Robust Control of Convective-Diffusion Systems

Matthias Schmid* and John L. Crassidis †

University at Buffalo, State University of New York, Amherst, NY, 14260-4400

In the presented control approach to fluid dynamics, the basal and primal motivation arises from laminar flow control. The key issues associated with this class of problems, distributed systems governed by nonlinear partial differential equations (Navier-Stokes), are identified, and a mathematical benchmark problem reflecting those properties (the Burgers equation with periodic boundary conditions and a non-homogeneous distributed forcing term) is created. A viscosity parameter κ being an analogue to the inverse Reynolds number is incorporated. In order to provide a suitable formulation for control purposes, a semi-discretization is performed using a Galerkin finite element method. The resulting state-space formulation is expanded to an unprecedented 'real world' control loop design, including process disturbance, measurement noise, model-error, and model-reduction. A Lyaponuv based proof for exponential stability of the origin (under certain initial conditions) can be established. For the nominal control, as well as for the required estimator, the linear quadratic regulator and the extended Kalman filter are applied. Additionally, model-error control synthesis is introduced in its one-step ahead prediction formulation for nonlinear distributed systems. This provides a computationally fast correction to cope with model-error and process disturbances. The derived and introduced techniques are subject to extensive numerical evaluation. Thereby, the combination of the linear quadratic regulator with model-error control synthesis reveals itself to be a powerful control tool, resulting in a fast attenuation of an initial distribution as well as a robust correction of process disturbance. Results hold in face of noisy measurements (additive white Gaussian noise) if the extended Kalman filter is added to the system. The problem is approached from a 'worst case' point of view, where the applied disturbance and noise by far exceeds 'real world' dimensions.

Nomenclature

H^p, C^p, L^p	Sobolev Space, Continuous Function Space, Lebesgue Space (each of Order p)
δ, δ_K	Dirac Distribution, Kronecker Delta
I	Identity Matrix
$L^p_{\mathbf{f}}(\mathbf{g})$	p^{th} Lie Derivative of $\mathbf{g}$ along $\mathbf{f}$
$z(\hat{\mathbf{x}}, \Delta t)$	Lie Derivative Expansion
Λ, S	Coefficient and Sensitivity Matrix, Taylor Series Expansion
$w(t,x)$	Solution to Burgers' Equation
$\mathbf{w}^N(t)$	FEM Solution to Burgers' Equation
$w_{st}(t,x)$	Steady-State Solution to Burgers' Equation
κ	Viscosity (Inverse Reynolds Number Analogue)
$f(t,x)$	Nonhomogeneous Forcing Term (Control Input)
p_i	FE Basis Functions
M, K	Mass Matrix, Stiffness Matrix of the FE Model
$\mathbf{N}(\hat{\mathbf{x}})$	Nonlinear Part of the FE Model
$\mathcal{M}$	Forcing Term Distribution Matrix of the FE Model
$\mathcal{K}, \mathcal{N}$	Accumulated Linear Part and Nonlinear Part of the FE Model
$\mathbf{x}(t), \hat{\mathbf{x}}(t), \hat{\mathbf{y}}(t)$	True State, Estimated State and Estimated Output Vector

*Graduate Student, Department of Mechanical & Aerospace Engineering, inbox@mschmid.com, Student Member AIAA.

†Professor, Department of Mechanical & Aerospace Engineering, johnc@eng.buffalo.edu, Associate Fellow AIAA.

$\tilde{\mathbf{y}}_k$	Discrete Measurements
C_p, C_m	Output Matrix, Plant and Model
$\mathbf{f}(\hat{\mathbf{x}}(t))$	Nonlinear Model
$\mathbf{g}(\mathbf{x}(t))$	Nonlinear Control Input Matrix
$\mathbf{u}(t), \hat{\mathbf{u}}(t), \bar{\mathbf{u}}(t)$	Accumulated Control, Model-Error Correction, Nominal Control
B_p, B_m	Control Input Matrix, Plant and Model
$\mathbf{d}(t), D$	Process Disturbance Vector and Covariance Matrix
$\mathbf{v}(t), V$	Measurement Noise Vector (AWGN) and Covariance Matrix
n, l, m, r	System Order, Number of Inputs and Outputs, (Partial) Relative Degree
L_L, H_L	LQR Gain Matrix, Final State Weighting Matrix
Q_L, R_L	LQR State and Control Weighting Matrices
Π	Solution of the Riccati Equation (LQR)
L_K, Q_K, R_K	Kalman Filter Gain, Process and Measurement Noise Weighting Matrix
P	Estimation Error Covariance Matrix
$\mathbf{e}$	Estimation Error (where indicated)
$\hat{G}(\hat{\mathbf{x}})$	Model-Error Distribution Matrix
$G_c(\hat{\mathbf{x}})$	Control Distribution Matrix (in Context of MECS)
$G_e(\hat{\mathbf{x}})$	External Disturbance Distribution Matrix
W_E, R_E	MECS: Correction Weighting Matrix and Measurement Noise Covariance
h	Optimization Interval of MECS (where indicated)
N_p, N_c	Number of Gridpoints, Plant and Model
N_t	Number of Discretization Points in Time
e	Performance Measure (where indicated)
t_{set}	Settling Time

I. Introduction

A. Motivation

Every discipline concerned with viscous fluids has to cope with characteristic frictional effects and its inherent nonlinearities. A very demanding example of these impacts is provided by aerodynamics: The turbulent flow appering at an airfoil surface is accompanied by a dramatical increase of frictional force, such that half of the fuel-consumption of an airplane during cruising condition is due to skin-friction.[1] In Laminar Flow Control (LFC), active devices are used to delay or possibly eliminate this turbulent flow, through either surface cooling or the suction of air through slots and porous surfaces. While Natural Laminar Flow[a] has its limits and has already been applied to actual aircraft design, LFC is predicted to dramatically reduce fuel-consumption by 30 percent for long-range flights in transport-type aircrafts. The 1980's Langley 8-foot transonic pressure tunnel tests even achieved full-chord laminar flow from 0.4 to 0.85 Mach and a drag reduction about 60 percent.[2]

Fluid dynamics provide an extremely demanding environment, requiring advanced and sophisticated control engineering. Results may be expanded to include a considerable variety of actual problems. Therefore, a created **benchmark** problem should reflect the key-features of these challenges, such as:

- Distributed parameter (Governed by partial differential equations)
- Inherent nonlinearity (non intentional)
- Recursively coupled states
- Unknown dynamics (model-error)
- Measurement noise
- Deterministic process disturbances (e. g. oscillatory)

As a matter of fact, model-error is not only created by neglected dynamics; the system being distributed leads to the need of semi-discretization in space yielding models of a tremendous order, and thus resulting in computing difficulties. And yet, it is necessary to employ low (reduced) order models for the control design so that the discretization error will add to the unmodeled dynamics. This issue is also addressed in this work. So, the necessary techniques associated with fluid-like control problems have to incorporate:

[a]Passive solution techniques are applied, like high altitude cruising, composite wing structure, et cetera.

- State-feedback control of distributed systems (semi-discretization of PDE's)
- Model-error detection and compensation
- Robustness
- Nonlinear noise filtering
- Nonlinear control-synthesis
- Lyapunov based stability analysis

The circumstances surrounding the transition from laminar to turbulent flow should thereby be of particular interest, when a quality measurement for the to be designed regulator systems is defined. Since simulation of fluid systems is extremely demanding, a benchmark problem for the control techniques under investigation has to be created. This is found in the one-dimensional Burgers' equation.

B. Previous Research

In the last 20 years, interest on Burgers' equation as a control problem has been picking up. Apparently, existing work is centered around J. A. Burnes, C. I. Byrnes, M. Krstić and B. King. Burns and Kang themselves consider their work[3] as 'a first step in the development of rigorous and practical computational algorithms for control of those nonlinear partial differential equations that describe physically interesting problems of this nature.'[b] Thereby, Burgers' equation on a finite domain with Dirichlet boundary conditions subject to distributed control is the matter under investigation (well-posedness and stability). Kang, Ito and Burns extend the results by imposing a control law (LQR) applying a one-sided Dirichlet boundary control.[4] Byrnes' and Gilliam's research parallels the previous results with slight modifications[5] (Neumann conditions), while Ito and Kang suggest a dissipative feedback control synthesis, employing nonlinear dynamic programming.[6] Gilliam, Lee, Martin and Shubov examine bifurcative behaviour of Burgers' equation with Neumann boundary conditions.[7] Ly, Mease and Titi summarise previous research and different boundary conditions in a comprehensive way while additionally limitations on the size of the initial data are partially removed.[8] Likewise, Kristić addresses the problem of global asymptotic stabilization for large initial conditions via boundary control laws. Extensions and different approaches for boundary control can also be found in the work by Byrnes and Gilliam,[9] by Balogh and Krstić[10] and by Krstić and Liu.[11]

Liu and Kristiṕrovide also the first work on the problem of adaptation for an unknown viscosity[12] (with boundary flux control and parameter estimator as dynamic components). In recent years, Smaoui published research addressing control of Burgers' equation by utilizing both, boundary and distributed control.[13] While the boundary part resembles the previous work of Krstić, the latter applies Karhunen-Loève decomposition, previously introduced by Chambers et alii[14] as a computationally efficient way to solve Burgers' equation with Dirichlet boundary conditions and a random forcing term. Smaoui reduces the problem to an approximation yielding only two fully-actuated nonlinear ordinary differential equation, making it not very usable for real applications. In the extensive work around J. A. Burns, B. B. King et alii, theorems for integral representations of the LQR feedback operator for hyperbolic PDE control problems are presented.[15] Those finding are implemented into the Karhunen-Loève decompostion by using the integral kernels as the required input collection. Atwell and King favor a 'design-then-reduce' approch for large-scale PDE problems claiming to yield robust low-order systems.[16] This design philosophy (inclusion of the controller dynamics into the reduction method) is finally extended by Atwell, Borggaard and King to Burgers' equation.[17] A MinMax regulator-filter method is developed for the problem in abstract form (Galerkin approximation of a perodic Burgers' equation with distributed control).

C. Comparison and Outline

So far neither model-error nor measurement noise have been addressed in a comprehensive approach for Burgers' equation with periodic boundary conditions and distributed control. In the presented work, a nominal state-feedback controller will be used (the linear-quadratic regulator derived from optimal control theory), accompanied by the frequently applied extended Kalman filter for state estimation. The resulting system is simulated to serve as a reference for the robust control approach, using model-error control synthesis. This method adopts the predictive filter, used before by Crassidis for assessing the model-error in nonlinear systems from measurements.[18] It is based on a predictive controller by Lu, implemented as a predictive error-estimator for the nonlinear tracking problem (given a desired response history).[19] Based on the predictive filter design, Crassidis utilized the error-estimate for a signal synthesis of the control input

[b]Here, it is referred to systems governed by the Navier-Stokes equations.

as a general approach to robust control problems.[20] The application of this technique to linear systems has been analyzed by Kim for stability,[21] and extended to a receding horizon approach of the model-error calculation.[22]

A different technique to tackle the underlying model-error and process disturbance problems would be disturbance accommodating control as described in the work of George, Singla and Crassidis.[23] Although this method is definitely worth to be investigated in its application to flow control, there are some potential drawbacks. Disturbance accommodating control uses the same (Kalman) filter for simultaneous state estimation and model-error (or disturbance) prediction. Thereby, the ODE system to be integrated is augmented by a prediction part in the dimension of the model. Since the time-varying Kalman filter in its use as a pure estimator already needs $\frac{n}{2} \cdot (n+1)$ equations, where n is the model's order, this combined approach would require $n \cdot (2n+1)$ ones. Obviously, the computational load is tremendously increased. As it will be seen later, the required semi-discretization of distributed parameter systems (governed by partial differential equations) results in very large order ODE systems. This clearly disqualifies the disturbance accommodating control for real-time implementation. Another weak point is revealed by the fact that the extended Kalman filter as a state-estimator for nonlinear systems is not directly derived from an optimality condition. Hence, an augmentation utilized for disturbance accommodating control pushes that technique further away from optimality, especially if small perturbations around the equilibrium are not guaranteed. Despite the difficulties associated with the realization of model-error control synthesis, its underlying schemata is still directly derived from an optimality condition.

II. Classification and Benchmark Problem

The Navier-Stokes equations form the fundamental physical model for the given motivation. But for the immediate design of prospective controller and estimator combinations, these are far to complicated in simulation. Hence, a simplified test environment containing the key issues arising from Newtonian flow has to be created. Such a problem is found in the one-dimensional Burgers equation: although simple in appearance, it reflects many of the (mathematical) difficulties associated with nonlinear flow (as well as with other nonlinear continuity problems).

