Disturbance Accommodating Controller for Uncertain Stochastic Systems with Controller Saturation

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Observer based stochastic disturbance accommodating control scheme utilizes an estimator to determine the necessary corrections to the nominal control input and thus minimizes the adverse effects of both model uncertainties and external disturbances on the controlled system. The total control input, which includes the nominal control as well as the control corrections to the nominal control to accommodate for the disturbances, could exceed the actuator saturation limits. The saturation of the disturbance accommodating control system is more violent than that of the nominal control system because of the positive feedback of the disturbance compensation. This paper presents the formulation of the stochastic disturbance accommodating controller with actuator saturation for a class of uncertain linear stochastic systems. A Lyapunov based stochastic adaptive approach is used to update the control gains and the process noise covariance online so that the closed-loop stability of the controlled system is guaranteed.

I. Introduction

External disturbances and system uncertainties can obscure the development of a stable control law. The main objective of disturbance accommodating controller is to make necessary corrections to the nominal control input to accommodate for external disturbances and system uncertainties.[1–4] The disturbance accommodating observer approach has shown to be extremely effective for disturbance attenuation.[5–7] However, due to observer gain sensitivity to the external disturbances, performance of the observer can significantly vary for different types of exogenous disturbances. In our previous work,[8,9] a robust control approach based on a significant extension of the observer based disturbance accommodating control concept is presented. The robust control approach compensates for model parameter uncertainties and external disturbances by estimating a model-error vector in real time that is used as a signal synthesis adaptive correction to the nominal control input to achieve maximum performance. Utilizing a Kalman filter in the feedback loop, this control approach simultaneously estimates the system states and the model-error vector or the disturbance term from noisy measurements.[10–13] The estimated states are then used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input to minimize the adverse effects of system uncertainties and the external disturbances.

One of the major disadvantages of the stochastic disturbance accommodating controller is that the total control input, which includes the nominal control as well as the control corrections to the nominal control to accommodate for the disturbances, could exceed the actuator saturation limits. When the desired input exceeds the actuator limits, the controller fails to accommodate for the disturbances and this could drive the system unstable. Dealing with actuator saturation has been well recognized to be practically imperative yet a theoretically challenging problem. When a control system has integral compensators, the control saturation could lead to an unstable response, which is called the integral wind-up phenomenon. There is very little research about the wind-up phenomenon for disturbance accommodating observer based controllers. The wind-up phenomenon of the disturbance observer system is more vulnerable than that of the integral control system because of the positive feedback of the disturbance compensation. A wind-up restraint disturbance accommodating controller using a saturation model in the disturbance observer is

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proposed in Ref. 14. Estimation and accommodation of fault induced disturbance using Proportional Integral Observers (PIO) is given in Ref. 15. A Proportional Integral Adaptive Observer (PIAO) is also presented in Ref. 15 for the accurate estimation of the disturbances caused by unknown actuator faults when the system parameters are unknown. Analysis and design of a disturbance accommodating controller with $\mathcal{L}_2$ gain and $\mathcal{L}_\infty$ performance under actuator saturation and bounded disturbance is presented in Ref. 16. In Ref. 17 a formulation of the disturbance accommodating gain scheduled control with known actuator saturation is presented. The approach presented in Ref. 17 assumes a known linear time-invariant disturbance model driven by a Gaussian white noise process. The known actuator saturation value and the assumed upper bound on external disturbance are introduced as two gain scheduling parameters. A stochastic disturbance rejecting control system with actuator saturation is presented in Ref. 18. The control formulation presented in Ref. 18 only considers disturbances of white noise type and assumes noise free measurements.

The presence of actuator saturation introduces additional challenges on the analysis and the design of disturbance accommodating controllers. This paper presents an adaptive stochastic disturbance accommodating control scheme for saturating actuators with known saturation. This is an extension of the work presented in Ref. 8. As shown in Refs. 8 and 9, the closed-loop performance of the stochastic disturbance accommodating control system depends on the estimator parameters such as the process noise covariance. Also in Ref. 9 an adaptive law is synthesized for the selection of a stabilizing process noise covariance. This paper focuses on developing a stochastic adaptive approach to update the control gains and the process noise covariance online so that the stability of closed-loop system under actuator saturation is guaranteed.

II. Adaptive Disturbance Accommodating Controller

The formulation of the adaptive disturbance accommodating controller for uncertain stochastic systems of linear time-invariant class is presented here. A detailed formulation and analysis of the stochastic disturbance accommodating controller can be found in Refs. 8 and 9. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ denote a complete filtered probability space, where $\mathcal{F}$ is a σ-field, $\{\mathcal{F}_t\}_{t\geq t_0}$ is a collection of sub-σ-fields called a filtration and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$. Additionally, the elements of Ω are denoted by ω and the members of $\mathcal{F}$ are called events. Now consider an n^{th}-order linear time-invariant stochastic system of the following form:

$$\begin{aligned}
\dot{\mathbf{X}}_1(t) &= \mathbf{X}_2(t), \quad \mathbf{X}_1(t_0) = \mathbf{x}_{1_0} \\
\dot{\mathbf{X}}_2(t) &= A_1\mathbf{X}_1(t) + A_2\mathbf{X}_2(t) + \mathbf{u}(t) + \mathbf{W}(t), \quad \mathbf{X}_2(t_0) = \mathbf{x}_{2_0} \\
\mathbf{Y}(t) &= C\mathbf{X}(t) + \boldsymbol{\mathcal{V}}(t)
\end{aligned} \tag{1}$$

