in the estimator design to estimate the true disturbance term. Construct the assumed augmented state vector, $\mathbf{Z}_m(t) = \begin{bmatrix} \mathbf{X}_m(t) \\ \boldsymbol{\mathcal{D}}_m(t) \end{bmatrix}$, now the assumed model in (7) can be written as

$$\dot{\mathbf{Z}}_m(t) = F_m \mathbf{Z}_m(t) + D_m \mathbf{u}(t) + G\boldsymbol{\mathcal{W}}(t) \tag{10}$$

where $F_m = \begin{bmatrix} A_m & I_{(n\times n)} \\ 0_{(n\times n)} & \mathrm{A}_{\mathcal{D}_m} \end{bmatrix}$ and $D_m = \begin{bmatrix} B_m \\ 0_{(n\times r)} \end{bmatrix}$. Notice that the uncertainty is now only associated with the dynamics of the disturbance term. The estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m \hat{\mathbf{Z}}(t) + D_m \mathbf{u}(t) + K(t)[\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)] \tag{11}$$

where $K(t)$ is the Kalman gain and $\hat{\mathbf{Y}} = H\hat{\mathbf{Z}}$ with $H = [C \;\; 0_{m\times n}]$. The estimator dynamics can be rewritten as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m \hat{\mathbf{Z}}(t) + D_m \mathbf{u}(t) + K(t)H[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)] + K(t)\mathbf{V}(t)$$

Notice that the estimator uses the assumed system model in (10) for the propagation stage and the true measurements for the update stage, i.e., $\hat{\mathbf{Z}}(t) = E[\mathbf{Z}_m(t)|\{\mathbf{Y}_t \ldots \mathbf{Y}_0\}]$. The Kalman gain can be calculated as $K(t) = P(t)H^T R^{-1}$, where $P(t) = E\big[(\mathbf{Z}_m(t) - \hat{\mathbf{Z}}(t))(\mathbf{Z}_m(t) - \hat{\mathbf{Z}}(t))^T\big]$ can be obtained by solving the continuous-time matrix differential Riccati equation [13]:

$$\dot{P} = F_m P + P{F_m}^T - PH^T R^{-1} HP + GQG^T \tag{12}$$

For the reference system, the nominal controller is given as

$$\bar{\mathbf{u}}(t) = -K_m \hat{\mathbf{X}}(t) \tag{13}$$

where $K_m \in \mathbb{R}^{r\times n}$ is the feedback gain. Now the total control law can be written in terms of the estimated system states and the estimated disturbance term as

$$\mathbf{u} = -(B_m^T B_m)^{-1} B_m^T \Big[B_m K_m \;\; I_{(n\times n)}\Big] \begin{bmatrix} \hat{\mathbf{X}} \\ \hat{\boldsymbol{\mathcal{D}}} \end{bmatrix} = S\hat{\mathbf{Z}}(t) \tag{14}$$

where $S = -(B_m^T B_m)^{-1} B_m^T \Big[B_m K_m \;\; I\Big]$. Notice that $(B_m^T B_m)$ is a nonsingular matrix since B_m is assumed to have linearly independent columns. A summary of the proposed control scheme is given Table. I.

TABLE I
SUMMARY OF DAC

Plant	$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) + D\mathbf{u}(t) + G\boldsymbol{\mathcal{V}}(t)$ $\mathbf{Y}(t) = H\mathbf{Z}(t) + \mathbf{V}(t)$
Observer	$\dot{P} = F_m P + P{F_m}^T - PH^T R^{-1} HP + GQG^T$ $K(t) = P(t)H^T R^{-1}$ $\dot{\hat{\mathbf{Z}}}(t) = F_m \hat{\mathbf{Z}}(t) + D_m \mathbf{u}(t) + K(t)[\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)]$
DAC	$\mathbf{u}(t) = (B_m^T B_m)^{-1} B_m^T \Big[-B_m K_m \;\; -I\Big]\hat{\mathbf{Z}}(t)$

It is important to note that if $Q = 0$, then $\boldsymbol{\mathcal{D}}_m(t) = \boldsymbol{\mathcal{D}}_m(t_0) = \mathbf{0}$ and the total control law becomes just the nominal control. If the nominal control, $\bar{\mathbf{u}}(t)$, on the true plant would result in an unstable system, then selecting a small Q would also result in an unstable system. On the other hand, selecting a large Q value would compel the estimator to completely rely upon the measurement signal and therefore the noise associated with the measurement signal is directly transmitted into the estimates. This could result in a noisy control signal which could lead to problems such as chattering and controller saturation. Also note that as R, the measurement noise covariance, increases, the observer gain decreases and thus the observer fails to update the propagated disturbance term based on measurements. For a highly uncertain system, if the nominal control action on the true plant would result in an unstable system, then selecting a small Q or a large R would also result in an unstable closed-loop system as shown in [11]. It is clear that the performance and stability of the conventional DAC depends upon judicious selection of the model disturbance covariance Q. In the next section, a rigorous stability analysis is presented which investigates the explicit dependency of the closed-loop system stability on Q and R.