Properties and a classification of Burgers' equation are regarded necessary to be reviewed. Since the solution to Burgers' equation is subject to its boundary conditions, an appropriate set has to be defined. For use as a benchmark problem in control engineering an infinite domain is not suitable; therefore, a finite domain formulation reflecting some 'infinite' properties has to be found. This is done by choosing periodic boundary conditions for function value and first derivatives (flux).

Burgers' equation can adopt certain key-features as an analytical model of different physical problems with varying analogies of the appearing terms. Hence, a generic function variable w and a generic coefficient κ is used when stating the general benchmark problem on the domain $\Omega = [0, L]$:

$$w : [0,\infty) \times [0,L] \to \mathbb{R};\ w(.,x)|_{[0,\infty)} \in C^1\ \forall\ x \in [0,L] \text{ and } w(t,.)|_{[0,L]} \in C^2\ \forall\ t \in [0,\infty)$$

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) + w(t,x)\ \frac{\partial w}{\partial x}(t,x) &= \kappa\ \frac{\partial^2 w}{\partial x^2}(t,x) + f(t,x) && && (1)\\ w(0,x) &= w_0(x) && 0 \le x \le L \\ w(t,0) &= w(t,L) && 0 \le t \le \infty \\ \frac{\partial w}{\partial x}(t,0) &= \frac{\partial w}{\partial x}(t,L) && 0 \le t \le \infty \end{aligned}$$

Problem (1) is called the forced viscous Burgers' equation and exhibits a nonlinear, inhomogeneous second order partial differential equation. It is of mixed form, containing a diffusion term $\kappa\ \frac{\partial^2 w}{\partial w^2}(x,t)$ and a nonlinear advection term $w(x,t)\ \frac{\partial w}{\partial x}(x,t)$. Since the time-derivative is also involved, eq. (1) is a hybrid form of a parabolic and a hyperbolic partial differential equation, where it is parabolic for $\kappa > 0$ and degenerates for $\kappa = 0$ to hyperbolic behavior. The nonlinear advection term tends to create discontinuities while the diffusion term rounds off steep descents so that pure discontinuity does not appear as long as $\kappa \neq 0$. The smaller κ is chosen when the solution comes closer to forming discontinuities, and in the case of $\kappa = 0$ the equation reduces to quasi-linear first-order advection generating shock waves.

The equation can be considered a 1-dimensional model of impulse conservation in solenoidal vector fields as well as an approximation of Euler's equation.[c] The nonlinear advection term mimics the nonlinearity due to the convective derivative in the Navier-Stokes equations and provides a one-dimensional approximation for channel flow of an incompressible Newtonian fluid without a pressure gradient (but including a forcing term). For disambiguation, it should be noted that the term convection is not used consistently in literature: in context of heat and mass transfer it refers to the sum of advective and diffusive transfer, and not to the convective derivative. In this first sense, Burgers' equation could be denoted as a convective equation. Furthermore, the benchmark problem provides even more sophisticated behavior of channel flow by functioning as the decisive part in J. M. Burgers' mathematical model for the creation of turbulence in incompressible fluids. It must be noted that this model emphasizes the energy dissipation between primary motion and secondary (turbulent) motion, while the benchmark problem does not dissipate energy due to the periodic boundary conditions. In a different interpretation, Burgers' equation describes the behaviour of traffic flow and - for $\kappa = 0$ - the one-dimensional pressure distribution of a compressible fluid obeying conservation princicples and neglecting internal friction. Hence, it is also the decisive part in the motion of a nonviscous compressible gas described by the Euler equation.

A meaningful control purpose and quality measure can now be defined: as an approximation of channel flow, attenuation of an initial disturbance is reasonable; whereas, from Burgers' point of view, turbulence should be eliminated. Either gives reason to drive the solution to zero. As mentioned before, energy or momentum, respectively, is conserved by the periodic boundary conditions; therefore, it will not be possible to drive the solution to zero without draining the total energy or impulse by means of the control input. But this is only an artificial property imposed by stating a solvable control problem (the state-space to be created has to be finite); the elimination of turbulence is also sufficiently achieved by driving the solution to its steady-state constant, without the loss of generality. Hence, the control problem will be the **attenuation of an initial disturbance to a steady-state constant**. The quality measure of the control is defined later in section V (average deviation from the target equilibrium).

A. Analytical Solution

The analytical solution to the benchmark problem can be found by using the Cole[24]-Hopf[25]-transformation. This two stage transformation merges to the following rule:

$$w(t,x) = c_x(t,x) = -2\kappa \, \frac{\phi_x}{\phi}(t,x) \tag{2}$$

The resulting heat equation can be solved by separation of variables in terms of ϕ which is omitted here. For a detailed derivation as well as for the nonviscous (shock) solution, the reader is referred to previous work by Schmid.[26] For the transformation of the boundary conditions, additional information is needed. This can be achieved by integration of the complete differential equation with subsequent application of the boundary conditions. This results in the following relation for both ends of the domain (as a consequence of the conserving nature of the problem): $\phi(t,0) = e^{\text{const}} \cdot \phi(t,L)$. The constant can be obtained from the initial condition $w_0(x)$ via

$$\text{const} \; = -\frac{1}{2\kappa} \int_0^L w_0(x) \; dx$$

Under the restriction that the total impulse or energy contained in the initial condition equals zero,[d] the general analytical solution can be formulated:

$$w(t,x) \;=\; -2\kappa \frac{\sum_{n=1}^{\infty} \frac{2n\pi}{L} e^{-\left(\frac{2n\pi}{L}\right)^2 \kappa t} \cdot \left(a_n \; \cos\left(\frac{2n\pi}{L}x\right) - b_n \; \sin\left(\frac{2n\pi}{L}x\right)\right)}{\frac{b_0}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{2n\pi}{L}\right)^2 \kappa t} \cdot \left(a_n \; \sin\left(\frac{2n\pi}{L}x\right) + b_n \; \cos\left(\frac{2n\pi}{L}x\right)\right)} \tag{3a}$$

$$a_n \;=\; \frac{2}{L} \int_0^L e^{-\frac{1}{2\kappa}\int_0^x w_0(x^*)dx^*} \cdot \sin\left(\frac{2n\pi x}{L}\right) \; dx \tag{3b}$$

$$b_n \;=\; \frac{2}{L} \int_0^L e^{-\frac{1}{2\kappa}\int_0^x w_0(x^*)dx^*} \cdot \cos\left(\frac{2n\pi x}{L}\right) \; dx \tag{3c}$$

[c] It shall be reminded that the conservation property only appears due to the periodic Neumann boundary conditions. Otherwise, parabolic equations - and hence Burgers' equation - do not show conservation property.

[d] $\frac{1}{2\kappa}\int_0^L w_0(x) \; dx = 0$

This solution is only valid for $\kappa \neq 0$; otherwise, the shock solution has to be calculated by the method of characteristics. It shall be noted that - under the mentioned prerequisites - eq. (3) is always valid; however, if evaluated numerically, it might lead to difficulties for small κ since the inverse of κ appears in the argument of the exponential function.

B. Steady-State Solution

As will become evident in section B, the steady-state solution is of great value when approaching the problem in control terms. By inspection, eq. (1) directly reveals that any constant function fulfills the partial differential equation and the periodic boundary conditions; obviously, a constant function does not change in time and therefore is a steady-state solution. But, do other steady-state solutions exist? In order to find those, the partial derivative in time has to be set to zero ($\frac{\partial}{\partial t} w_{st}(t,x) \equiv 0$), and a nonlinear ordinary differential equation of second order results:

$$\frac{d}{dx}\frac{1}{2}w_{st}^2(x) = \kappa \frac{d^2 w_{st}(x)}{x^2}$$

Since constant functions are already known to be steady-state solutions, they can be eliminated by integration of both sides and necglecting the resulting constant of integration. The remaining first-order differential equation can then be solved straightforward:

$$w_{st}(x) = \frac{\kappa}{\text{const} - \frac{x}{2}}$$

But the steady-state solutions have to obey the periodic boundary conditions. The Dirichlet condition can only be fulfilled for $L = 0$ opposing the problem statement. Hence, it has been proven by contradiction that constant functions are the only steady-state solutions of problem (1). The integral on the whole domain of $w(t,x)$ does not change in time due to the conservative property, $\frac{\partial}{\partial t}\int_0^L w(t,x) = 0$. Thus, the constant of the steady-state solution is related to the initial distribution $w_0(x)$ by

$$w_{st}(x) = \text{const} = \frac{1}{L}\int_0^L w_0(x)dx \tag{4}$$

C. Finite Element Approximation

In order to provide a suitable formulation of Burgers' equation for control purposes, a Galerkin finite-element approximation will be applied. Thereby, the partial differential equation is semi-discretized in the spatial domain, resulting in a system of coupled first-order ordinary differential equations in time (i.e., a state-space representation).[e]

1. Bilinear form

Before stating the weak form of eq. (1), the problem has to be embedded into the right solution space: since in the FE approach discontinuities may appear (finite jumps), the concept of weak derivatives is applied. Therefore, the solution space becomes a Sobolev space $H^{k,p}$, which denotes that subset of L^p, whose functions, and their derivatives up to the order of k, have a finite L^p-norm (for $p \geq 1$). In the FEM the L^2-norm is considered.[f] Since problem (1) is a second order partial differential equation, only first-order weak derivatives will be considered, resulting in $k = 1$ and $p = 2$. The Sobolov embedding theorem states

$$\begin{aligned} H^{k,p} &\hookrightarrow C^{j,\beta} \\ \Omega &\in \mathbb{R}^n \\ k - j - \beta &> \frac{n}{p} \end{aligned}$$

As eq. (1) is a one-dimensional ($n = 1$) problem, $j = 0$ and $\beta = 0$ follow; therefore, $H^1 \hookrightarrow C^0$. Hence, the Neumann boundary condition (periodicity of flux) has to be omitted. Only the periodic Dirichlet condition

[e] The control approach is not limited to Galerkin approximations, in fact, any method resulting in an abstract form can be utilized.

[f] $L^2(\Omega, B) := \{f : \Omega \to \mathbb{R}^m : \exists \int_\Omega \|f(x)\|^2 dx \; < \infty\}$, f measurable with respect to Lebesque measure.

will be valid (and necessary) for the FE approximation. Therefore, the continuous solution space becomes the Sobolev space of first order combined with the periodic boundary conditions: $\tilde{H}^1 := \{w \in H^1 : \; w(t,0) = w(t,L)\}$.