Here, the stochastic state vector is given as $\mathbf{X}(t) = \begin{bmatrix}\mathbf{X}_1^T(t) & \mathbf{X}_2^T(t)\end{bmatrix}^T$, where $\mathbf{X}_1(t) \triangleq \mathbf{X}_1(t,\omega) : [t_0, t_f]\times\Omega \mapsto \mathbb{R}^{\frac{n}{2}}$ and $\mathbf{X}_2(t) \triangleq \mathbf{X}_2(t,\omega) : [t_0, t_f]\times\Omega \mapsto \mathbb{R}^{\frac{n}{2}}$. The matrices, $A_1 \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ and $A_2 \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ are unknown state matrices. The stochastic measurement vector is given as $\mathbf{Y}(t) \triangleq \mathbf{Y}(t,\omega) : [t_0, t_f]\times\Omega \mapsto \mathbb{R}^m$. The output matrix, $C \in \mathbb{R}^{m\times n}$, is assumed to be known. The measurement noise, $\boldsymbol{\mathcal{V}}(t) \triangleq \boldsymbol{\mathcal{V}}(t,\omega) : [t_0, t_f]\times\Omega \mapsto \mathbb{R}^m$, is assumed to be zero mean Gaussian white noise process with

$$E\left[\boldsymbol{\mathcal{V}}^T(t+\tau)\boldsymbol{\mathcal{V}}(t)\right] = R\delta(\tau)$$

The stochastic external disturbance $\mathbf{W}(t) \triangleq \mathbf{W}(t,\omega) : [t_0, t_f]\times\Omega \mapsto \mathbb{R}^{\frac{n}{2}}$ is mean square integrable, i.e.,

$$E\Big[\int_{t_0}^{t_f} \mathbf{W}^T(\tau)\mathbf{W}(\tau)d\tau\Big] < \infty$$

and can be modeled as a linear time-invariant system driven by a Gaussian white noise process, i.e.,

$$\dot{\mathbf{W}}(t) = \mathbb{L}_1(\mathbf{X}(t), \mathbf{W}(t)) + \boldsymbol{\mathcal{W}}(t), \quad \mathbf{W}(t_0) = \mathbf{0} \tag{2}$$

where $\mathbb{L}_1(\cdot)$ is an unknown linear operator and $\boldsymbol{\mathcal{W}}(t) \triangleq \boldsymbol{\mathcal{W}}(t,\omega) : [t_0, t_f]\times\Omega \mapsto \mathbb{R}^{\frac{n}{2}}$, is assumed to be zero mean Gaussian white noise process with

$$E\left[\boldsymbol{\mathcal{W}}^T(t+\tau)\boldsymbol{\mathcal{W}}(t)\right] = \mathcal{Q}\delta(\tau)$$

The assumed (known) model of the above system is

$$\begin{aligned}
\dot{\mathbf{x}}_{1_m}(t) &= \mathbf{x}_{2_m}(t), \quad \mathbf{x}_{1_m}(t_0) = \mathbf{x}_{1_0} \\
\dot{\mathbf{x}}_{2_m}(t) &= A_{1_m}\mathbf{x}_{1_m}(t) + A_{2_m}\mathbf{x}_{2_m}(t) + \bar{\mathbf{u}}(t), \quad \mathbf{x}_2(t_0) = \mathbf{x}_{2_0}
\end{aligned} \tag{3}$$

The nominal controller $\bar{\mathbf{u}}(t)$ is selected so that the assumed system would have the following closed-loop form

$$\begin{aligned}
\dot{\mathbf{x}}_{1_m}(t) &= \mathbf{x}_{2_m}(t) \\
\dot{\mathbf{x}}_{2_m}(t) &= \left[A_{1_m} - K_{1_m}\right]\mathbf{x}_{1_m}(t) + \left[A_{2_m} - K_{2_m}\right]\mathbf{x}_{2_m}(t), \quad \mathbf{x}_2(t_0) = \mathbf{x}_{2_0}
\end{aligned} \tag{4}$$

Let the disturbance term $\boldsymbol{\mathcal{D}}(t) = (A_1 - A_{1_m})\,\mathbf{X}_1(t) + (A_2 - A_{2_m})\,\mathbf{X}_2(t) + \mathbf{W}(t)$, then the system in Eq. (1) can be written in terms of the known system parameters as

$$\begin{aligned}
\dot{\mathbf{X}}_1(t) &= \mathbf{X}_2(t), \quad \mathbf{X}_1(t_0) = \mathbf{x}_{1_0} \\
\dot{\mathbf{X}}_2(t) &= A_{1_m}\mathbf{X}_1(t) + A_{2_m}\mathbf{X}_2(t) + \mathbf{u}(t) + \boldsymbol{\mathcal{D}}(t), \quad \mathbf{X}_2(t_0) = \mathbf{x}_{2_0}
\end{aligned} \tag{5}$$

Now an estimator is designed to estimate the system states and the disturbance term from noisy measurements. The estimator dynamics may be written as

$$\begin{aligned}
\dot{\hat{\mathbf{X}}}_1(t) &= \hat{\mathbf{X}}_2(t) + L_1(t)\left(\mathbf{Y}(t) - C\hat{\mathbf{X}}(t)\right), \quad \hat{\mathbf{X}}_1(t_0) = \mathbf{x}_{1_0} \\
\dot{\hat{\mathbf{X}}}_2(t) &= A_{1_m}\hat{\mathbf{X}}_1(t) + A_{2_m}\hat{\mathbf{X}}_2(t) + \mathbf{u}(t) + \hat{\boldsymbol{\mathcal{D}}}(t) + L_2(t)\left(\mathbf{Y}(t) - C\hat{\mathbf{X}}(t)\right), \quad \hat{\mathbf{X}}_2(t_0) = \mathbf{x}_{2_0} \\
\dot{\hat{\boldsymbol{\mathcal{D}}}}(t) &= A_{\boldsymbol{\mathcal{D}}}\hat{\boldsymbol{\mathcal{D}}}(t) + L_3(t)\left(\mathbf{Y}(t) - C\hat{\mathbf{X}}(t)\right), \quad \hat{\boldsymbol{\mathcal{D}}}(t_0) = \mathbf{0}_{\frac{n}{2}\times 1}
\end{aligned} \tag{6}$$