III. Stability Analysis

Substituting the control law, (14), into the plant dynamics, (8), the true system can be written as

$$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) + DS\hat{\mathbf{Z}}(t) + G\boldsymbol{\mathcal{V}}(t), \;\; \mathbf{Y}(t) = H\mathbf{Z}(t) + \mathbf{V}(t)$$

From hereon the explicit notation for time varying quantities is omitted when there is no risk of confusion. The estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}} = F_m\hat{\mathbf{Z}} + D_mS\hat{\mathbf{Z}} + KH[\mathbf{Z} - \hat{\mathbf{Z}}] + K\mathbf{V}$$

Let $\tilde{\mathbf{Z}} = \mathbf{Z} - \hat{\mathbf{Z}}$ be the estimation error, then the error dynamics can be written as

$$\dot{\tilde{\mathbf{Z}}} = [F_m - KH + \Delta F]\tilde{\mathbf{Z}} + [\Delta F + \Delta DS]\hat{\mathbf{Z}} + G\boldsymbol{\mathcal{V}} - K\mathbf{V}$$

where $\triangle F = F - F_m$ and $\triangle D = D - D_m$. Combining the error dynamics and the estimator dynamics we could write,

$$\begin{bmatrix} \dot{\tilde{\mathbf{Z}}} \\ \dot{\hat{\mathbf{Z}}} \end{bmatrix} = \begin{bmatrix} (F_m - KH + \Delta F) & (\Delta F + \Delta DS) \\ KH & (F_m + D_mS) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Z}} \\ \hat{\mathbf{Z}} \end{bmatrix} + \begin{bmatrix} G & -K \\ 0 & K \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{V}} \\ \mathbf{V} \end{bmatrix}$$

or in a more compact form as

$$\dot{\boldsymbol{\mathcal{Z}}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t) \tag{15}$$

where $\boldsymbol{\mathcal{Z}} = \begin{bmatrix} \tilde{\mathbf{Z}}^T & \hat{\mathbf{Z}}^T \end{bmatrix}^T$ and $\boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{V}}^T & \mathbf{V}^T \end{bmatrix}^T$.

A. Closed-Loop Stability and Transient Response Bound for Systems with No Uncertainties

In this subsection a detailed analysis of the closed-loop system's asymptotic stability in the mean is given when there are no uncertainties. As it is shown here, a transient bound on the system mean response can be obtained in terms of the time varying correlation matrix.

Consider a case where there is no model error, i.e., $F = F_m$, $D = D_m$, and $\boldsymbol{\mathcal{V}}(t) = \boldsymbol{\mathcal{W}}(t)$. If there is no model error, then the estimator is unbiased, i.e., $E[\bar{\tilde{\mathbf{Z}}}] \equiv \boldsymbol{\mu}_{\bar{\tilde{\mathbf{Z}}}} = \mathbf{0}$. Now we can write

$$\begin{bmatrix} \dot{\bar{\tilde{\mathbf{Z}}}} \\ \dot{\bar{\hat{\mathbf{Z}}}} \end{bmatrix} = \begin{bmatrix} F_m - KH & 0 \\ KH & F_m + D_mS \end{bmatrix} \begin{bmatrix} \bar{\tilde{\mathbf{Z}}} \\ \bar{\hat{\mathbf{Z}}} \end{bmatrix} + \begin{bmatrix} G & -K \\ 0 & K \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{W}} \\ \mathbf{V} \end{bmatrix}$$

where $\bar{\tilde{\mathbf{Z}}}$ and $\bar{\hat{\mathbf{Z}}}$ denote the estimation error and estimated states when there is no model error, respectively. The above equation can be written in a more compact form as

$$\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t) \tag{16}$$

A few definitions regarding the stability of a stochastic process are now presented.

Definition 1. *Given $M \geq 1$ and $\beta \in \mathbb{R}$, the system in* (16) *is said to be (M, β)-stable in the mean if*

$$\| \bar{\Phi}(t, t_0)\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t_0) \| \leq Me^{\beta(t-t_0)} \| \boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t_0) \| \tag{17}$$

where $\bar{\Phi}(t, t_0)$ is the evolution operator generated by $\bar{\Upsilon}(t)$, $\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t) = E[\bar{\boldsymbol{\mathcal{Z}}}(t)]$, and $\| \cdot \|$ indicates the appropriate two norm.