2. Semi-Discretization:

In the process of the FE approximation, the spatial domain is discretized while the time domain remains continuous. Therefore, the solution space becomes $\tilde{H}^1_n$, a finite-dimensional space with dimension n, where $\tilde{H}^1_n \subset \tilde{H}^1$ and $\tilde{H}^1_n \to \tilde{H}^1$ for $n \to \infty$ holds. If $p_0, ..., p_{n-1}$, with $x \mapsto p_i(x) \in H^1$ and fulfilled boundary conditions $p_i(0) = p_i(L)$, generate a basis of $\tilde{H}^1_n$, the approximative solution can be stated as a linear combination:

$$w_h(t,x) = \sum_{j=0}^{n-1} w_j(t)\; p_j(x) \tag{5}$$

where $w_h \in \tilde{H}^1_n$, since $w_h(t,0) = \sum_{j=0}^{n-1} w_j(t)\; p_j(0) = \sum_{j=0}^{n-1} w_j(t)\; p_j(L) = w_h(t,L)$. In order to identify the unknown node parameters, the following discrete FE statement results:

$$\begin{aligned}
&\text{Find} && w_h \in \tilde{H}^1_n = \{w_h(t,x) : w_h(t,x) = \sum_{j=0}^{n} w_j(t)\; p_j(x) \;\wedge\; w_h(t,0) = w_h(t,L)\} \qquad (6)\\
&\text{where} && a(\dot{w_h}, v_h) =< f, v_h > \\
& && \forall\; v_n \in \tilde{H}^1_n = \{v_h(t,x) : \; v_h(t,x) = \sum_{i=0}^{n} v_i(t)\; p_i(x)\;,\; v_h(t,0) = v_h(t,L)\} \\
&\text{with} && a(\dot{w_h}, v_h) = \int_0^L (v_h \frac{\partial w_h}{\partial t} + v_h\; w_h \frac{\partial w_h}{\partial x} + \kappa \frac{\partial v_h}{\partial x}\frac{\partial w_h}{\partial x})\; dx \\
& && < f, v_h >= \int_0^L (f \cdot v_h)\; dx \\
&\text{and} && \tilde{H}^1_n \subset \tilde{H}^1
\end{aligned}$$

The testfunctions v will be substituted by the basis functions in order to identify the unknown node parameters w_j. So combining the node parameters in a vector and arranging for $\dot{\mathbf{w}}^N(t)$ results in a system of first order ODE's:

$$\begin{aligned}
\dot{\mathbf{w}}^N(t) &= -M^{-1}\mathbf{N}(\mathbf{w}^N(t)) - \kappa M^{-1} K \mathbf{w}^N(t) + M^{-1}\mathbf{b} \\
\dot{\mathbf{w}}^N(t) &= \mathcal{N}(\mathbf{w}^N(t)) + \mathcal{A}\mathbf{w}^N(t) + \mathcal{M}\mathbf{b}
\end{aligned} \tag{7}$$

Where $\mathbf{w}^N \in \mathbb{R}^{(n+1)}$ and $M : \mathbb{R}^{(n+1)} \to \mathbb{R}^{(n+1)}$, $K : \mathbb{R}^{(n+1)} \to \mathbb{R}^{(n+1)}$, $\mathbf{N}(w^N) \in \mathbb{R}^{(n+1)}$ are defined as

$$M_{i,j} = \int_0^L p_i(x) p_j(x) dx \tag{8a}$$

$$K_{i,j} = \int_0^L p_i'(x) p_j'(x) dx \tag{8b}$$

$$N(\mathbf{w}^N)_i = \sum_{k=0}^{n}\sum_{l=0}^{n} w_k w_l \int_0^L p_l' p_k p_i dx \tag{8c}$$

$$b_i = \int_0^L f(t,x) p_i(x) dx \tag{8d}$$

3. Linear basis functions:

Equations (8) hold for any choice of basis functions fulfilling the discussed prerequisists (and, for any way of discretizing the spatial domain). For the purposes of this work, linear basis functions and a linear spatial domain will be sufficient. This is only partially due to the desire for simplicity; furthermore, this choice results from the properties of the FEM: the art of placing the nodes is based upon the knowledge of where the discretized problem is well-suited and where it needs further refinement. This information can be achieved through physical investigation, or by testing different settings. A close look at Burgers' equation suggests that areas around very steep descent have to be regarded carefully, due to the tendency of the equation to create shocks: one should refine the grid in those areas.

Since an initial disturbance travels across the complete domain, every region requires the same attention. Therefore, only an adaptive algorithm would be reasonable; however, a state space representation cannot be achieved in that way. Secondly, higher order basis functions tend to 'round off' and might converge faster to the exact solution, but this will occur only in regions where the solution is smooth. Again, the shock nature requires the capability to incorporate steep discontinuities, and constitutes the challenging area. As it will be shown , linear basis functions also converge quite fast, and show few errors in smooth areas, so the use of equally distributed linear basis functions appears as absolutely sufficient. The impact of sophisticated basis functions (e.g., Karhunen-Loève decomposition) is discussed in section V.

4. Evaluation:

In order to be verified, the proposed finite element approximation is tested in comparsion with the analytical solution. Therefore, the error

$$e = \frac{1}{N_t N_x} \sum_{n=1}^{N_x} \sum_{k=1}^{N_t} |w^{N_x}(t_k, x_n) - w(t_k, x_n)|$$

is computed for different quantities of nodes, where N_t and N_x represent the number of gridpoints in time and space, respectively; $w^{N_x}(t_k, x_n)$ denotes the FE solution of order N_x and $w(t_k, x_n)$ the exact solution evaluated at the corresponding points. The expected decrease of the error for refined grids is shown in figure 1(a). Also, finite element approximations of order N_x are compared to the approximations resulting from the double amount of nodes, hence the error

$$e = \frac{1}{N_t N_x} \sum_{n=1}^{N_x} \sum_{k=1}^{N_t} |w^{N_x}(t_k, x_n) - w^{2N_x}(t_k, x_n)|$$

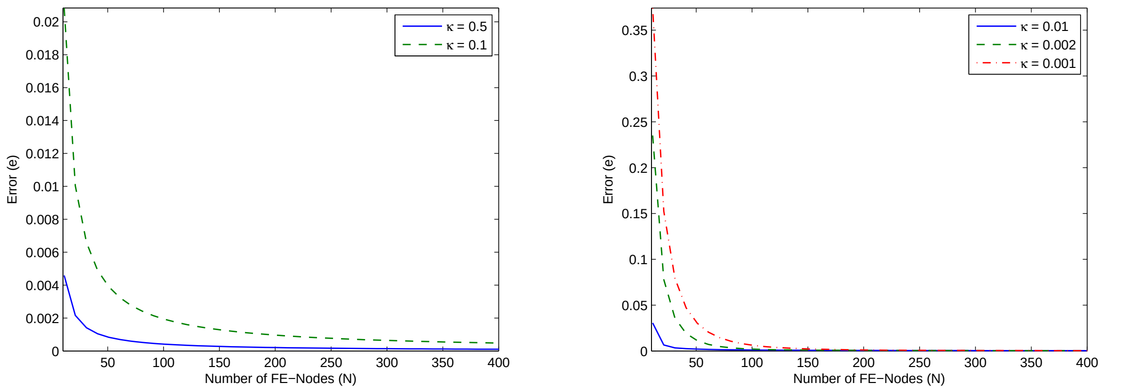
is regarded in figure 1(b). From both diagrams it can be assessed that the proposed finite element approximation converges even for (reasonably) small κ. The error between refined grids exponentially decreases; additionally, the limiting function is the analytical solution at least for large κ. Both evaluations have been executed for a spatial domain of $[0, 1]$. The comparison to the analytical solution used an ending time T_{end} of 1 second with 101 discrete time-steps, while the figure 1(b) is based on an ending time T_{end} of 2 seconds with 201 discrete time-steps. Furthermore, smaller values of κ lead - not surprisingly - to larger errors in general, since the nonlinear part of the equation prevails the diffusion term; hence, the solution is 'less smooth,' leading to numerical difficulties. The reader should be reminded that figure 1 only shows *averaged* errors and that the numerical error in the 'shock' area reveals to be significantly higher.

III. Robust Nonlinear Control

A. Control Problem Statement

For the remainder of this work, the benchmark problem, defined in the previous section II, has to be reformulated in terms of a realistic control statement. Therefore, additional constraints on measurement dynamics, measurement availability, time-delays, disturbance, and noise will have to be made. In order to comply with common notation, the functions $\mathbf{w}^N(t)$ of eq. (7) are replaced by the functions $\mathbf{x}(t)$, the conventional notation used for state-space systems. The forcing term (or control input, respectively) is denoted by $\mathbf{u}(t)$; the output is denoted by $\mathbf{y}(t)$. Figure 2 illustrates the total control loop utilized for further considerations and simulations; the optional model-error compensation is already included.

The exact equations governing the plant are ought to remain unknown as even the full Navier-Stokes equations are only a theoretical model, and do not embrace the complete physical truth. For simulation purposes, the use of a general analytical solution of Burgers' equation would be ideal; but since a closed-form solution depends on the *a priori* known analytical function of the forcing term, it is not feasible for the control problem. The finite element approximation converges to the exact solution for large N. Hence, a



(a) Averaged Error per Gridpoint between FEM and Analytical Solution

(b) Averaged Error per Gridpoint between w^N and w^{2N}

Figure 1. FE Evaluation for Different Numbers of Nodes

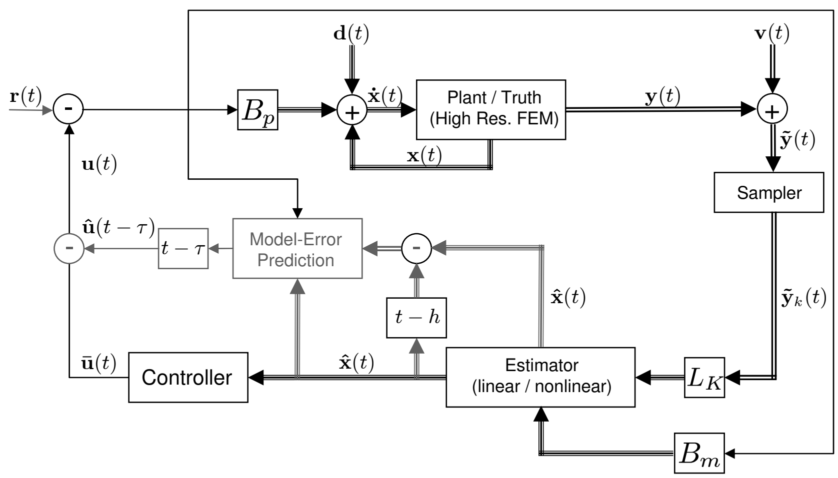


Figure 2. General Closed-Loop Setting

high-definition mesh ($N = 101$) is applied for implementing the plant in MATLAB, while the nominal (model) equations are based on a mesh of $N = 21$ for the reduced model-order approach. Thus, some model error is introduced into the system. The state-space representation of the applied nonlinear system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + B_p \mathbf{u}(t) + \mathbf{d}(t) \tag{9a}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}) = C_p \ \mathbf{x}(t) \tag{9b}$$

Thereby, the order of the ODE system is n ($\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$), the number of inputs is given by l ($B_p : \mathbb{R}^l \rightarrow \mathbb{R}^n$) and the number of outputs is m ($\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$). This is a generic formulation of a multi-input, multi-output system. Again, the plant is treated as consisting of an unknown vector function of the states combined with a constant control coefficient B_p and an unknown disturbance $\mathbf{d}(t)$. The output function reduces to the one-to-one reproduction of certain states (representing sensor locations). The nominal equations for the estimated states and the estimated outputs become (based on the finite element approximation)

$$\dot{\hat{\mathbf{x}}}(t) = -M^{-1}\mathbf{N}(\hat{\mathbf{x}}(t)) - \kappa M^{-1} K \hat{\mathbf{x}}(t) + M^{-1}\mathbf{b}(\mathbf{u}(t)) \tag{10a}$$

$$\hat{\mathbf{y}}(t) = C_m \ \mathbf{x}(t) \tag{10b}$$

The control loop - and hence the estimator - are regarded as a continuous-time system since nonlinear systems (in particular when resulting from PDE's) are continuous in nature. But the measurements are sampled and only available at certain intervals. Furthermore, white Gaussian noise is added. It should be noted that it does not matter if the noise is added before - as shown in figure 2 - or after the sampler. Additional measurement dynamics are neglected. As the remaining system is continuous in time, a correct representation of the sampled (and hold) measurements is given by

$$\begin{aligned}
\tilde{\mathbf{y}}_k(t) &= \int_{-\infty}^{+\infty} (\mathbf{y}(\xi) + \mathbf{v}(\xi)) \cdot \delta\left(\xi - \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t\right) \, d\xi \\
E\{\mathbf{v}(t)\} &= \mathbf{0} \\
E\{\mathbf{v}(t)\mathbf{v}^T(t-\xi)\} &= V(t) \; \delta(t-\xi)
\end{aligned}$$

For the sake of simplicity, the above expressions will be abbreviated and formulated in a discrete version:

$$\tilde{\mathbf{y}}_k = \mathbf{y}_k + \mathbf{v}_k \tag{11a}$$

$$E\{\mathbf{v}_k\} = \mathbf{0} \tag{11b}$$

$$E\{\mathbf{v}_i \mathbf{v}_j^T)\} = V_k \; \delta_{K\ ij} \tag{11c}$$

Some Remarks: In the introduction, airfoil flow was one of the possible applications of laminar flow control. Hence, the underlying control idea could be the design of an area with pinholes which inject or suck flow from either a centralized control valve or through independently controlled injectors. The first version corresponds to a (spatially distributed) single-input system, while the second one obviously constitutes a multi-input system. But a pinhole area forms a discrete control setup, so how can this be realized in mathematical terms? In control engineering, Delta functions (Dirac or Kronecker) are often used in this context. But the reader should be *warned* against careless use of Dirac distributions! This is even more important because the forcing term's integral of eq. (8) allows for it, to be formally evaluated for Dirac distributions:[g]

$$\begin{aligned}
b_i &= \int_0^L f(t,x) p_i(x) \; dx = \int_0^L \left(\sum_{j=1}^{N_C} \delta(x - j \cdot h_c) u_j(t) \right) p_i(x) \; dx \\
&= \sum_{j=1}^{N_C} u_j(t) \int_0^L \delta(x - j \cdot h_c) p_i(x) \; dx = \begin{cases} u_i(t) & \text{for } i = j\frac{h_c}{h} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where N_c denotes the number of control inputs or actuator locations, while h_c is the associated spatial distance. It has been assumed without loss of generality that the actuator locations coincide with the FE gridpoints. But this operation is *not* allowed at all in the context of the presented finite element approximation. Formally, this can be substantiated by a looking at the FE problem formulation in eq. (6): the used functional space for the basis- and testfunctions (and hence, for the solution) is the Sobolev Space, here $\tilde{H}^1$. Thus, it is required that any participating function, including the forcing function, lies in the same functional space. The Dirac delta as a distribution does not. In the FE formulation, every function is mapped on the testfunctions. Although they can be chosen freely, they have to obey the requirements stated in (6), i.e., being elements of $\tilde{H}^1_n$. Clearly, a function (or distribution) cannot be mapped onto basis functions which are not in the same functional space. This results in a divergence of the whole FE system for $N \to \infty$ since the control gain goes to infinity.