where $L(t) = \begin{bmatrix} L_1^T(t) & L_2^T(t) & L_3^T(t) \end{bmatrix}^T$ is the observer gain and the Hurwitz matrix $A_{\boldsymbol{\mathcal{D}}}$ represents the assumed dynamics of the disturbance term. The observer gain is calculated as $L(t) = P(t)H^T R^{-1}$, where $P(t)$ is obtained from

$$\dot{P}(t) = F_m^T P(t) + P(t)F_m - P(t)H^T R^{-1} H P(t) + GQG^T \tag{7}$$

where $F_m = \begin{bmatrix} 0_{\frac{n}{2}\times\frac{n}{2}} & I_{\frac{n}{2}\times\frac{n}{2}} & 0_{\frac{n}{2}\times\frac{n}{2}} \\ A_{1_m} & A_{2_m} & I_{\frac{n}{2}\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times\frac{n}{2}} & 0_{\frac{n}{2}\times\frac{n}{2}} & A_{\boldsymbol{\mathcal{D}}} \end{bmatrix}$, $H = \begin{bmatrix} C & 0_{m\times\frac{n}{2}} \end{bmatrix}$, $G = \begin{bmatrix} 0_{n\times\frac{n}{2}} \\ I_{\frac{n}{2}\times\frac{n}{2}} \end{bmatrix}$, and Q is the assumed process noise covariance associated with the disturbance term model. Let $\hat{\mathbf{Z}}(t) = \begin{bmatrix} \hat{\mathbf{X}}_1^T(t) & \hat{\mathbf{X}}_2^T(t) & \hat{\boldsymbol{\mathcal{D}}}^T(t) \end{bmatrix}^T$, now the estimator in Eq. (6) can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + B_m\mathbf{u}(t) + L(t)\left(\mathbf{Y}(t) - H\hat{\mathbf{Z}}(t)\right) \tag{8}$$

where $B_m = \begin{bmatrix} 0_{\frac{n}{2}\times\frac{n}{2}} \\ I_{\frac{n}{2}\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times\frac{n}{2}} \end{bmatrix}$. In order to obtain the desired closed-loop dynamics given in Eq. (4), the disturbance accommodating control, $\mathbf{u}(t)$ is selected as

$$\mathbf{u}(t) = -K_{1_m}\hat{\mathbf{X}}_1(t) - K_{2_m}\hat{\mathbf{X}}_2(t) - \hat{\boldsymbol{\mathcal{D}}}(t) = -\mathcal{K}_m\hat{\mathbf{Z}}(t) \tag{9}$$

where $\mathcal{K}_m = \begin{bmatrix} K_{1_m} & K_{2_m} & I_{\frac{n}{2}\times\frac{n}{2}} \end{bmatrix}$. Note that the true disturbance term dynamics can be written as

$$\begin{aligned}
\dot{\boldsymbol{\mathcal{D}}}(t) &= (A_1 - A_{1_m})\,\mathbf{X}_2(t) + (A_2 - A_{2_m})\,\dot{\mathbf{X}}_2(t) + \mathbb{L}_1(\mathbf{X}(t), \mathbf{W}(t)) + \boldsymbol{\mathcal{W}}(t) \\
&= \mathbb{L}_2\left(\boldsymbol{\mathcal{D}}(t), \mathbf{X}(t), \mathbf{u}(t)\right) + \boldsymbol{\mathcal{W}}(t)
\end{aligned} \tag{10}$$

where $\mathbb{L}_2(\cdot)$ is an unknown linear operator. Now the true extended system dynamics can now be written as

$$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) + B\mathbf{u}(t) + G\boldsymbol{\mathcal{W}}(t) \tag{11}$$

where $F = \begin{bmatrix} 0_{\frac{n}{2}\times\frac{n}{2}} & I_{\frac{n}{2}\times\frac{n}{2}} & 0_{\frac{n}{2}\times\frac{n}{2}} \\ A_{1_m} & A_{2_m} & I_{\frac{n}{2}\times\frac{n}{2}} \\ \mathbb{L}_{2_{\mathbf{X}_1}} & \mathbb{L}_{2_{\mathbf{X}_2}} & \mathbb{L}_{2_{\boldsymbol{\mathcal{D}}}} \end{bmatrix}$ and $B = \begin{bmatrix} 0_{\frac{n}{2}\times\frac{n}{2}} \\ I_{\frac{n}{2}\times\frac{n}{2}} \\ \mathbb{L}_{2_{\mathbf{u}}} \end{bmatrix}$. Operators $\mathbb{L}_{2_{\mathbf{X}_1}}$, $\mathbb{L}_{2_{\mathbf{X}_2}}$, $\mathbb{L}_{2_{\mathbf{u}}}$ and $\mathbb{L}_{2_{\boldsymbol{\mathcal{D}}}}$ are partitions on $\mathbb{L}_2(\cdot)$ that are acting on $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{u}$ and $\boldsymbol{\mathcal{D}}$, respectively. After substituting for the control law, the plant dynamics in Eq. (5) can be written as

$$\begin{aligned} \dot{\mathbf{X}}_1(t) &= \mathbf{X}_2(t) \\ \dot{\mathbf{X}}_2(t) &= A_{1_m}\mathbf{X}_1(t) + A_{2_m}\mathbf{X}_2(t) + \boldsymbol{\mathcal{D}}(t) - K_{1_m}\hat{\mathbf{X}}_1(t) - K_{2_m}\hat{\mathbf{X}}_2(t) - \hat{\boldsymbol{\mathcal{D}}}(t) \end{aligned} \tag{12}$$