Since most applications involve the case where $\beta \leq 0$, (M, β)-stability guarantees both a specific decay rate of the mean (given by β) and a specific bound on the transient behavior of the mean (given by M).

Definition 2. *If the stochastic system in* (16) *is (M, β)-stable in the mean, then the transient bound of the system mean response for the exponential rate β is defined to be*

$$M_\beta = \inf\Big\{ M \in \mathbb{R}; \forall t \geq t_0 : \| \bar{\Phi}(t, t_0) \| \leq Me^{\beta(t-t_0)} \Big\} \tag{18}$$

The optimal transient bound $M_\beta = 1$ can be achieved by choosing a sufficiently large β, i.e.,

$$\beta(t - t_0) \geq \int_{t_0}^{t} \| \bar{\Upsilon}(\tau) \| \, d\tau \quad \Longrightarrow$$
$$\| \bar{\Phi}(t, t_0) \| \leq e^{\int_{t_0}^{t} \| \bar{\Upsilon}(\tau) \| d\tau} \leq e^{\beta(t-t_0)}, \quad t \geq t_0$$

Therefore it is of interest to know the smallest $\beta \in \mathbb{R}$ such that $\| \bar{\Phi}(t, t_0) \| \leq e^{\beta(t-t_0)}$, $t \geq t_0$. Given a system, which is (M, β)-stable in the mean, the transient bound M_β of the system mean can be readily obtained based on the premises of the following theorem.

Theorem 1. *Let $\bar{\mathcal{P}}(t) = E[\bar{\boldsymbol{\mathcal{Z}}}(t)\bar{\boldsymbol{\mathcal{Z}}}^T(t)]$ and $\bar{\Lambda}\delta(\tau) = E[\bar{\boldsymbol{\mathcal{G}}}(t)\bar{\boldsymbol{\mathcal{G}}}^T(t - \tau)]$. Now suppose the system in* (16) *is (M, β)-stable in the mean, then there exists a continuously differentiable positive definite matrix function $\bar{\mathcal{P}}(t)$ satisfying the Lyapunov matrix differential equation*

$$\dot{\bar{\mathcal{P}}}(t) = \bar{\Upsilon}(t)\bar{\mathcal{P}}(t) + \bar{\mathcal{P}}(t)\bar{\Upsilon}^T(t) + \Gamma(t)\bar{\Lambda}\Gamma^T(t) \tag{19}$$

such that

$$M_\beta^2 \leq \sup_{t\geq t_0} \sigma_{\max}(\bar{\mathcal{P}}(t))/\sigma_{\min}(\bar{\mathcal{P}}(t_0)) \tag{20}$$

where $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ denotes the maximum and minimum singular values, respectively.

Proof: See proof of Theorem 1 in [12]. ■

B. Closed-Loop Stability and Transient Response Bound for Uncertain Systems

Consider the system (15) where model error is present, i.e.,

$$\dot{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t) \tag{21}$$

where

$$\Delta\Upsilon(t) = \begin{bmatrix} \Delta F & (\Delta F + \Delta DS) \\ 0 & 0 \end{bmatrix}$$

The correlation matrix $\mathcal{P}(t) = E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)]$ satisfies the following Lyapunov matrix differential equation:

$$\dot{\mathcal{P}} = (\bar{\Upsilon} + \Delta\Upsilon)\mathcal{P} + \mathcal{P}(\bar{\Upsilon} + \Delta\Upsilon)^T + \Gamma\Lambda\Gamma^T \tag{22}$$

where $\Lambda\delta(\tau) = E[\boldsymbol{\mathcal{G}}(t)\boldsymbol{\mathcal{G}}^T(t-\tau)]$. Note that $\Gamma(t)\Lambda\Gamma^T(t)$ can be factored as shown below:

$$\Gamma(t)\Lambda\Gamma^T(t) = \begin{bmatrix} (G\mathcal{Q}G^T + KRK^T) & -KRK^T \\ -KRK^T & KRK^T \end{bmatrix}$$
$$= \begin{bmatrix} G \\ 0 \end{bmatrix} \mathcal{Q} \begin{bmatrix} G^T & 0 \end{bmatrix} + \begin{bmatrix} PH^T \\ -PH^T \end{bmatrix} R^{-1} \begin{bmatrix} HP & -HP \end{bmatrix}$$
$$= L\mathcal{Q}L^T + N(t)R^{-1}N^T(t)$$