Since the basis functions cover the whole domain, the used finite element method assumes the control (forcing term) to be distributed, even if discrete pulses are applied in the originating PDE. Hence, for the purposes of this work, a distributed control using stepfunctions in space is applied from the start:

$$f(x,t) = \sum_{j=1}^{N_C} u_j(t) \cdot \sqcap(x - j \cdot h_c) \qquad \text{where} \quad \sqcap(x) = \begin{cases} 1 & \text{for } -\frac{h_c}{2} < x \le \frac{h_c}{2} \\ 0 & \text{otherwise} \end{cases}$$

Thereby, the number of control inputs is still limited, but they are equally distributed on the corresponding subdomain or interval. Figure 3 displays an example of a possible control input at a particular time and shows how it is mapped on the basis for the under-actuated plant.

[g]Obviously, the use of a Kronecker delta function makes no sense since the integral neglects perturbations at singular points.

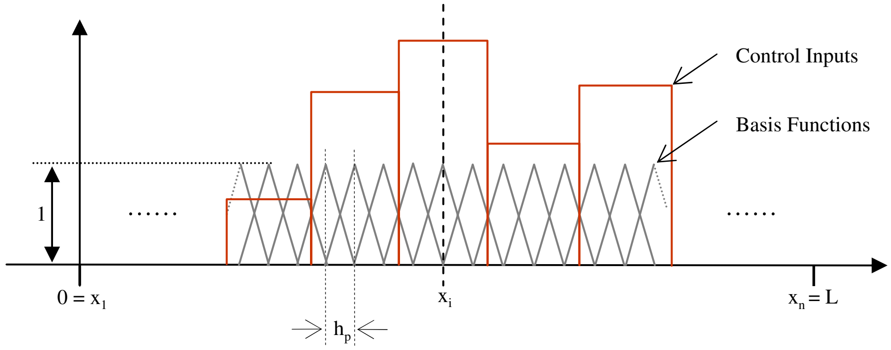


Figure 3. Distributed Control with High Order Basis Functions

The system's outputs are the exact reproductions of certain states, i.e., the solution of Burgers' equation at specific sensor locations. C_p and C_m have been chosen so that the spatial points coincide with the model ones.[h] The viscosity coefficient κ is varied between 0.01, 0.002, and 0.001 in order to provide nonlinear effects for the range and speed of the motions considered. Especially unmodeled dynamics appear due to model reduction (limiting the amount of computation and reducing the FEM grid) and an additive process disturbances.

Only discrete and noisy measurements are possible, therefore necessitating the presence of an observer. The control approach in the following sections separates the controller from the estimator problem. The problem-specific controllers are designed assuming full state-knowledge disregarding any output function or noise (i.e., deterministic controller). At the same time, the filter (or estimator) is developed, providing an, in a certain sense, optimal estimation of the states based on measurements, the nominal model, and noise assumptions. It shall be noted that a general separation principle (certainty equivalence principle[27]) does *not* hold, in general, for nonlinear systems as it does for LTI systems. However, breaking the system into two separate parts has shown to be a reasonable approach to stochastic nonlinear control problems; furthermore, this is the only way to perform a necessary facilitation for conventional control.

B. Analysis

A linearization of the system empowers local stability analysis: if the linearized system is strictly stable, the nonlinear system is (locally) asymptotically stable; if the linearized system is unstable, the same is (locally) true for the nonlinear system. Marginal stability of the linearized system does not allow to draw conclusions on the nonlinear systems since the neglected high-order terms may play a decisive role. Unfortunately, the latter applies to the system considered. This is not surprising, since the system demonstrates a conserving nature due to the periodic boundary conditions.

Fortunately, the fact that the state-space system is the finite element approximation of a partial differential equation allows to focus on the originating PDE. Let the expected system behavior be reviewed: the analytical solution given by eq. (3) suggests that the open-loop solution of an initial disturbance actually decays exponentially in time. It has already been stated that any initial disturbance converges to a corresponding steady-state solution. Since the used benchmark problem has conserving properties, each initial distribution corresponds to a certain final constant, and is linked by the integral $\int_0^L w_0(x)dx = \text{const}$ (the system is not losing impulse or energy). This fact makes it particularly difficult to use energy based Lyapunov techniques. However, without loss of generality, only initial distributions are regarded which fulfill

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[h] E.g.: $B_p = \begin{bmatrix} 1 & 1 & 1 & .5 & 0 & 0 & \dots & & .5 \\ 0 & 0 & 0 & .5 & 1 & 1 & \vdots & 0 & 0 \\ \vdots & & \vdots & & & \ddots & \vdots & \vdots & \\ .5 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \end{bmatrix}^T$ and $C_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \vdots & 0 \\ \vdots & & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

$\int_0^L w_0(x)dx = 0$.[i] The steady-state is then expected to be the origin ($w_e(t,x) = 0$). Let

$$V(t) = \frac{1}{2}\int_0^L w^2(t,x)\ dx \tag{12}$$

be a Lyapunov function. Then, taking the time derivative and subsequent substitution of the original partial differential equation leads to

$$\begin{aligned}\dot{V}(t) &= \frac{d}{dt}\frac{1}{2}\int_0^L w^2(t,x)\ dx = \frac{1}{2}\int_0^L \frac{\partial}{\partial t}w^2(t,x)\ dx = \int_0^L w(t,x)\cdot w_t(t,x)\ dx \\ &= \kappa\int_0^L w\ w_{xx}\ dx - \int_0^L w^2 w_x\ dx + \int_0^L fw\ dx\end{aligned}$$

Integration by parts, and applying the boundary conditions to the first term, yields

$$\dot{V}(t) = -\kappa\int_0^L w_x^2\ dx - \int_0^L w^2 w_x\ dx + \int_0^L fw\ dx$$

The second term can be eliminated be using $\int_0^L w^2 w_x\ dx = \int_0^L \frac{\partial}{\partial x}\frac{1}{3}w^3\ dx = \left[\frac{1}{3}w^3\right]_0^L = 0$, so that

$$\dot{V}(t) = -\kappa\int_0^L w_x^2\ dx + \int_0^L fw\ dx$$

This already reveals that the nonlinear part does not contribute to this particular Lyapunov function (for the given boundary conditions), and that the choice of $\kappa = 0$ (the shock case) renders this Lyapunov function dependent only on the forcing term (or control input). Application of the Poincaré inequality[j] to the first term on the right-hand side leads to

$$\begin{aligned}\|w(t,x)\|_{L^2(\Omega)} &\le C\|w_x(t,x)\|_{L^2(\Omega)} \\ \int_0^L w^2\ dx &\le C\int_0^L w_x^2(t,x)\ dx \\ \dot{V}(t) &\le -\kappa\frac{L}{\pi}\int_0^L w^2(t,x)\ dx + \int_0^L f(t,x)w(t,x)\ dx\end{aligned} \tag{13}$$

The fact has been used that C for $p = 2$ and Ω bounded and convex can be computed as $\frac{d}{\pi}$ (d being the diameter of Ω). Here, the above limitation is useful since the correction by the average in the Poincaré inequality vanishes. So far, there are three main outcomes of eq. (13):

1. For the uncontrolled (open-loop) system, eq. (13) leads to $\dot{V}(t) \le -\alpha V(t)$, where $\alpha = \kappa\frac{2L}{\pi}$, so that $V(t)$ converges exponentially to zero for $t \to \infty$. Since the Lyapunov function has been chosen to be the continuous correspondent to the norm used in the definition of exponential stability,[k] global exponential stability of the equilibrium at the origin under the discussed prerequisites has been shown. Furthermore, the rate of the exponential convergence is directly proportional to the viscosity factor κ.

2. Equation (13) becomes $\dot{V}(t) \le -\alpha V(t) - \int_0^L [\xi(x)w(t,x)]\, w(t,x)\ dx$ if a pure feedback control law (in continuous representation) is applied. At least for a strictly positive kernel or functional gain $\xi(x)$, the system is stable and the rate of convergence (to the origin) is higher than in the open-loop case. If the kernel is not strictly positive, closer investigation is required.

3. In a shock situation (κ very close or equal to zero), the closed-loop system can always be stabilized by requiring the feedback gain to be strictly positive, although the open-loop system is only asymptotically stable for $\kappa = 0$.

[i] In the case of a transformation of variables, the consistency of the presented stability results (forcing different initial values to obey the integral condition) has not yet been investigated.

[j] Be $1 \le p \le \infty$ and Ω a bounded open subset of n-dimensional Euclidean space $\mathbb{R}^n$ having a Lipschitz boundary, then there exists a constant C, depending only on Ω and p, such that for every function f in the Sobolev space $W^{1,p}(\Omega)$: $\|f - f_\Omega\|_{L^p(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)}$ with the average value of f over Ω given by $f_\Omega = \frac{1}{|\Omega|}\int_\Omega f(y)dy$ where $|\Omega|$ is the Lebesque measure of the domain Ω.

[k] Euclidean norm of vector space versus L^2-norm of functions. Exponential stability: An equilibrium point $\mathbf{0}$ is exponentially stable if there exist two strictly positive numbers α and λ such that $\forall\ t > 0$, $\|\mathbf{x}(t)\| \le \alpha\|\mathbf{x}(0)\|e^{-\lambda t}$ in some ball $\mathbf{B}_{\mathcal{R}}$ around the origin.

C. An Argument towards Controllability

It remains to be seen if the higher order plant is controllable under the assumption of the given control gain matrix (expanded control impulses in space, section A). Again, the fundamental behavior of the originating PDE helps: clearly, every steady-state's constant value depends only on the impulse (or energy) contained in the initial distribution and the impulse (or energy) drained or added by the control. As far as the controllability of equilibria is concerned, the system will finally settle on the one determined by the overall control feed. If the general definition of contrabillity is taken as a basis,[1] it may be justified from a practical point of view to argue that even the higher order model is controllable about any constant equilibrium (under the assumption of distributed control). A state-feedback control law with strictly positive functional gains, as shown in the previous section B, for example, causes every initial distribution to finally settle at $\mathbf{x_0} = \mathbf{0}$. For an extended argument towards controllability, the reader is referred to previous work of Schmid.[26]

D. Control Design

At first, this section applies standard techniques to the benchmark problem to create the nominal controller used for the subsequent combination with the model-error corrector. In doing so, the linear quadratic regulator has been chosen as a state-feedback technique. The LQR has been selected because of its widespread use and relatively simple implementation providing computational efficiency. Once the optimal gain has been computed, the control input simply feeds back the (observed) states. Furthermore, due to the sampled output and the additive white Gaussian noise, the regulator has to be provided with state estimates. Hence, the popular Kalman filter is used in its extended version.

1. Nominal Controller

The general LQR problem unfolds to

$$\begin{aligned}
\text{Minimize} \qquad & J(\mathbf{u}) = \frac{1}{2}\mathbf{x}^T(t_f)H_L\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[\mathbf{x}^T(t)Q_L\mathbf{x}(t) + \mathbf{u}^T(t)R_L\mathbf{u}(t)\right] dt \qquad (14)\\
& \text{subject to} \qquad \dot{\hat{\mathbf{x}}}(t) = A(t)\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\\
& \text{over all} \qquad \mathbf{u} \in L^2([0,T];\mathbb{R})
\end{aligned}$$

where $A(t)$ is the *a priori* computed linearization at a certain equilibrium. The control law results in

$$\mathbf{u}^*(t) = -R_L^{-1}(t)B^T(t)\Pi(t)\mathbf{x}(t) \qquad (15)$$

$$\mathbf{0} = -\Pi A - A^T\Pi - Q_L + \Pi B R_L^{-1} B^T \Pi \qquad (16)$$

Different numerical methods exist to solve eq. (16) for Π; within the scope of this work, however, the built-in MATLAB function LQR is used to identify the optimal gain. The linearization is performed at the origin, resulting in only the linear (diffusion) part of the original equation (yielding the heat equation).