Define the estimation errors as $\tilde{\mathbf{X}}_1(t) = \mathbf{X}_1(t) - \hat{\mathbf{X}}_1(t)$, $\tilde{\mathbf{X}}_2(t) = \mathbf{X}_2(t) - \hat{\mathbf{X}}_2(t)$, and $\tilde{\boldsymbol{\mathcal{D}}}(t) = \boldsymbol{\mathcal{D}}(t) - \hat{\boldsymbol{\mathcal{D}}}(t)$. Now subtracting Eq. (4) from Eq. (12), the error dynamics can be written as

$$\begin{aligned} \dot{\mathbf{e}}_1(t) &= \mathbf{X}_2(t) - \mathbf{x}_{2_m}(t) = \mathbf{e}_2(t) \\ \dot{\mathbf{e}}_2(t) &= [A_{1_m} - K_{1_m}]\,\mathbf{e}_1(t) + [A_{2_m} - K_{2_m}]\,\mathbf{e}_2(t) + K_{1_m}\tilde{\mathbf{X}}_1(t) + K_{2_m}\tilde{\mathbf{X}}_2(t) + \tilde{\boldsymbol{\mathcal{D}}}(t) \end{aligned} \tag{13}$$

Note that the desired error dynamics is given as

$$\begin{aligned} \dot{\bar{\boldsymbol{e}}}_1(t) &= \bar{\boldsymbol{e}}_2(t) \\ \dot{\bar{\boldsymbol{e}}}_2(t) &= [A_{1_m} - K_{1_m}]\,\bar{\boldsymbol{e}}_1(t) + [A_{2_m} - K_{2_m}]\,\bar{\boldsymbol{e}}_2(t) \end{aligned} \tag{14}$$

Now the error between the desired and the actual error dynamics can be written as

$$\begin{aligned} \dot{\mathcal{E}}_1(t) &= \dot{\bar{\boldsymbol{e}}}_1(t) - \dot{\mathbf{e}}_1(t) = \mathcal{E}_2(t) \\ \dot{\mathcal{E}}_2(t) &= [A_{1_m} - K_{1_m}]\,\mathcal{E}_1(t) + [A_{2_m} - K_{2_m}]\,\mathcal{E}_2(t) - K_{1_m}\tilde{\mathbf{X}}_1(t) - K_{2_m}\tilde{\mathbf{X}}_2(t) - \tilde{\boldsymbol{\mathcal{D}}}(t) \end{aligned} \tag{15}$$

It is important to note that if $Q = 0$, then the disturbance accommodating control law given in Eq. (9) becomes just the nominal control. If the nominal control, $\bar{\mathbf{u}}(t)$, on the true plant would result in an unstable system, then selecting a small Q would also result in an unstable system. It is shown in Refs. 8 and 9 that the performance of the estimator depends on the choice of the process noise covariance Q. Based on the premises of the following theorem, Q is updated online so that $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ are asymptotically stable in the mean, i.e.,

$$\lim_{t\to\infty} E[\mathcal{E}_1(t)] = 0 \quad \text{and} \quad \lim_{t\to\infty} E[\mathcal{E}_2(t)] = 0$$

and bounded stable in higher order moments.

Theorem 1. *For a linear extended controlled system of the form*

$$\dot{\mathbf{Z}}(t) = F\mathbf{Z}(t) - B\mathcal{K}_m\hat{\mathbf{Z}}(t) + G\boldsymbol{\mathcal{W}}(t) \tag{16}$$

and an estimator of the form

$$\dot{\hat{\mathbf{Z}}}(t) = \{F_m - B_m\mathcal{K}_m\}\,\hat{\mathbf{Z}}(t) + L(t)H\left[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)\right] + L(t)\boldsymbol{\mathcal{V}}(t) \tag{17}$$

the estimation error $\tilde{\mathbf{Z}}(t) = \mathbf{Z}(t) - \hat{\mathbf{Z}}(t)$ is exponentially stable in the first moment and bounded in all higher order moments if the process noise covariance is updated online using the adaptive law

$$\dot{Q}(t) = \gamma G^T[L_{ss}\tilde{\boldsymbol{y}}(t)\tilde{\boldsymbol{y}}^T(t)L_{ss}^T]G \tag{18}$$

where $\gamma > (\frac{3n}{2})$ is the adaptive gain and L_{ss} is the steady-state Kalman gain corresponding to the initial process noise covariance, $Q(t_0)$. Also, the exponential stability of $\tilde{\mathbf{Z}}(t)$ in the first moment implies $\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_1(t) \\ \mathcal{E}_2(t) \end{bmatrix}$ is exponentially stable in the first moment and bounded in all higher order moments.

Proof of this theorem is based on the following two lemmas. The first lemma is based on a well known supermartingale property and it is used to construct the adaptive law for $Q(t)$ such that the controlled plant in Eq. (16) is exponentially stable in the first moment. The second lemma proves that, for a linear system with affine disturbances of white noise type, exponential stability in the first moment implies bounded stability in the higher order moments.