Theorem 2. *The uncertain system in* (21) *is (M,β)-stable in the mean if*

$$\| \Delta\Upsilon(t)\bar{\mathcal{P}}(t) \|^2 \leq \sigma_{\min}(Q)\sigma_{\min}(R^{-1}) \| L \|^2 \| N(t) \|^2 \tag{23}$$

where $\bar{\mathcal{P}}(t)$ satisfies

$$\dot{\bar{\mathcal{P}}} = \bar{\Upsilon}\bar{\mathcal{P}} + \bar{\mathcal{P}}\bar{\Upsilon}^T + LQL^T + NR^{-1}N^T \tag{24}$$

Proof: See proof of Theorem 2 in [12]. ■

Therefore (M,β)-stability in the mean is guaranteed if the inequality equation (23) is satisfied. Let Q^* and R^* be chosen so that the above inequality is satisfied. Now substituting Q^* and R^* into (22) we have

$$\dot{\mathcal{P}^*} = (\bar{\Upsilon} + \Delta\Upsilon)\mathcal{P}^* + \mathcal{P}^*(\bar{\Upsilon} + \Delta\Upsilon)^T + LQ^*L^T + NR^{*-1}N^T \tag{25}$$

The solution of the above equation is

$$\begin{aligned}\mathcal{P}^*(t) &= [\bar{\Phi}(t,t_0) + \Phi_\Delta(t,t_0)]\mathcal{P}^*(t_0)[\bar{\Phi}(t,t_0) + \\ &\Phi_\Delta(t,t_0)]^T + \int_{t_0}^t [\bar{\Phi}(t,\tau) + \Phi_\Delta(t,\tau)]\big\{LQ^*L^T + \\ &N(\tau)R^{*-1}N^T(\tau)\big\}[\bar{\Phi}(t,\tau) + \Phi_\Delta(t,\tau)]^T d\tau\end{aligned} \tag{26}$$

where $[\bar{\Phi}(t,t_0)+\Phi_\Delta(t,t_0)]$ represents the evolution operator corresponding to $(\bar{\Upsilon} + \Delta\Upsilon)$.

Corollary 1. *If the system given in* (21) *is (M,β)-stable in the mean, then there exists a continuously differentiable positive definite symmetric matrix function $\mathcal{P}^*(t)$ given by* (26) *such that*

$$M_\beta^2 \leq \sup_{t\geq t_0} \sigma_{\max}(\mathcal{P}^*(t))/\sigma_{\min}(\mathcal{P}^*(t_0)) \tag{27}$$

here M_β represents the transient bound of the perturbed system's mean response.

C. Mean Square Stability

Previously we analyzed stability in the mean. Here, it is shown that the (M,β)-stability in the mean implies mean square stability. More details on mean square stability can be found in [14] and [15].

Definition 3. *A stochastic system of the form $\dot{\boldsymbol{\mathcal{Z}}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t)$ is mean square stable if*

$$\lim_{t\to\infty} E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)] < \mathrm{C} \tag{28}$$

where C is a constant square matrix whose elements are finite.

Note that

$$\frac{d}{dt}E[\boldsymbol{\mathcal{Z}}\boldsymbol{\mathcal{Z}}^T] = \dot{\mathcal{P}} = \Upsilon\mathcal{P} + \mathcal{P}\Upsilon^T + \Gamma\Lambda\Gamma^T$$

and the solution to the above equation can be written as

$$\mathcal{P}(t) = \int_{-\infty}^t \Phi(t,\tau)\Gamma(\tau)\Lambda\Gamma^T(\tau)\Phi^T(t,\tau)d\tau$$

(M,β)-stable in the mean implies the system matrix, $\Upsilon(t) = \bar{\Upsilon}(t) + \Delta\Upsilon(t)$, generates a stable evolution operator, $\Phi(t,t_0)$, therefore $\mathcal{P}(t)$ has a bounded solution [16].

D. Almost Sure Asymptotic Stability

The solution to the stochastic system given in (21) cannot be based on the ordinary mean square calculus because the integral involved in the solution depends on $\boldsymbol{\mathcal{G}}(t)$, which is of unbounded variation. For the treatment of this class of problems, the stochastic differential equation can be rewritten in Itô form as [17]

$$d\boldsymbol{\mathcal{Z}}(t) = [\bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t)]dt + \Gamma(t)\Lambda^{1/2}d\mathcal{B}(t)$$

or simply as

$$d\boldsymbol{\mathcal{Z}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t)dt + \Gamma(t)\Lambda^{1/2}d\mathcal{B}(t) \tag{29}$$

where $d\mathcal{B}(t)$ is an increment of Brownian motion process with zero-mean, Gaussian distribution and covariance

$$E[d\mathcal{B}(t)d\mathcal{B}^T(t)] = Idt \tag{30}$$

The solution $\boldsymbol{\mathcal{Z}}(t)$ of (29) is a semimartingale process that is also Markov [18].