2. Kalman Filter

The optimal filter problem reveals to be the following estimate update problem using filtered measurements:

$$\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{f}}(\hat{\mathbf{x}}(t)) + B_m(t)\mathbf{u}(t) + L_K(t)\left[\tilde{\mathbf{y}}(t) - C_m(t)\hat{\mathbf{x}}(t)\right]\\
\hat{\mathbf{y}}(t) &= \mathbf{h}(\hat{\mathbf{x}}(t)) = C_m(t)\hat{\mathbf{x}}(t)\\
\tilde{\mathbf{y}}_k &= C_p(t)\mathbf{x}_k + \mathbf{v}_k
\end{aligned}$$

where $\mathbf{v}$ is a vector of additive white Gaussian noise, as previously defined. The measurement noise covariance is given by $V(t)\delta(t-\tau)$. The Kalman filter assumes additive white process noise (covariance given by $D(t)\delta(t-\tau)$) for the plant:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + B_p(t)\mathbf{u}(t) + \mathbf{d}(t)$$

[1] A nonlinear system (eq. (9)) is said to be controllable if, for any two points $\mathbf{x}_0$, $\mathbf{x}_1$, there exists a time T and an admissible control defined on $[0,T]$ such that for $\mathbf{x}(0) = \mathbf{x}_0$ we have $\mathbf{x}(T) = \mathbf{x}_1$.

The process disturbance and the measurement noise are uncorrelated, i.e., the cross-covariance is zero ($E\{\mathbf{v}(t)\mathbf{d}(t-\tau)\} = 0$). The objective is to determine the correction gain $L_K(t)$ in an optimal way. The Kalman filter statement constitutes the analogon or the dual problem of the linear quadratic regulator. The functional to be minimized with respect to $L_K(t)$ becomes

$$J(L_K(t)) = E\left\{\mathbf{e}^T(t) Q_K \mathbf{e}(t)\right\} \tag{17}$$

Here, the weighted quadratic estimation error, $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, has to be minimized, resulting in the following update relation whenever a measurement becomes available:

$$\begin{aligned}
L_{Kk} &= P_k^- C_{pk}^T \left[C_{pk} P_k^- C_{pk}^T + V_k\right]^{-1} \\
\hat{\mathbf{x}}_k^+ &= \hat{\mathbf{x}}_k^- + L_{Kk}\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)\right] \\
P_k^+ &= \left[I - L_{Kk} C_{pk}\right] P_k^- \\
C_{pk}(\hat{\mathbf{x}}_k^-) &\equiv \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_k^-}
\end{aligned}$$

Within the measurement sampling interval, the propagation phase, the estimates are integrated forward subject to the nominal model. Also, the estimation error covariance matrix P is integrated forward using piecewise linearization:

$$\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= \mathbf{f}(\hat{\mathbf{x}}(t)) + B_m \mathbf{u}(t) \\
\dot{P}(t) &= A(t)P(t) + P(t)A^T(t) + D(t) \\
A(\hat{\mathbf{x}}(t), t) &\equiv \left.\frac{\partial \hat{\mathbf{f}}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t)}
\end{aligned}$$

Note that the *extended* Kalman filter is not precisely derived from an *optimality* solution, i.e., from the minimization of a functional subject to nonlinear differential equation constraints, even though, experience has shown its successful application for many years. Since the estimates have to stay sufficiently close to the true states, tuning via the weighting matrices (as performed in the section on numerical simulation V) may be required. This is especially necessary for highly nonlinear systems, and for a Non-Gaussian process disturbance $\mathbf{d}(t)$. For deeper insight into this roughly sketched derivation, the reader is referred to the book by Crassidis on optimal estimation.[28]

IV. Model-Error Control Synthesis

A. Concept

The so-called model-error control synthesis uses a real-time nonlinear estimator to provide robustness (and better performance) in the presence of model uncertainties or unmodeled dynamics. Therefore, an adaptive correction is synthesized with the control signal. The design is built upon the predictive filter, so as to provide an estimate of the model error present in the system. This prediction is then utilized as an additional control input to correct the nominal control signal. For the motivating problem in this work, real-time implementation is an objective. Therefore, a fast-controller is needed. This is achieved by a one-step ahead prediction (OSAP) version of the model-error. The reader is referred to the extensive work of Kim[29,22] to review different approaches to the predictive estimate. The model-error control synthesis assumes the real plant to be of the following extended model-error form:

$$\dot{\mathbf{x}}(t) = \hat{\mathbf{f}}(\mathbf{x}(t)) + G_c(\mathbf{x}(t))\mathbf{u}(t) + \hat{G}(\mathbf{x}(t))\hat{\mathbf{u}}(t)$$
$$\hat{G}(\mathbf{x}(t))\hat{\mathbf{u}}(t) \equiv \Delta\mathbf{f}(\mathbf{x}(t)) + \Delta G_c(\mathbf{x}(t))\mathbf{u}(t) + G_e(\mathbf{x}(t))\mathbf{d}(t)$$

Note that $G_c(\mathbf{x}(t))$ is the control input distribution matrix, and $G_e(\mathbf{x}(t))$ is the distribution matrix associated with the external disturbance $\mathbf{d}(t)$. The expressions for the output and the state measurements remain the same as in eq. (9) and eq. (11), as does the assumed model denoted by eq. (10). $\Delta\mathbf{f}(\mathbf{x}(t))$ and $\Delta G_c(\mathbf{x}(t))$ are the assumed model errors in the corresponding terms.[m]

[m] Model-Error in the plaint: $\mathbf{f}(\mathbf{x}(t)) \equiv \hat{\mathbf{f}}(\mathbf{x}(t)) + \Delta\mathbf{f}(\mathbf{x}(t))$, and model-error in the control gain: $G_c(\mathbf{x}(t)) \equiv \hat{G}_c(\mathbf{x}(t)) + \Delta G_c(\mathbf{x}(t))$

Here, $\hat{\mathbf{u}}(t)$ is the model-error associated with the corresponding model-error distribution matrix $\hat{G}(\mathbf{x}(t)$. Note that the expression $\hat{G}(\mathbf{x}(t))\hat{\mathbf{u}}(t)$ reflects the accumulation of the error in the nominal open-loop model, the error in the control-distribution matrix, and the external disturbance. The basic idea of model-error control synthesis is quite simple: one tries to predict the future model-error by applying any kind of predictive filter, then feeding that (negative) model-error back into the system. Hence, the control signal synthesis becomes

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) - \hat{\mathbf{u}}_c(t-\tau) \tag{18}$$

where $\bar{\mathbf{u}}(t)$ is the nominal controller's output at time t and $\hat{\mathbf{u}}(t-\tau)$ is the delayed estimated model-error vector. The delay is always necessary because of computational requirements before the model-error (at the current time) can be predicted. Since most real systems are under-actuated, or the number of actuators is less than or equal to the dimension of external disturbances, the term $\hat{\mathbf{u}}_c$ is used for correction instead of $\hat{\mathbf{u}}$. Then, $\hat{\mathbf{u}}_c$ can be determined via a (Moore-Penrose) pseudo-inverse. If the control distribution matrix has full rank, $\hat{\mathbf{u}}_\mathbf{c}(t)$ equals $\hat{\mathbf{u}}(t)$.

B. One-Step Ahead Prediction

For the purpose of this work, the original one-step ahead prediction approach (see the work of Crassidis[18,20]) is chosen, since it provides the capability of being implemented in real-time. The output estimate can be expanded into a multi-dimensional Taylor series:

$$\hat{\mathbf{y}}(t+h) \approx \hat{\mathbf{y}}(t) + \mathbf{z}(\hat{\mathbf{x}}(t), h) + \Lambda(h)S(\hat{\mathbf{x}}(t))\mathbf{u}(t) \tag{19}$$

Let p_i be the relative degree of each output,[n] i.e., the lowest order of differentiation in which any component of the input $\mathbf{u}(t)$ first appears.[o] Then, the elements of the Taylor series expansion become

$$z_i(\hat{\mathbf{x}}(t), h) = \sum_{k=1}^{p_i} \frac{h^k}{k!} L^k_{\hat{\mathbf{f}}}(h_i)$$

$$\text{here:} \quad \mathbf{z}(\hat{\mathbf{x}}(t), h) = h \cdot C_m \cdot \left(\hat{\mathbf{f}}(\hat{\mathbf{x}}(t)) + B_m \mathbf{u}(t)\right)$$

Where $\mathbf{z}(\hat{\mathbf{x}}(t))$ is a vector and $\mathbf{z} \in \mathbb{R}^{m\times 1}$, $L^k_{\hat{\mathbf{f}}}(h_i)$ is the k-th order Lie derivative of $h_i(\hat{\mathbf{x}}(t))$ (elements of the output function $\mathbf{h}(\hat{\mathbf{x}}(t))$) with respect to $\hat{\mathbf{f}}$. The generalized sensitivity matrix $S(\hat{\mathbf{x}}(t)) \in \mathbb{R}^{m\times l}$ consists of the following rows:

$$s_i = \left\{L_{g_1}\left[L^{p_i-1}_{\hat{\mathbf{f}}}(h_i)\right], \ldots, L_{g_q}\left[L^{p_i-1}_{\hat{\mathbf{f}}}(h_i)\right]\right\}$$

$$\text{here:} \quad S = C_m \cdot B_m$$

And the diagonal matrices $\Lambda(h) \in \mathbb{R}^{m\times m}$ elements are given by

$$\lambda_{ii} = \frac{h^{p_i}}{p_i!}$$

$$\text{here:} \quad \Lambda = h \cdot I_{m\times m}$$

Now, a cost functional can be formulated: the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square for the model correction term (part of the control input) shall be minimized:

$$J(\hat{\mathbf{u}}(t)) = \frac{1}{2}\left\{\tilde{\mathbf{y}}(t+h) - \hat{\mathbf{y}}(t+h)\right\}^T R_E^{-1}\left\{\tilde{\mathbf{y}}(t+h) - \hat{\mathbf{y}}(t+h)\right\} + \frac{1}{2}\hat{\mathbf{u}}^T(t) W_E \hat{\mathbf{u}}(t) \tag{20}$$

Thereby, two weighting matrices are introduced: W_E, being positive semidefinite, determines the effort of the correction term (meaning the more W_E decreases the more model correction is added). In the original predictive filter approach, R_E is assumed to be the measurement error covariance matrix caused by the additive

[n] $i = 1, 2, ...m$, where m is the number of outputs.

[o] For the system regarded in this work, the partial relative degree for each output becomes 1, for a detailed discussion see previous work by Schmid.[26]

white Gaussian noise. Therefore, Crassidis includes a rule to determine the sample measurement covariance from a recursive relationship 'on-the-fly,' based on a test for whiteness.[20] Hence, the filter dynamics become variant and need a certain time to converge to a stochastic steady-state. However, if the predictive filter is extended to model-error control, experience has shown high sensitivity against measurement noise. Thus, this work will neglect the model-error approach for compensating the additive white Gaussian noise. Instead, the extended Kalman filter is applied and the model-error prediction will be built on top of the resulting estimates. The general optimal solution to eq. (20) can be derived as

$$\begin{aligned} \hat{\mathbf{u}}(t) &= -M(t)\left[\mathbf{z}(\hat{\mathbf{x}}(t),h) - \tilde{\mathbf{y}}(t+h) + \hat{\mathbf{y}}(t)\right] \\ \text{where:}\quad M(t) &= \left\{\left[\Lambda(h)S(\hat{\mathbf{x}}(t))\right]^T R_E^{-1}\Lambda(h)S(\hat{\mathbf{x}}(t)) + W_E\right\}^{-1}\left[\Lambda(h)S(\hat{\mathbf{x}}(t))\right]^T R_E^{-1} \\ \text{here:}\quad M(t) &= \left\{h^2 + W_E\right\}^{-1} h \end{aligned} \tag{21}$$

Note that the resulting gain matrix for the benchmark problem (obtained by using the system's matrices and, additionally, the assumption of $R_E = I$ for above reasons) has been expressed independent of time.