Lemma 1. *Assume there is a nonnegative function* $V(\mathbf{Y},t):\mathbb{R}^n\times[t_0,\infty)\mapsto\mathbb{R}_+$ *such that*

$$\mathfrak{L}V(\mathbf{y},t)\le 0 \tag{19}$$

for all $(\mathbf{y},t)\in\mathbb{R}^n\times[t_0,\infty)$. *The operator* $\mathfrak{L}\{\cdot\}$ *acting on* $V(\mathbf{Y},t)$ *is given by*

$$\mathfrak{L}V(\mathbf{y},t)=\lim_{dt\to 0}\frac{1}{dt}E\big[dV(\mathbf{Y}(t),t)|\mathbf{Y}(t)=\mathbf{y}\big] \tag{20}$$

where $dV(\mathbf{Y}(t),t)$ *can be calculated using the Itô Formula*[19–21] *and* $\mathbf{y}$ *indicates a sample path of* $\mathbf{Y}(t,\omega)$, *i.e.,* $\mathbf{y}(t)=\mathbf{Y}(t,\omega_\imath)\,|_{\omega_\imath\in\Omega}$. *Then,* $V(\mathbf{Y},t)$ *is a nonnegative supermartingale process and, for any initial condition* $\mathbf{Y}(t_0)=\mathbf{y}_0$,

$$\mathbb{P}\Big(\sup_{\infty>t\ge t_0}V(\mathbf{Y},t)\ge\lambda\Big)\le\frac{V(\mathbf{y}_0,t_0)}{\lambda} \tag{21}$$

where λ *is any positive constant.*

Proof. If $\mathfrak{L}V(\mathbf{y},t)\le 0$, then Dynkin's formula[22,23] can be used:

$$E\big[V(\mathbf{Y},t)\big]-V(\mathbf{y}_0,t_0)=E\Big[\int_{t_0}^{t}\mathfrak{L}V(\mathbf{Y},\tau)d\tau\Big]\le 0$$

Thus $E\big[V(\mathbf{Y},t)\big]\le V(\mathbf{y}_0,t_0)$ and $E\big[V(\mathbf{Y},t)\big]\to V(\mathbf{y}_0,t_0)$ as $t\to t_0$. These two facts imply supermartingale property and (21) is the supermartingale probability inequality. □

Lemma 1 will be used to construct an update law for the process noise covariance $Q(t)$ such that the controlled plant in Eq. (16) is exponentially stable in the first moment. The lemma given proves that, for a linear system with affine disturbances of white noise type, exponential stability in the first moment implies bounded stability in the higher order moments.

Lemma 2. *For an affine stochastic system of the following form*

$$\dot{\mathbf{Z}}(t)=F\mathbf{Z}(t)+G\boldsymbol{\mathcal{W}}(t),\quad \mathbf{Z}(t_0)=\mathbf{Z}_0 \tag{22}$$

where the affine term $\boldsymbol{\mathcal{W}}(t)$ *is a zero mean Gaussian white noise process with*

$$E\left[\boldsymbol{\mathcal{W}}^T(t+\tau)\boldsymbol{\mathcal{W}}(t)\right]=\mathcal{Q}\delta(\tau)$$

stability in first moment implies stability in all the higher order moments.

Proof. In order to analyse the stability in the first moment, we consider the mean dynamics of the system given in Eq. (22)

$$\dot{\boldsymbol{\mu}}_{\mathbf{Z}}(t)=F\boldsymbol{\mu}_{\mathbf{Z}}(t)$$

where $\boldsymbol{\mu}_{\mathbf{Z}}(t)=E\left[\mathbf{Z}(t)\right]$. Stability in the first moment implies the state matrix F generates a stable state evolution operator, $\Phi_F(t,t_0)$. Now the second moment of $\mathbf{Z}(t)$ or the correlation matrix $\mathcal{P}(t)=E[\mathbf{Z}(t)\mathbf{Z}^T(t)]$ follows the following matrix Lyapunov differential equation:

$$\dot{\mathcal{P}}(t)=F\mathcal{P}(t)+\mathcal{P}(t)F^T+G\mathcal{Q}G^T,\quad \mathcal{P}(t_0)=E[\mathbf{Z}_0\mathbf{Z}_0^T]$$

and the solution to the above equation can be written as

$$\mathcal{P}(t)=\int_{-\infty}^{t}\Phi_F(t,\tau)G\mathcal{Q}G^T\Phi_F^T(t,\tau)d\tau$$

Since $\Phi_F(t,t_0)$ is a stable evolution, $\mathcal{P}(t)$ has a bounded solution.[24] Notice that $\mathbf{Z}(t)$ is a Gaussian process and the higher order moments of $\mathbf{Z}(t)$ can be written as a function of the first two moments of $\mathbf{Z}(t)$. This completes the proof. □

Notice that due to the persistently acting external disturbances, $\boldsymbol{\mathcal{W}}(t)$ and $\boldsymbol{\mathcal{V}}(t)$, the extended states $\mathbf{Z}(t)$ and the estimated states $\hat{\mathbf{Z}}(t)$ are not almost surely (a.s.) asymptotically stable.[25] Almost sure asymptotic stability is defined as

Definition 1. *A stochastic process, $\boldsymbol{\mathcal{Z}}(t)$, is asymptotically stable with probability* 1*, or almost surely asymptotically stable, if*

$$\mathbb{P}\big(\boldsymbol{\mathcal{Z}}(t) \to \boldsymbol{0} \quad \text{as} \quad t \to \infty\big) = 1 \tag{23}$$

Here, an adaptive law is developed to update the selected process noise covariance $Q(t)$ online so that the controlled system in Eq. (16) is bounded-input bounded-output (BIBO) stable in the mean. For a linear time invariant system, assuming the system is controllable and observable, it can be shown that the BIBO stability in the mean implies exponential stability in the mean.