Definition 4. *The linear stochastic system given in* (29) *is asymptotically stable with probability* 1*, or almost surely asymptotically stable, if*

$$\mathbb{P}\big(\boldsymbol{\mathcal{Z}}(t) \to \boldsymbol{0} \quad \text{as} \quad t\to\infty\big) = 1 \tag{31}$$

A stochastic Lyapunov stability approach is usually employed to analyze almost sure stability. Given below is the well-known classical result on the global asymptotic stability for stochastic systems [14], [19]:

Theorem 3. *Assume that there are functions* $V(\boldsymbol{z},t)\in\mathbb{C}^{2,1}$, *i.e., twice continuously differentiable in* $\boldsymbol{z}$ *and once in* t, *and* $\kappa_1,\kappa_2,\kappa_3\in$ class-$\mathcal{K}$ *such that*

$$\kappa_1(\|\,\boldsymbol{z}\,\|)\le V(\boldsymbol{z},t)\le\kappa_2(\|\,\boldsymbol{z}\,\|) \tag{32a}$$

$$\mathfrak{L}V(\boldsymbol{z},t)\le-\kappa_3(\|\,\boldsymbol{z}\,\|) \tag{32b}$$

for all $(\boldsymbol{z},t)$, *where* $\boldsymbol{z}$ *indicates a sample path of* $\boldsymbol{\mathcal{Z}}(t,\omega)$, *i.e.,* $\boldsymbol{z}(t)=\boldsymbol{\mathcal{Z}}(t,\omega_\imath)\mid_{\omega_\imath\in\Omega}$. *Then, for every initial value* $\boldsymbol{z}_0$, *the solution of* (29) *has the property that*

$$\boldsymbol{\mathcal{Z}}(t)\to\boldsymbol{0}\quad\text{almost surely as } t\to\infty \tag{33}$$

The operator $\mathfrak{L}\{\cdot\}$ *acting on* $V(\boldsymbol{z},t)$ *is given by*

$$\mathfrak{L}V(\boldsymbol{z},t)=\lim_{dt\to0}\frac{1}{dt}E\big[dV(\boldsymbol{\mathcal{Z}}(t),t)|\boldsymbol{\mathcal{Z}}(t)=\boldsymbol{z}\big] \tag{34}$$

where $dV(\boldsymbol{\mathcal{Z}}(t),t)$ *can be calculated using the Itô Formula.*

Note that in some literature an explicit notation for $\mathfrak{L}V(\boldsymbol{z},t)$ is given as

$$\mathfrak{L}V(\boldsymbol{z},t)=\frac{\partial V(\boldsymbol{z},t)}{\partial t}+\left[\frac{\partial V(\boldsymbol{z},t)}{\partial\boldsymbol{z}}\right]^T\Upsilon(t)\boldsymbol{z}+\frac{1}{2}\mathrm{Tr}\Big\{\Lambda^{1/2}\Gamma^T(t)\Big(\frac{\partial^2V}{\partial\boldsymbol{z}\partial\boldsymbol{z}^T}\Big)\Gamma(t)\Lambda^{1/2}\Big\}$$

which is the same as (34). (M,β)-stability in the mean response implies that $\Upsilon(t)$ generates an asymptotically stable evolution for the linear system in (29), but it does not imply almost sure asymptotic stability due to the persistently acting disturbance. In fact, given $\Upsilon(t)$ generates an asymptotically stable evolution, the necessary and sufficent condition for almost sure asymptotic stability is

$$\lim_{t\to\infty}\|\,\Gamma(t)\,\|^2\log(t)=0 \tag{35}$$

A detailed proof of this argument can be found in [20]. Equation (35) constitutes the sufficent condition for the almost sure asymptotic stability of a linear stochastic system given (M,β)-stability in the mean.

IV. Adaptive Q

In this section an adaptive law is developed to update the disturbance term process noise covariance online so that the controlled system is (M,β)-stable in the mean. Consider the Itô version of the linear stochastic system given in (8):

$$d\mathbf{Z}(t)=\big\{F\mathbf{Z}(t)+DS\hat{\mathbf{Z}}(t)\big\}dt+Gd\boldsymbol{\mathcal{B}}_1(t) \tag{36}$$

where $d\boldsymbol{\mathcal{B}}_1(t)$ is an increment of Brownian motion process with zero-mean, Gaussian distribution and $E[d\boldsymbol{\mathcal{B}}_1(t)d\boldsymbol{\mathcal{B}}_1^T(t)]=\mathcal{Q}\,dt$. The Itô version of the assumed linear stochastic system given in (10) is

$$d\mathbf{Z}_m=\big\{F_m\mathbf{Z}_m+D_mS\hat{\mathbf{Z}}\big\}dt+Gd\boldsymbol{\mathcal{B}}_2 \tag{37}$$