C. Realization

So far, eq. (21) is the exact and analytically derived solution to the stated model-error correction problem using the one-step ahead prediction approach. But when implementation of this technique is considered, some problematic properties reveal themselves immediately: first, eq. (21) is an implicit equation, since the model-error correction $\hat{\mathbf{u}}(t)$ is appearing in the control term $\mathbf{u}(t)$ on the right-hand side in $\mathbf{z}(\hat{\mathbf{x}}(t), h)$; it is not guaranteed that an explicit relation can be found. Second, for computing the current model-error prediction (or correction), knowledge about a future measurement $\tilde{\mathbf{y}}(t + h)$ is required. To cope with these issues, the model-error solution has to be time-shifted, leading to a certain delay.

Until now, three different time-related parameters, or coefficients, have been introduced: the measurement sampling time Δt, the time-delay effect of the model-error control synthesis τ, and the optimization interval h. The distinction of these parameters can be quite important. The measurement sampling interval is an external factor which has to be taken for granted. The extent of the optimization interval h, however, is more or less up to the user: in a previous approach to tracking-error prediction by Lu,,[19] the state equations and also the *a priori* known reference trajectory are expanded using Taylor series,[19] so that the right-hand side only depends on t, h, and $\mathbf{x}(t)$. Then, the optimization interval h truly becomes a design parameter, simply expressing how far in time the tracking error is projected. However, when model-error control synthesis is regarded, those pure design properties of h are lost.

Since a future measurement is needed, the model error obviously cannot be computed directly, so a time-shift appears due to the availability of all necessary information. This time-shift indeed influences the overall time-delay, but does not coincide with it. The overall time-shift depends on the specific incorporation of the future measurement problem. A *backward realization*[26] is applied in this work: Given a current measurement $\tilde{\mathbf{y}}(t)$, a previous output estimate $\hat{\mathbf{y}}(t - dt)$, and the previously applied control $\mathbf{u}(t - dt)$, then, the previous model-error $\hat{\mathbf{u}}(t - dt)$ is computed at the current time. It is applied at the next possible numerical moment depending on the numerical speed of the controller; again τ reflects the purely computational time-delay. This is illustrated in figure 4 in case of measurement availability at every numerical integration stepsize dt. The update rule can be brought to a short notation:

$$\hat{\mathbf{u}}(t) = -M(t-\tau-h)\left[\mathbf{z}(\hat{\mathbf{x}}(t-\tau-h),h) - \tilde{\mathbf{y}}(t-\tau) + \hat{\mathbf{y}}(t-\tau-h)\right] \tag{22}$$

The advantage of this realization lies in the fact that the optimal solution rule of eq. (21) is not violated in comparison to other realization techniques. But this advantage has to be compensated by a significantly higher time-lack of the correction term. However, if the integration step-size is quite small, and thus the system quite fast, this becomes negligible. In the case of 'continuous-discrete' systems where measurements are sampled at a slower rate than the integration step-size, the model correction term can be hold constant (within the sampling interval), as it has been done in the context of this work.

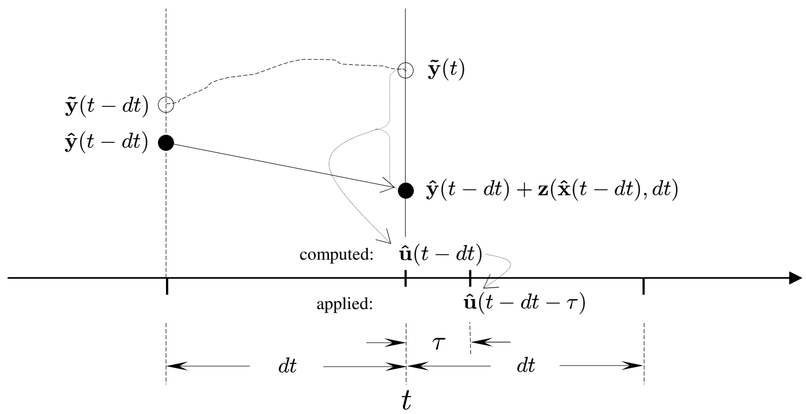


Figure 4. Model-Error Prediction, Backward Realization

V. Numerical Simulation

A. Parameters and Simulation Setting

The previously introduced control loop setting (together with the presented control techniques) is subject to numerical evaluation. The system is implemented in MATLAB. The time-delay in the model-error correction term requires a constant integration step-size. The simulation is divided into a propagation and an update phase. Within the integration (propagation) interval dt, the states, the estimates, and (if applicable) the covariance matrix is propagated forward in time using the MATLAB function ODE23 (with a setting of 10^{-6} for relative and 10^{-8} for absolute tolerance). Within the propagation, the nominal control input and the model-error compensation are held constant, whereas the linearization necessary for the covariance's integration is updated within the interval. There are no measurements available within dt. Once every variable has been propagated to the next time-step, a measurement is generated from the propagated states. Then, the nominal control input, the state estimate, the error covariance, and the model-error compensation are updated for use in the next interval.

The control objective is the attenuation of an initial distribution, where the target equilibrium is the origin ($\mathbf{x}_{final} = \mathbf{0}$). The inital disturbance is given in terms of $w(t,x)$ by

$$w(0,x) = w_0(x) = \begin{cases} 0.5 \cdot \sin(\frac{2\pi x}{L}) & 0 < x \leq 0.5 \\ 0 & 0.5 < x \leq 1 \end{cases}$$

Note that the spatial domain is taken to have the length $L = 1$ ($\Omega = [0,1]$). Additionally, a deterministic disturbance $\mathbf{d}(t) = 0.75\cos(10t)$ is added as process noise (representing, for example, unmodeled dynamics or vibrations). The initial condition of the state estimate is chosen to equal the true initial state. So as to provide comparability between the different control designs, a performance measure has to be introduced: here, the absolute value of the deviation from zero is averaged:[p]

$$e = \frac{1}{N_t N_p} \sum_{i=1}^{N_p} \sum_{j=1}^{N_t} |x_i(j \cdot dt)| \tag{23}$$

Also, the settling time can be regarded, i.e., the minimal time from which the states remain bounded:

$$t_{set} = \min_{t_0} \{t_0 : \forall t > t_0 \rightarrow x_i(t) \in [l_l, l_u]\} \tag{24}$$

The limits are chosen to be $l_u = 0.055$ for the upper bound and $l_l = -0.055$ for the lower bound. As a reference, the open-loop simulation from 0 to 10 seconds is shown in picture 5. The associated performance measures become $e = 0.1589$, $e = 0.1608$ and $e = 0.1612$ for $\kappa = 0.01$, $\kappa = 0.002$ and $\kappa = 0.001$, respectively. Note that the settling time criteria is not met within the simulation interval.

[p] N_t being the number of discretization points in time with the associated integration interval dt, and N_p being the number of spatial gridpoints for the plant.

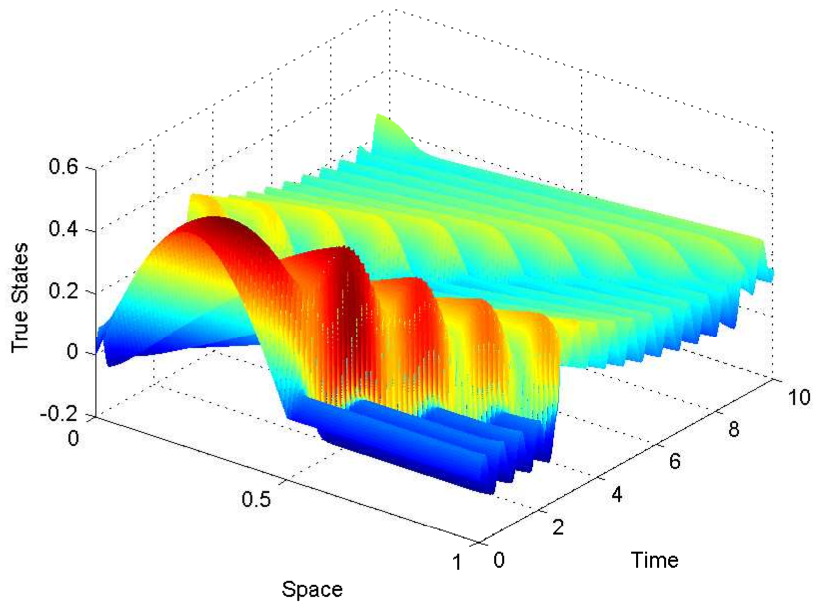


Figure 5. Open-Loop Simulation Including Disturbance; $N = 101$**,** $\kappa = 0.001$

As described in section A, both, the plant and the model, are realized by the means of a finite element approximation with different orders: N_p for the plant and N_m for the model. The model is fully controlled, as well as fully observed, so $B_m = C_m = I$. The input and output matrices for the plant reveal to be identity ($B_p = I$, $C_p = I$), in case of the full model-order control, and in case of the reduced-order control as given as in section A. Two different simulation setups are conducted: a full-order model run with $N_m = N_p = 101$ and a reduced-order model run with $N_p = 101$ and $N_m = 21$. Both realize the following propagation rule within the integration interval dt ($\mathbf{x}_k^-$ and $\hat{\mathbf{x}}_k^-$ at $k \cdot dt$ provided by the propagation):

$$\begin{bmatrix} \dot{\mathbf{x}}^{N_p}(t) \\ \dot{\hat{\mathbf{x}}}^{N_m}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{N_p} & 0 \\ 0 & \mathcal{A}^{N_m} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{N_m}(t) \\ \hat{\mathbf{x}}^{N_p}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{N}^{N_p}(\mathbf{x}^{N_p}(t)) \\ \mathcal{N}^{N_m}(\hat{\mathbf{x}}^{N_m}(t)) \end{bmatrix} + \begin{bmatrix} \mathbf{d}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} C_p \\ C_m \end{bmatrix} (\bar{\mathbf{u}}_{\mathbf{k}} - \hat{\mathbf{u}}_k)$$

The measurements are provided as aforementioned:

$$\tilde{\mathbf{y}}_k = C_p \mathbf{x}_k + \mathbf{v}_k, \ \mathbf{v}_k \sim N(\mathbf{0}, V_k)$$

B. Full-Order Model Control

The linear quadratic regulator is employed as the nominal controller so that the associated control law

$$\bar{\mathbf{u}}_k = -L_L \hat{\mathbf{x}}_k^+$$

contains the solution to the steady-state Riccati equation L_L. The functional gain is computed beforehand based on the linearization of the system.[q] The solution only depends on the ratio of the weighting matrices Q_L and R_L. Due to the symmetric and diagonal dominant structure of the system, these have been chosen to be weighted diagonal matrices: $R_L = I$ and $Q_L = q(x)I$, with $q(x) = 10$ on $0.7 \leq x \leq 0.9$ and $q(x) = 1$ on the rest of the domain. For demonstration purposes, a higher weight is put on the interval $[0.7, 0.9]$. The nominal control remains constant during function evaluations performed by the integrator (ODE23). Table 1 indicates the resulting performance measures for different values of κ and different noise and filter combinations applied to the full-order system without and with model-error correction.

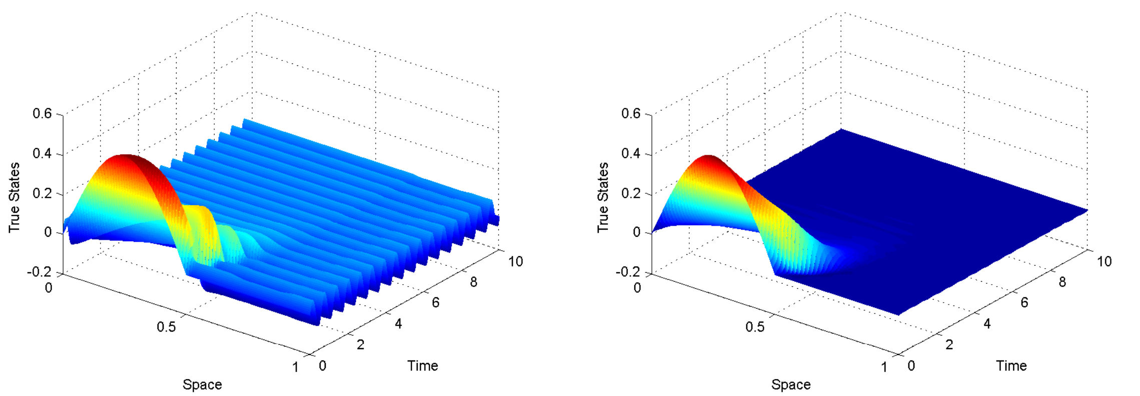
[q]The system's linearization is not updated for the linear quadratic regulator. Since the system is linearized at the origin, the Jacobian $\left.\frac{\partial \mathbf{N}(\mathbf{x}(t))}{\partial \mathbf{x}(t)}\right|_{x^*}$ actually equals zero and only the linear part $\kappa M^{-1} K$ comes into action.