The plant in Eq. (16) is BIBO stable in the mean if the estimated states, $\hat{\mathbf{Z}}(t)$, and the output $\mathbf{Y}(t)$ are bounded in the mean. Since $\{F_m - B_m\mathcal{K}_m\}$ generates an exponentially stable state evolution operator, the plant is BIBO stable in the mean if and only if the estimated states are bounded in the mean. That is, if $\hat{\mathbf{Z}}(t)$ is bounded in the mean, then the input $\mathcal{K}_m\hat{\mathbf{Z}}(t)$ is bounded in the mean. Note that the estimator gain $L(t)$ is obtained from Eq. (7) and it is bounded. If $\hat{\mathbf{Z}}(t)$ is bounded in the mean, it implies $H\tilde{\mathbf{Z}}(t)$ is bounded in the mean and thus the plant in Eq. (16) is BIBO stable in the mean.

Since $\{F_m - B_m\mathcal{K}_m\}$ generates an exponentially stable state evolution operator, it is the residual, $H\tilde{\mathbf{Z}}(t)$, that drives the estimated mean response unbounded. Ignoring this residual, the estimator given in Eq. (17) may be written as

$$\dot{\hat{\mathbf{Z}}}_m(t) = \{F_m - B_m\mathcal{K}_m\}\,\hat{\mathbf{Z}}_m(t) + L(t)\boldsymbol{\mathcal{V}}(t) \tag{24}$$

Notice that the stochastic process $\hat{\mathbf{Z}}_m(t)$ is exponentially stable in the mean. Therefore, if $\tilde{\mathbf{Y}}_m(t) = H\left[\hat{\mathbf{Z}}_m(t) - \hat{\mathbf{Z}}(t)\right]$ is bounded in the mean, then $H\hat{\mathbf{Z}}(t)$ is also bounded in the mean. Based on the observability assumption, if $H\hat{\mathbf{Z}}(t)$ is bounded in the mean, then $\hat{\mathbf{Z}}(t)$ is bounded in the mean and thus the plant in Eq. (17) is BIBO stable in the mean. Now the proof of Theorem 1 is given based on the results obtained from the previous two lemmas.

Proof. Consider the following nonnegative function:

$$V(\tilde{\mathbf{Y}}_m, t) = \int_{t_0}^{t} \tilde{\mathbf{Y}}_m^T(\tau) L_{ss}^T L_{ss} \tilde{\mathbf{Y}}_m(\tau) d\tau + \mathrm{Tr}\{G\Delta Q(t) G^T\}$$

where $\Delta Q(t) = Q^* - Q(t)$ and $Q^* \geq Q(t)$, $\forall t \geq t_0$. Note that Q^* would result in exponentially decaying estimation errors and $L_{ss} \in \mathbb{R}^{(\frac{3n}{2})\times m}$ is an arbitrary matrix such that $L_{ss}^T L_{ss} > 0$. For convenience, the steady-state Kalman gain corresponding to the initial process noise covariance, $Q(t_0)$, may be selected as L_{ss} as long as $L_{ss}^T L_{ss} > 0$. Now $\mathfrak{L}V(\tilde{\mathbf{y}}_m, t)$ can be calculated as

$$\mathfrak{L}V(\tilde{\mathbf{y}}_m, t) = \tilde{\mathbf{y}}_m^T(t) L_{ss}^T L_{ss} \tilde{\mathbf{y}}_m(t) - \mathrm{Tr}\{G\dot{Q}(t)G^T\}$$

Select the adaptive law for $Q(t)$ as

$$\dot{Q}(t) = \gamma G^T [L_{ss}\tilde{\mathbf{y}}_m(t)\tilde{\mathbf{y}}_m^T(t)L_{ss}^T] G \tag{25}$$

where $\gamma > (\frac{3n}{2})$ is the adaptive gain. Substituting Eq. (25) into $\mathfrak{L}V(\tilde{\mathbf{x}}, t)$ yields

$$\begin{aligned} \mathfrak{L}V(\tilde{\mathbf{y}}_m, t) &= -\mathrm{Tr}\left\{[L_{ss}\tilde{\mathbf{y}}_m(t)\tilde{\mathbf{y}}_m^T(t)L_{ss}^T][\gamma G G^T - I_{\frac{3n}{2}\times\frac{3n}{2}}]\right\} \\ &\leq -\mathrm{Tr}\left\{L_{ss}\tilde{\mathbf{y}}_m(t)\tilde{\mathbf{y}}_m^T(t)L_{ss}^T\right\}\mathrm{Tr}\left\{\gamma G G^T - I_{\frac{3n}{2}\times\frac{3n}{2}}\right\} \leq 0 \end{aligned}$$

Thus we have

$$\mathfrak{L}V(\tilde{\mathbf{y}}_m, t) \leq 0$$

i.e., $V(\tilde{\mathbf{Y}}_m, t)$ is a supermartingale and

$$\mathbb{P}\Big(\sup_{\infty > t \geq t_0} V(\tilde{\mathbf{Y}}_m, t) \geq \lambda\Big) \leq \frac{\mathrm{Tr}\{G\Delta Q(t_0)G^T\}}{\lambda}$$

where $\lambda > 0$ is an arbitrary constant. Selecting sufficiently large λ yields

$$\lim_{t\to\infty} \int_{t_0}^{t} \tilde{\mathbf{Y}}_m^T(\tau) K_{ss}^T K_{ss} \tilde{\mathbf{Y}}_m(\tau) d\tau < \infty \quad \text{a.s.} \Rightarrow \lim_{t\to\infty} \tilde{\mathbf{Y}}_m(t) = 0 \quad \text{a.s.}$$

Thus the true plant is also BIBO stable in the mean and because we assume the plant is controllable and observable, the BIBO stability in the mean implies exponential stability in the mean. Therefore the adaptive law given in (18) will guarantee the plant is exponentially stable in the mean and bounded in all higher moments. Since the controlled plant in Eq. (16) and the estimator in Eq. (17) are exponentially stable in the mean, the estimation error is also exponentially stable in the mean. Thus $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ are also exponentially stable in the first moment and the bounded higher order moments follows from Lemma 2. □

Note that one of main assumptions behind the formulation of the presented adaptive disturbance accommodating control is that the total control input $\mathbf{u}(t) = -\mathcal{K}_m \hat{\mathbf{Z}}$ never exceeds the actuator saturation limits. The saturation of the disturbance accommodating control is more violent than that of a nominal control system because of the positive feedback of the disturbance compensation. Next the formulation of the disturbance accommodating controller for a saturating system with known saturation is given.