where $d\boldsymbol{\mathcal{B}}_2(t)$ is also an increment of Brownian motion process with zero-mean, Gaussian distribution and $E[d\boldsymbol{\mathcal{B}}_2(t)d\boldsymbol{\mathcal{B}}_2^T(t)]=Q\,dt$. Now the estimator can be written as

$$d\hat{\mathbf{Z}}=\big\{F_m+D_mS\big\}\hat{\mathbf{Z}}dt+KH[\mathbf{Z}-\hat{\mathbf{Z}}]dt+Kd\boldsymbol{\mathcal{B}}_3$$

where the measurement nose, $\mathbf{V}(t)$, is formally approximated as $\frac{d\boldsymbol{\mathcal{B}}_3(t)}{dt}$ and $E[d\boldsymbol{\mathcal{B}}_3(t)d\boldsymbol{\mathcal{B}}_3^T(t)]=R\,dt$. Let $H[\mathbf{Z}(t)-\hat{\mathbf{Z}}(t)]=\tilde{\mathbf{Y}}(t)$, i.e.,

$$d\hat{\mathbf{Z}}=\big\{F_m+D_mS\big\}\hat{\mathbf{Z}}dt+K\tilde{\mathbf{Y}}dt+Kd\boldsymbol{\mathcal{B}}_3 \tag{38}$$

Notice that the estimator in (38) is bounded-input bounded-output (BIBO) stable in the mean, i.e., if the measurement minus estimate residual, $\tilde{\mathbf{Y}}(t)$, is bounded in the mean, then $\hat{\mathbf{Z}}(t)$ is bounded in the mean. The adaptive law developed here will update the assumed process noise covariance, $Q(t)$, so that the measurement residual is bounded. A bounded measurement residual implies that the plant given in (36) is also BIBO stable in the mean. An update law for the assumed process noise covariance is obtained on the premises of the following theorem [14].

Theorem 4. *Assume there is a nonnegative function* $V(\mathbf{Y},t)$ *such that*

$$\mathfrak{L}V(\mathbf{y},t)\le0 \tag{39}$$

for all $(\boldsymbol{y},t)\in R^m\times R_+$, *where* $\boldsymbol{y}$ *indicates a sample path of* $\mathbf{Y}(t,\omega)$, *i.e.,* $\boldsymbol{y}(t)=\mathbf{Y}(t,\omega_\imath)\mid_{\omega_\imath\in\Omega}$. *Then,* $V(\mathbf{Y},t)$ *is a nonnegative supermartingale process and, for any initial condition* $\mathbf{Y}(t_0)=\boldsymbol{y}_0$,

$$\mathbb{P}\Big(\sup_{\infty>t\ge t_0}V(\mathbf{Y},t)\ge\lambda\Big)\le\frac{V(\boldsymbol{y}_0,t_0)}{\lambda} \tag{40}$$

where λ *is any positive constant.*

Proof: If $\mathfrak{L}V(\mathbf{y},t)\le0$, then Dynkin's formula [14] can be used:

$$E\big[V(\mathbf{Y},t)\big]-V(\mathbf{y}_0,t_0)=E\Big[\int_{t_0}^t\mathfrak{L}V(\mathbf{Y},\tau)d\tau\Big]\le0$$

Thus $E\big[V(\mathbf{Y},t)\big]\le V(\mathbf{y}_0,t_0)$ and $E\big[V(\mathbf{Y},t)\big]\to V(\mathbf{y}_0,t_0)$ as $t\to t_0$. These two facts imply supermartingale property and (40) is the supermartingale probability inequality. ∎

Now consider the following nonnegative function:

$$V(\tilde{\mathbf{Y}},t)=\int_{t_0}^t\tilde{\mathbf{Y}}^T(\tau)K_{ss}^TK_{ss}\tilde{\mathbf{Y}}(\tau)d\tau+\mathrm{Tr}\{G\Delta Q(t)G^T\}$$

where $\Delta Q(t)=Q^*-Q(t)$ and $Q^*\ge Q(t),\forall t\ge t_0$. Note that Q^* will satisfy the inequality equation (23) and K_{ss} is the steady-state Kalman gain corresponding to the initial process noise covariance, $Q(t_0)$. Now $\mathfrak{L}V(\tilde{\mathbf{y}},t)$ can be calculated as

$$\mathfrak{L}V(\tilde{\mathbf{y}},t)=\tilde{\mathbf{y}}^T(t)K_{ss}^TK_{ss}\tilde{\mathbf{y}}(t)-\mathrm{Tr}\{G\dot{Q}(t)G^T\}$$