[r]Since noise is carried into the states through the model-error corrector the upper and lower bound of the settling criteria are sometimes exceeded by stochastic noise peaks. Hence the settling time showed a maximal deviation of 1.58.

Table 1. **Control Performance, Full-Order LQR**

	Viscosity Coefficient					
	$\kappa = 0.01$		$\kappa = 0.002$		$\kappa = 0.001$	
	e	t_{set}	e	t_{set}	e	t_{set}
Without Model-Error Correction						
$\mathbf{v}_k = 0$						
Direct Meas.	0.0562	-	0.0567	-	0.0567	-
$\mathbf{v}_k \sim N(\mathbf{0}, 0.05)$						
Direct Meas.	0.0563	-	0.0567	-	0.0567	-
Kalman Filter	0.0571	-	0.0575	-	0.0576	-
With Model-Error Correction						
$\mathbf{v}_k = 0$						
Direct Meas.	0.0166	1.93	0.0165	2.11	0.0165	2.15
Kalman Filter	0.0211	2.00	0.0212	2.08	0.0212	2.10
$\mathbf{v}_k \sim N(\mathbf{0}, 0.05)$						
Direct Meas.	0.0474	-	0.0477	-	0.0488	-
Kalman Filter[r]	0.0218	2.25	0.0229	2.93	0.0234	3.09

Results without MECS: The linear quadratic regulator with direct measurements performs well in terms of attenuation of the initial disturbance, but it cannot cope with process disturbance (here, the harmonic oscillations). Since the system's damping depends on the viscosity parameter κ, one would expect a worse response for smaller values. But interestingly the performance value only changes slightly: this is due to the fact that the performance measure is an averaged factor. The inital disturbance is still attenuated fast in comparison to the simulation interval, so that the shock tendency is not covered by the performance measure. Though a slight shock tendency is still depicted in figure 6(a) for $\kappa = 0.001$. The linear quadratic regulator has an integrative property. This makes it especially robust against (white Gaussian) measurement noise and numerical instabilities. Hence, the efficiency of the noisy case exhibited in table 1 is indistinctive from the noise-free situation. The linear quadratic regulator 'averages' the weighted sum of several states, so that part of the noise cancels out. But the downside of this integrative property is given by the fact that



(a) True States; No Noise; No Model-Error Correction (b) True States; No Noise; Active Model-Error Correction

Figure 6. Full-Order Control; $N = 101$, $\kappa = 0.001$

areas of step descent are not as potentially recognized or smoothed as by a differentiating controller (for example). Surprisingly, the addition of the Kalman filter is not improving the control quality (in the noisy case), it is indeed slightly downgraded. The reader is reminded that the extended Kalman filter approach is not directly derived from an optimality condition.

The process noise in the system under investigation is not additive white Gaussian noise, yet, it is deterministic. Therefore, the weighting matrices loose somehow their meaning and become merely tuning parameter. Nevertheless, the desired behavior can be defined to consist of solely filtering out the measurement noise and closely following the truth. There is no reason for choosing R_K not to equal the real (or applied) measurement noise covariance matrix used for simulation: here $R_K = (0.05)^2\ I_{N_m \times N_m}$.[s] But Q_L still has to be optimized for the desired behavior: too high values cause the estimator to closely follow the model and to neglect the disturbance[t] while too low values carry over measurement noise into the estimates.[u] Choosing the discrete summation in space and time of the truth-minus-estimate as a loss function, a parameter optimization can be performed:

$$J(Q_L) = \sum_{i=1}^{N_p} \sum_{j=1}^{N_t} |x_i(j \cdot dt) - \hat{x}_i(j \cdot dt)|$$

The system's matrices being diagonal dominant and especially consisting of the same entries along the main and each secondary diagonal gives reason to chose Q_L of the form 'scalar times identity:' $Q_L = s \cdot I_{N_m \times N_m}$. The resulting one parameter loss function has been evaluated for different values of viscosity as well as the open- and closed-loop settings. Thereby distinctive minima being very close to each other have been revealed. Since the loss function is not varying significantly in the neighbourhood of these minima, the weighting matrix for the simulation purposes in this work is selected to be $Q_L = 0.025 \cdot I_{N_m \times N_m}$.

Results with MECS: One of this work's purposes is to demonstrate the robustness and superiority of model-error control synthesis as a computationally efficient compensation technique for process disturbance in nonlinear distributed systems (illustrated in figure 2). At this point it has to be noted that every efficiency measurement presented in this work has not only be obtained from one simulation run. Especially when stochastic noise generation is involved, the result has been obtained via the mean of several iterations.[v]

As table 1 exhibits, there is a tremendous improvement when the model-error corrector is added in the case of direct measurements in the absence of noise. Actually, it reveals the best performance within the evaluations of this work by complete attenuation of the process disturbance as depicted by figure 6(b). But model-error control synthesis is essentially a differentiating control. Thus, it has to cope with the difficulties every numerical differentiator has to face. It is not astonishing, that there is a huge kickback in the presence of noise or *numerical* errors. The fatal noise covariance[w] applied in this work even causes the system to become unstable (within the simulation interval).

Although thought to cope with both, noise and model-error, experience has shown instabilities and several problems of the model-error control synthesis in the presence of noise. Once more, the benchmark problem depicts that behavior. However, for the sake of completeness it has to mentioned that the original approach did not just take direct measurements as an input collection: it still included a decoupled estimator for forward integration of the model. Thereby, the model-error predictor has been used for both, estimator update and control correction.

But in case of noise, the addition of the tuned Kalman filter comes in handy. Figure 7 supplementary exhibits the great improvement of the results. There is only a minor disturbance residue left in the states and the model-error corrector carries some noise to the system. But in the face of the rough conditions applied, the system's performance is remarkable. Figures 7(b) and 7(d) additionally show the estimates and the resulting model-error prediction. Figure 7(d) displays a lot of chattering, and, on first sight, one might interpret it as noise. But there is still a considerable harmonic contained in the signal (which could be revealed by the mean of the model-error correction in space).

C. Reduced-Order Model Control

It is also one objective to evaluate the robustness and performance of control based on reduced-order models. The benchmark problem in eq. (1) represents only a small one-dimensional domain of a simplified problem. But already a high order is needed when the distributed PDE is replaced by a system of ODE's. One can easily realize the tremendous computational demand that the application to a spread-out multidimensional

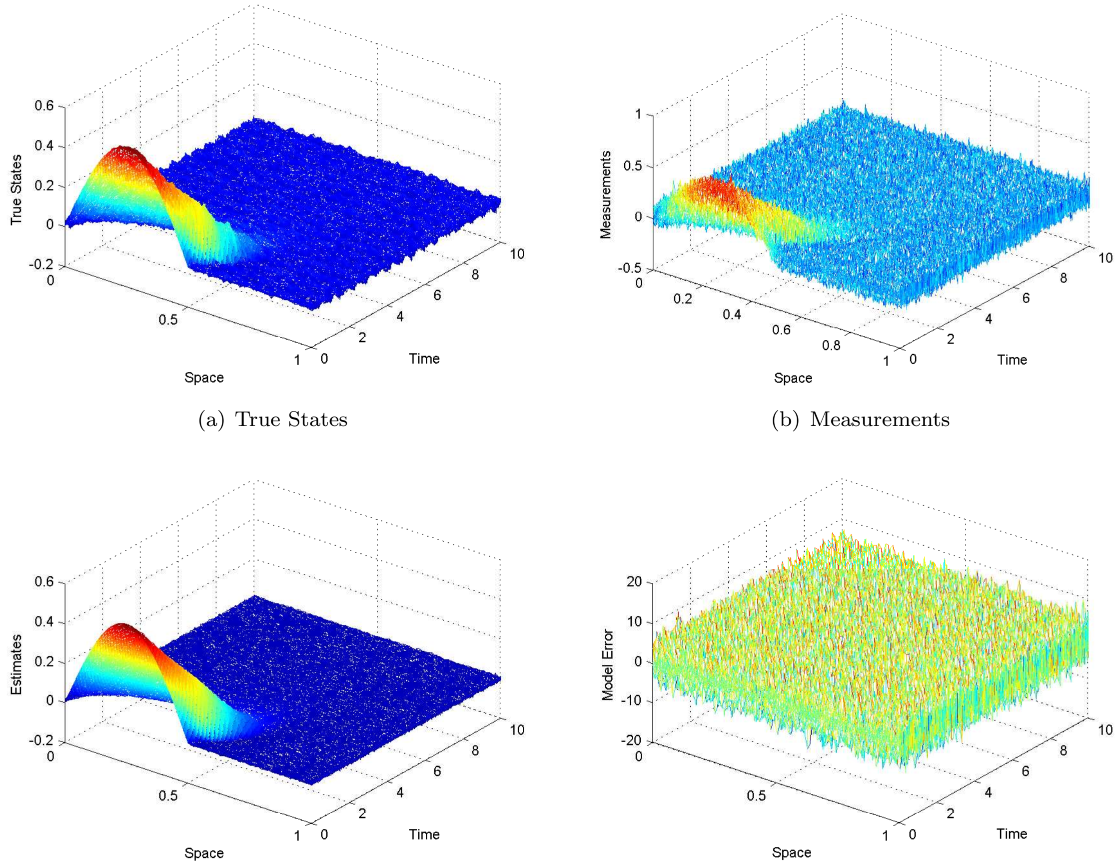
[s]Although the Kalman filter is the dual problem to the linear quadratic regulator, the Kalman gain matrix does *not* only depend on the ratio of the two weighting matrices, but also on the chosen absolute values. In comparison to 'real world' sensor quality, the measurement variance has been chosen to be a worst case condition: the standard deviation (here, the mean of the absolute value) is 10 percent of the initial distribution.

[t]Then, the estimates are close to zero and do not contain model-error anymore. So when the model-error corrector is fed with the estimates, the resulting predictions are way too low.

[u]This potentially causes high chattering in the differentiating model-error predictor.

[v]Thereby, the variance as an indicator of consistency has been found to be negligible.

[w]Again, in comparison to the quality of 'real world' sensors, the measurement noise applied in this work is quite high.



(a) True States (b) Measurements

(c) Estimates (d) Model-Error Correction Term

Figure 7. Full-Order Control; Noise, with Model-Error Correction, Kalman Filter, $N_m = 101$**,** $\kappa = 0.001$

problem, governed by the Navier-Stokes equations, would create. In order to meet these challenges, the control design has to be reduced. This objective has partially been addressed in previous research. J. Atwell et alii place their focus on 'intelligent' techniques of model reduction. Furthermore, they suggest to first design the controller (and also the estimator) for the high-order model, and then reduce the order of the augmented system (instead of reducing the model and then designing the controller). The proposed reduction technique is Karhunen-Loève decomposition. The Karhunen-Loève decomposition requires an input collection to construct the basis functions. By choosing the functional gains of the linear quadratic regulator as such a collection, Atwell et alii were able to incorporate the closed-loop (controller) dynamics into the reduction process. But, nonlinear control layout requires the us of immensely sophisticated mathematical tools for advanced control design; so even a comparatively low order (e.g., order of 10) might not be addressable. A detailed discussion can be found in previous work by Schmid.[26] Here, the emphasis lies not on well-developed reduction methods but on the design of robust controllers with sufficient performance in the face of coarse models. Therefore, the 'reduce-then-design' approach chosen in this work is based on a very rude model reduction, which is arrived at just truncating the order of the linear B-spline finite-element basis, neglecting any additional knowledge. This gives reason to expect that a significant higher performance could be achieved if 'intelligent' methods were to be applied. The model-error control synthesis is the preferred choice when coping with the process disturbances.

Table 2 for the reduced model-order case reveals the efficiency of the same controller filter combination as in table 1. Similar performance and behavior is shown: the linear quadratic regulator exhibits surprisingly good efficiency in damping the initial distribution. The process disturbance is still obviously not addressed as expected (figure 8(a)).

[x]Here, a larger deviation in the performance measure between different simulation runs was observed.