III. Adaptive Disturbance Accommodating Controller with Saturation

A major drawback of the disturbance accommodating control is that the control synthesis given in Eq. (9) could saturate the actuators. In this section an adaptive approach to resolve the saturation issue in the disturbance accommodating control is derived. When there is actuator saturation, the applied control input is different from the calculated control input, i.e.,

$$\mathbf{u}_a(t) = \mathbf{u}(t) - \boldsymbol{\delta}(t) \tag{26}$$

When there is no saturation, i.e., $\boldsymbol{\delta}(t) = 0$, the control gains are selected to be constants. When there is saturation time varying control gains are selected to compensate for nonzero $\boldsymbol{\delta}(t)$, i.e.,

$$\mathbf{u}(t) = -K_1(t)\hat{\mathbf{X}}_{1_s}(t) - K_2(t)\hat{\mathbf{X}}_{2_s}(t) - K_{\boldsymbol{\mathcal{D}}}(t)\hat{\boldsymbol{\mathcal{D}}}_s(t) \tag{27}$$

Note that the applied control is $\mathbf{u}_a = \mathbf{u} - \boldsymbol{\delta}$. Now substituting this control law, the plant dynamics can be written as

$$\begin{aligned} \dot{\mathbf{X}}_{1_s}(t) &= \mathbf{X}_{2_s}(t) \\ \dot{\mathbf{X}}_{2_s}(t) &= A_{1_m}\mathbf{X}_{1_s}(t) + A_{2_m}\mathbf{X}_{2_s}(t) + \boldsymbol{\mathcal{D}}_s(t) - K_1(t)\hat{\mathbf{X}}_{1_s}(t) - K_2(t)\hat{\mathbf{X}}_{2_s}(t) - K_{\boldsymbol{\mathcal{D}}}(t)\hat{\boldsymbol{\mathcal{D}}}_s(t) - \boldsymbol{\delta}(t) \end{aligned} \tag{28}$$

Let $\tilde{\mathbf{Y}}_s(t) = \left[\mathbf{Y}_s(t) - C\hat{\mathbf{X}}_s(t)\right]$, now the estimator dynamics is given as

$$\begin{aligned} \dot{\hat{\mathbf{X}}}_{1_s}(t) &= \hat{\mathbf{X}}_{2_s}(t) + L_1(t)\tilde{\mathbf{Y}}_s(t) \\ \dot{\hat{\mathbf{X}}}_{2_s}(t) &= \left[A_{1_m} - K_1(t)\right]\hat{\mathbf{X}}_{1_s}(t) + \left[A_{2_m} - K_2(t)\right]\hat{\mathbf{X}}_{2_s}(t) + \mathbf{u}(t) + \left[I - K_3(t)\right]\hat{\boldsymbol{\mathcal{D}}}_s(t) - \boldsymbol{\delta}(t) + L_2(t)\tilde{\mathbf{Y}}_s(t) \\ \dot{\hat{\boldsymbol{\mathcal{D}}}}_s(t) &= A_{\boldsymbol{\mathcal{D}}}\hat{\boldsymbol{\mathcal{D}}}_s(t) + L_3(t)\tilde{\mathbf{Y}}_s(t) \end{aligned} \tag{29}$$