Select the adaptive law for $Q(t)$ as

$$\dot{Q}(t)=\gamma(G^TG)^{-1}G^T[K_{ss}\tilde{\mathbf{y}}(t)\tilde{\mathbf{y}}^T(t)K_{ss}^T]G(G^TG)^{-1} \tag{41}$$

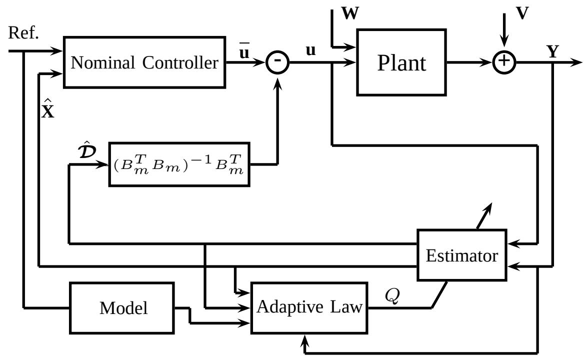


Fig. 1. Adaptive DAC Block Diagram

where $\gamma > 1$ is the adaptive gain. Now we have

$$\mathfrak{L}V(\tilde{\mathbf{y}}, t) \leq 0$$

i.e., $V(\tilde{\mathbf{Y}}, t)$ is a supermartingale and

$$\mathbb{P}\Big(\sup_{\infty > t \geq t_0} V(\tilde{\mathbf{Y}}, t) \geq \lambda\Big) \leq \frac{V(\tilde{\mathbf{Y}}_0, t_0)}{\lambda}$$

Therefore

$$\lim_{t\to\infty} \int_{t_0}^{t} \tilde{\mathbf{Y}}^T(\tau) K_{ss}^T K_{ss} \tilde{\mathbf{Y}}(\tau) d\tau < \infty \quad \text{a.s.} \Rightarrow$$
$$\lim_{t\to\infty} \tilde{\mathbf{Y}}(t) = 0 \quad \text{a.s.}$$

Thus we have bounded measurement residual and the plant is BIBO stable in the mean. Since we assume the plant is controllable and observable, the BIBO stability in the mean implies exponential stability in the mean. Therefore the adaptive law given in (41) will guarantee the plant is (M, β)-stable in the mean. A schematic representation of the proposed adaptive controller is given in Fig. 1

V. RESULTS

For simulation purposes, we consider a two degree of freedom helicopter that pivots about the pitch axis by angle θ and about the yaw axis by angle ψ. As shown in Fig. 2, there is a thrust force F_p acting on the pitch axis that is normal to the plane of the front propeller and a thrust force F_y acting on the yaw axis that is normal to the rear propeller. Therefore a pitch torque is being applied at a distance r_p from the pitch axis and a yaw torque is applied at a distance r_y from the yaw axis. The gravitational force, F_g, generates a torque at the helicopter center of mass that pulls down on the helicopter nose. As shown in Fig. 2, the center of mass is a distance of l_{cm} from the pitch axis along the helicopter body length.

After linearizing about $\theta_0 = \psi_0 = \dot{\theta}_0 = \dot{\psi}_0 = 0$, the helicopter equations of motion can be written as

$$(J_{eq,p} + m_{heli} l_{cm}^2)\ddot{\theta} = K_{pp} V_{m,p} + K_{py} V_{m,y} - B_p \dot{\theta} - m_{heli} g l_{cm}$$
$$(J_{eq,y} + m_{heli} l_{cm}^2)\ddot{\psi} = K_{yy} V_{m,y} + K_{yp} V_{m,p} - B_y \dot{\psi}$$

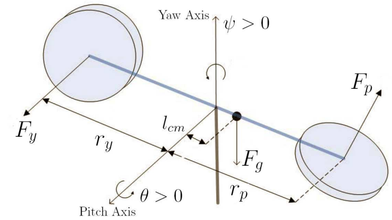


Fig. 2. Two Degree of Freedom Helicopter

The control input to the system are the input voltages of the pitch and yaw motors, $V_{m,p}$ and $V_{m,y}$, respectively. Let $\mathbf{u} = [u_1 \quad u_2]^T = [V_{m,p} \quad V_{m,y}]^T$ and $\mathbf{x} = [\theta \quad \psi \quad \dot{\theta} \quad \dot{\psi}]^T$, now the state-space representation of the above system is