Table 2. Control Performance, Reduced-Order LQR without MECS

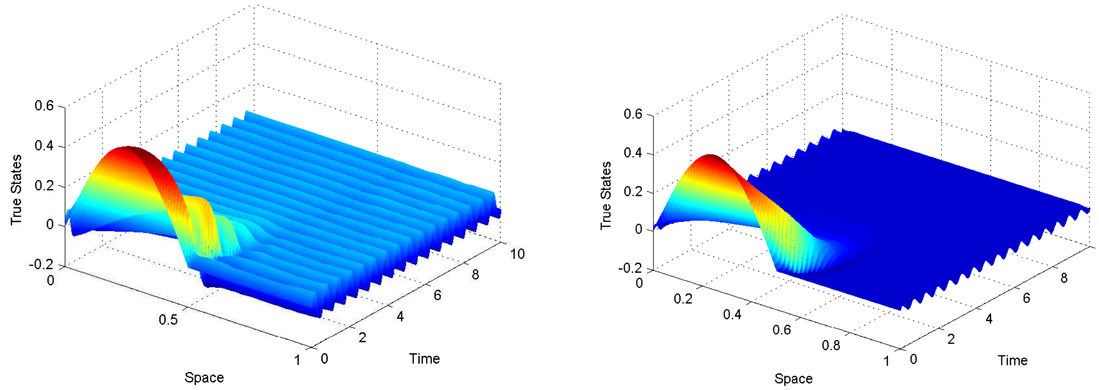
	Viscosity Coefficient					
	$\kappa = 0.01$		$\kappa = 0.002$		$\kappa = 0.001$	
	e	t_{set}	e	t_{set}	e	t_{set}
Without MECS						
$\mathbf{v}_k = 0$						
Direct Meas.	0.0562	-	0.0567	-	0.0568	-
$\mathbf{v}_k \sim N(\mathbf{0}, 0.05)$						
Direct Meas.	0.0563	-	0.0567	-	0.0568	-
Kalman Filter	0.0571	-	0.0576	-	0.0577	-
With MECS						
$\mathbf{v}_k = 0$						
Kalman Filter	0.0207	1.99	0.0538	3.26	0.0785	-
$\mathbf{v}_k \sim N(\mathbf{0}, 0.05)$						
Kalman Filter[x]	0.0244	-	0.0568	-	0.0807	-
With 'Filtered' MECS						
$\mathbf{v}_k = 0$						
Direct Meas.	0.0168	1.92	0.0172	2.09	0.0174	2.12
Kalman Filter	0.0209	2.00	0.0212	2.07	0.0212	2.08
$\mathbf{v}_k \sim N(\mathbf{0}, 0.05)$						
Direct Meas.	0.0223	2.50	0.0229	2.93	0.0229	3.37
Kalman Filter	0.0210	2.03	0.0217	2.06	0.0214	2.09

The difficulty associated with the reduced-order model for Burgers' equation as applied in this work is the appearance of numerical (model) instabilities: the Galerkin approximation leads to creation of overshooting and chattering in areas of steep descent (jump discontinuities or shocks, respectively). This becomes even worse for less damped systems, i.e., for lower values of viscosity. But the integrative property of the linear quadratic regulator helps again by disregarding parts of the discontinuities (fundamental property of the integral). In contrast to the full-order setting, the addition of a Kalman filter improves dynamics for both, the noisy and noise-free case. This could be explained by the additional 'smoothing' capability of the estimator (in presence of numerical instabilities).

Some Comparison to Previous Research: The displayed results differ tremendously from the ones exhibited by Atwell and King.[17] Despite several trials, their results could not be reproduced, especially not their conclusions on the poor performance in both cases, the full-order and the reduced-order LQG control. It has to be mentioned that they have used the steady-state Kalman filter in the linear quadratic Gaussian approach, derived from the steady-state linear Riccati equation. The fact that the proof of the existence of the steady-state solution is only limited to a certain class of *linear* systems (being both, stable and fully observable) has been ignored. One cannot be sure that the estimation error covariance matrix converges in case of the nonlinear Burgers' equation. Furthermore, their use of the weighting matrices is more than confusing: it has been pointed out that the weighting matrix of the process disturbance, W_K, has to be optimized (in a certain sense) for the Non-Gaussian case. Despite the fact that the applied disturbance is not of unit dimension, $\mathbf{d}(t) = 0.75\ \cos(10t)$, the disturbance gain matrix has been used as weighting factor (being the identity matrix). Also, their test setting did not contain any measurement noise, questioning the use of an estimator at all. Even if not every state has been measured (which is unfortunately not clearly specified in their work), a regular Luenberger observer should have been applied. Nevertheless, the LQG approach has been implemented without measurement noise; but instead of choosing the measurement noise covariance to be of very small magnitude, the identity matrix was again employed. These facts make it particularly difficult to reproduce their results and to follow their conclusions. A similar confusing implementation of LQG control was described in the other work of King and Atwell.[30] For comparative reasons the true states for the noisy LQG control are depicted in figure 8(a).

The addition of model-error correction suffers from the instabilities of the reduced-order model: the differentiating character of the model-error control synthesis cannot cope with the appearing overshoots and chattering, so that the closed-loop system becomes unstable.[y] Only the addition of the Kalman filter with its 'smoothing property' helps to stabilize the control. But this only holds for higher damped systems ($\kappa = 0.01$) where no shock region is formed (illustrated by table 2).

[y] This appears for direct measurements in case of any of the applied viscosities.



(a) True States; Noise; Kalman Filter; No Model-Error Correction (b) True States; No Noise; Active 'Filtered' Model-Error Correction

Figure 8. Reduced-Order Control; $N_m = 21$, $\kappa = 0.001$

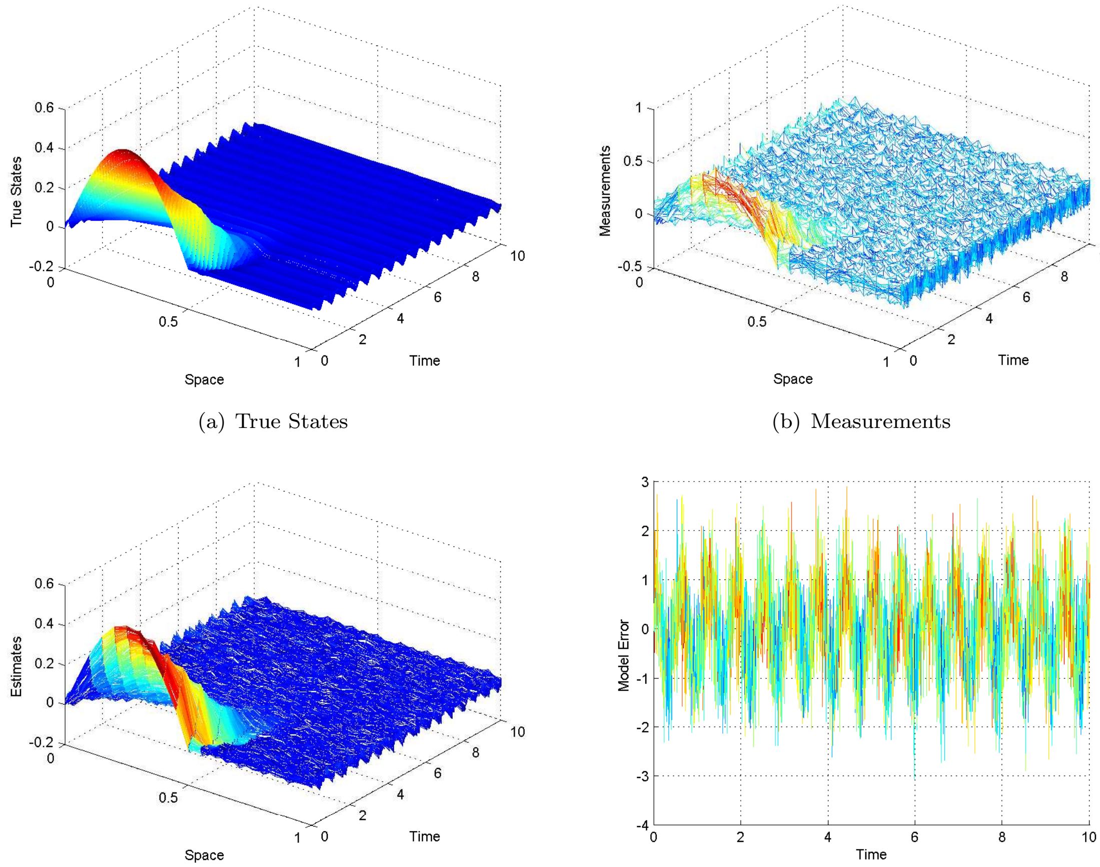
But as it has been mentioned, the test setting in this work is a worst case approach with one of the most coarse model truncation techniques possible. It is strongly believed that the model-error control synthesis shows superior performance (even for a much lower model-order), as soon as any advanced reduction method is applied. For the present truncation, the output of the model-error predictor could also be filtered (Fourier analyzed) to avoid numerical chattering. But the knowledge on the process disturbance can be exploited: the disturbance is equally applied to every state, so that a summation and division by N_m constitutes a coarse filter and could possibly lead to an improvement. Some simple other refinements as using a linearized model for the estimator or bounding the estimators output by upper and lower limits have been tested in previous work,[26] but showed instabilities and very poor performance.

The suggestions should only serve as first ideas within the scope of this work. Many other refinements, filter alterations or combinations could be possible. Nevertheless, the reader should be reminded that computational efficiency is a key issue in coping with distributed parameter systems; thus, expensive sophisticated adaptations might work in theory, but could turn out to be infeasible. So far, the averaging approach has turned out to be by far the best suggested refinement. Hence, figure 8(b) exhibits the true states for this approach in case of noise-free direct measurements. Additionally, figure 9 shows the true states, measurements, estimates and model-error correction for the reduced-order linear quadratic Gaussian controller in the presence of noise. Note that the oscillations in figure 8(b) *only* appear at the boundaries while the interior of the domain is highly damped. The reasons for this effect have yet to be investigated. Obviously, the capability to control individual model-errors appearing at different spatial locations has been lost by averaging. If one wishes to preserved that property, moving averages in time could be regarded.

VI. Contributions

A control approach to the problem of fluid flow has been presented using a benchmark problem incorporating Burgers' equation. This benchmark problem represents a one-dimensional mimicry of mathematical key-issues associated with nonlinear continuity problems: nonlinear convection with associated dissipation, diffusion, energy or impulse conservation, and distributed parameters. By applying a Galerkin linear finite element method, the benchmark problem has been converted into an ODE system. This state-space representation has been expanded to a test setting, additionally addressing process disturbance and noise. Standard control and filter techniques have been reviewed briefly, while model-error control synthesis has been introduced as a sophisticated approach to robustness. The derived methods and inaugurated settings have been tested numerically, and the expect system behavior has been affirmed. The contributions of this work are in detail:

1. A benchmark, or model problem, is created using periodic boundary conditions (conservation) and distributed control. Neither have been adequately addressed in previous research. The model is embedded in an unprecedented and augmented 'real world' test setting, including considerations for



(a) True States

(b) Measurements

(c) Estimates

(d) Model-Error Correction Term

Figure 9. Reduced-Order Control; Noise, with 'Filtered' Model-Error Correction, Kalman Filter, $N_m = 21$, $\kappa = 0.001$

measurement noise (filtering challenge) and external process disturbances (robustness). Additionally, model-order reduction is addressed.

2. Exponential stability for the open-loop (unforced) case is proven using a Lyapunov function and the Poincaré inequality (limited to the origin as target equilibrium and initial conditions following the above integral constraint). Also, conditions for the stability of feedback control are presented. It is shown that a feedback with a strict positive kernel improves the rate of convergence. An argumentation towards controllability is disclosed.

3. Model-Error Control Synthesis is introduced for nonlinear distributed systems of Burgers' class: it provides robustness and performance improvements in the case of model-uncertainties, unmodeled dynamics, process disturbances, and, in its modified version, reduced-order models. The model-error correction is not used in its original configuration, since it is both, a regulator and an estimator. Rather, it is combined with the extended Kalman filter, efficiently taking care of measurement noise.

4. The model problem is tackled from a worst case point of view: the nominal controller is a standard, simple (state) feedback law, neglecting the linearization of the nonlinear model parts. The model-order reduction is coarsely performed by truncating a linear Galerkin finite-element approximation. Hence, even better results are expected if more sophisticated reduction procedures are applied. The measurement noise covariance, the process disturbance's amplitude, and the bounds on the performance measure are chosen to exceed 'real world' demands.

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