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + \mathbf{w} \tag{43}$$

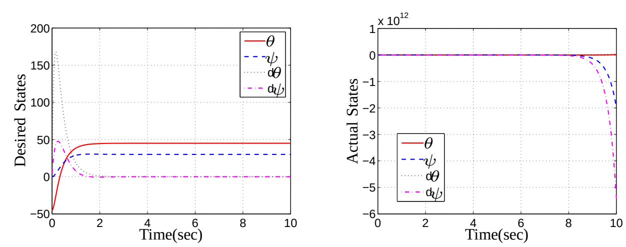
where the constant gravitational torque, $m_{heli} g l_{cm}$, is considered as external disturbance, $\mathbf{w}$. For simulation purposes, the viscous damping coefficient about the yaw axis is selected so that the nominal control action on the true plant is unstable. The measured output equation is given as

$$\mathbf{Y} = C\mathbf{x} + \mathbf{V} \tag{44}$$

where $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $\mathbf{V}$ is the Gaussian measurement noise with covariance $R = 1 \times 10^{-5}\delta(\tau)I$. Notice that the disturbance term can be written as $\mathbf{d} = [0 \quad 0 \quad d_{\ddot{\theta}} \quad d_{\ddot{\psi}}]^T$. The first two zero elements in the disturbance-term indicate the perfect knowledge of the system kinematics. The disturbance-term dynamics is modeled as

$$\dot{d}_{\ddot{\theta}_m}(t) = -d_{\ddot{\theta}_m}(t) + \mathcal{W}_1(t), \; \dot{d}_{\ddot{\psi}_m}(t) = -3d_{\ddot{\psi}_m}(t) + \mathcal{W}_2(t)$$

The nominal controller is selected to be an infinite time

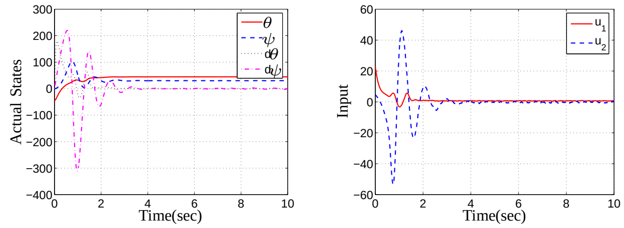


(a) (b)

Fig. 3. Desired and Actual States: $Q = 1 \times 10^{-3} I_{2\times 2}$

horizon LQR. For simulation purposes the initial states are selected to be $[\theta_0 \quad \psi_0 \quad \dot{\theta}_0 \quad \dot{\psi}_0]^T = [-45^o \quad 0 \quad 0 \quad 0]^T$ and the desired states θ_d and ψ_d are selected to be 45^o and 30^o, respectively. More details on the numerical values of the system parameters and the simulation setup can be found in [12]. The desired response given in Fig. 3(a) is the system response to nominal control when there is no model error and external disturbance. Figure 3(b) shows the unstable system response obtained for the first simulation where $Q = 1 \times 10^{-3} I_{2\times 2}$. The unstable response given in

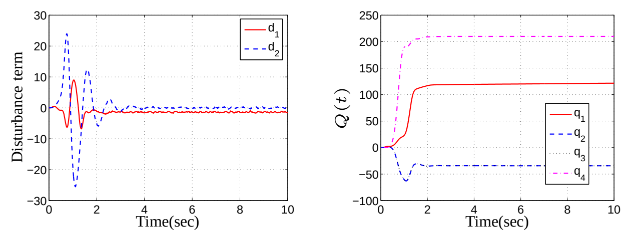
Fig. 3(b) indicates that the selected Q does not satisfy the inequality in (23).



(a) (b)

Fig. 4. Actual States and Input: Adaptive Q

Figure 4 shows the stable system response and the control input obtained using the adaptive DAC approach where $Q_0 = 1 \times 10^{-3} I_{2\times 2}$. Figure 4(a) indicates that the adaptive controller is able to stabilize the system despite the low initial Q_0. In Fig. 5, the estimated disturbance term and the time varying $Q(t)$ are given.



(a) (b)

Fig. 5. Disturbance term and $Q(t)$

VI. Conclusion

This paper presents the formulation of an observer-based stochastic DAC approach for LTI-MIMO systems which automatically detects and minimizes the adverse effects of both model uncertainties and external disturbances. Assuming all system uncertainties and external disturbance can be lumped in a disturbance term, this control approach utilizes a Kalman estimator in the feedback loop for simultaneously estimating the system states and the disturbance term from measurements. The estimated states are then used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input to minimize the effect of system uncertainties and the external disturbance. The stochastic stability analysis conducted on the controlled system reveals a lower bound requirement on the covariance matrices, Q and R^{-1}, to ensure controlled system stability. Since the measurement noise covariance can be obtained from sensor calibration, the process noise matrix Q is treated as a tuning parameter. Based on the stochastic Lyapunov analysis, an adaptive law is developed for updating the selected process noise covariance online so that the controlled system is stable. The simulation results reveal that if the selected Q is below the lower bound, then the controlled system is unstable. The controlled system is stabilized after implementing the adaptive approach for updating $Q(t)$ online.

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