Now subtracting Eq. (4) from the Eq. (28) yields

$$\begin{aligned}
\dot{\mathbf{e}}_{1_s}(t) &= \mathbf{X}_{2_s}(t) - \mathbf{x}_{2_m}(t) = \mathbf{e}_{2_s}(t) \\
\dot{\mathbf{e}}_{2_s}(t) &= A_{1_m}\mathbf{X}_{1_s}(t) + A_{2_m}\mathbf{X}_{2_s}(t) + \boldsymbol{\mathcal{D}}_s(t) - K_1(t)\hat{\mathbf{X}}_{1_s}(t) - K_2(t)\hat{\mathbf{X}}_{2_s}(t) - K_{\boldsymbol{\mathcal{D}}}(t)\hat{\boldsymbol{\mathcal{D}}}_s(t) - \boldsymbol{\delta}(t) \\
&\qquad - \left[A_{1_m} - K_{1_m}\right]\mathbf{x}_{1_m}(t) - \left[A_{2_m} - K_{2_m}\right]\mathbf{x}_{2_m}(t) \\
&= A_{1_m}\mathbf{e}_{1_s}(t) + A_{2_m}\mathbf{e}_{2_s}(t) + \boldsymbol{\mathcal{D}}_s(t) - K_1(t)\hat{\mathbf{X}}_{1_s}(t) - K_2(t)\hat{\mathbf{X}}_{2_s}(t) \\
&- K_{\boldsymbol{\mathcal{D}}}(t)\hat{\boldsymbol{\mathcal{D}}}_s(t) - \boldsymbol{\delta}(t) + K_{1_m}\mathbf{x}_{1_m}(t) + K_{2_m}\mathbf{x}_{2_m}(t) + K_{1_m}\mathbf{X}_{1_s}(t) + K_{2_m}\mathbf{X}_{2_s}(t) - K_{1_m}\mathbf{X}_{1_s}(t) \\
&- K_{2_m}\mathbf{X}_{2_s}(t) + K_{1_m}\hat{\mathbf{X}}_{1_s}(t) + K_{2_m}\hat{\mathbf{X}}_{2_s}(t) - K_{1_m}\hat{\mathbf{X}}_{1_s}(t) - K_{2_m}\hat{\mathbf{X}}_{2_s}(t) \\
&= \left[A_{1_m} - K_{1_m}\right]\mathbf{e}_{1_s}(t) + \left[A_{2_m} - K_{2_m}\right]\mathbf{e}_{2_s}(t) + K_{1_m}\tilde{\mathbf{X}}_{1_s}(t) + K_{2_m}\tilde{\mathbf{X}}_{2_s}(t) - \tilde{K}_1(t)\hat{\mathbf{X}}_{1_s}(t) - \tilde{K}_2(t)\hat{\mathbf{X}}_{2_s}(t) \\
&\qquad + \boldsymbol{\mathcal{D}}_s(t) - K_{\boldsymbol{\mathcal{D}}}(t)\hat{\boldsymbol{\mathcal{D}}}_s(t) - \boldsymbol{\delta}(t) + \hat{\boldsymbol{\mathcal{D}}}_s(t) - \hat{\boldsymbol{\mathcal{D}}}_s(t) \\
&= \left[A_{1_m} - K_{1_m}\right]\mathbf{e}_{1_s}(t) + \left[A_{2_m} - K_{2_m}\right]\mathbf{e}_{2_s}(t) + K_{1_m}\tilde{\mathbf{X}}_{1_s}(t) + K_{2_m}\tilde{\mathbf{X}}_{2_s}(t) - \tilde{K}_1(t)\hat{\mathbf{X}}_{1_s}(t) - \tilde{K}_2(t)\hat{\mathbf{X}}_{2_s}(t) \\
&\qquad + \tilde{\boldsymbol{\mathcal{D}}}_s(t) - \tilde{K}_{\boldsymbol{\mathcal{D}}}(t)\hat{\boldsymbol{\mathcal{D}}}_s(t) - \boldsymbol{\delta}(t)
\end{aligned}$$

Let $\mathcal{K}(t) = \begin{bmatrix} K_1(t) & K_2(t) & K_{\boldsymbol{\mathcal{D}}}(t) \end{bmatrix}$, now the above error dynamics can be simplified to

$$\begin{aligned}
\dot{\mathbf{e}}_{1_s}(t) &= \mathbf{e}_{2_s}(t) \\
\dot{\mathbf{e}}_{2_s}(t) &= \left[A_{1_m} - K_{1_m}\right]\mathbf{e}_{1_s}(t) + \left[A_{2_m} - K_{2_m}\right]\mathbf{e}_{2_s}(t) + \mathcal{K}_m\tilde{\mathbf{Z}}_s(t) - \tilde{\mathcal{K}}(t)\hat{\mathbf{Z}}_s(t) - \boldsymbol{\delta}(t)
\end{aligned} \tag{30}$$

where $\tilde{\mathcal{K}}(t) = \mathcal{K}(t) - \mathcal{K}_m$, $\hat{\mathbf{Z}}_s(t) = \begin{bmatrix} \hat{\mathbf{X}}_{1_s}^T(t) & \hat{\mathbf{X}}_{2_s}^T(t) & \hat{\boldsymbol{\mathcal{D}}}_s^T(t) \end{bmatrix}^T$, and

$$\tilde{\mathbf{Z}}_s(t) = \begin{bmatrix} \{\mathbf{X}_{1_s}(t) - \hat{\mathbf{X}}_{1_s}(t)\}^T & \{\mathbf{X}_{2_s}(t) - \hat{\mathbf{X}}_{2_s}(t)\}^T & \{\boldsymbol{\mathcal{D}}_s(t) - \hat{\boldsymbol{\mathcal{D}}}_s(t)\}^T \end{bmatrix}^T$$

Given next is the stochastic Lyapunov based approach to adapt for the process noise covariance and the control gains so that the error dynamics given in Eq. (30) is exponentially stable in the first moment and bounded stable in higher order moments. The proposed stochastic Lyapunov based approach is based on the stable adaptive reference trajectory modification scheme presented in Ref. 26.

Notice that the error dynamics is exponentially stable in the mean if the estimation error $\tilde{\mathbf{Z}}_s(t)$, estimated states $\hat{\mathbf{Z}}_s(t)$, $\tilde{\mathcal{K}}(t)$, and $\boldsymbol{\delta}(t)$ are exponentially stable in the mean. Consider the following desired estimator dynamics given in Eq. (24)

$$\dot{\hat{\mathbf{Z}}}_m(t) = \{F_m - B_m\mathcal{K}_m\}\hat{\mathbf{Z}}_m(t) + L(t)\boldsymbol{\mathcal{V}}(t)$$

Notice that the stochastic process $\hat{\mathbf{Z}}_m(t)$ is exponentially stable in the mean. Therefore, if $\tilde{\mathbf{Z}}_{m_s}(t) = \begin{bmatrix}\hat{\mathbf{Z}}_m(t) - \hat{\mathbf{Z}}_s(t)\end{bmatrix}$ is bounded in the mean, then $\hat{\mathbf{Z}}_s(t)$ is also bounded in the mean. Now if $\tilde{\mathbf{Y}}_s(t)$ can be shown to be stable in the mean, then the plant in Eq. (28) is BIBO stable in the mean and thus the exponential stability of the controlled plant's mean response can be obtained based on the controllability and observability assumption. Finally, along with the exponential mean stability of $\mathbf{Z}_s(t)$ and $\hat{\mathbf{Z}}_s(t)$, if the exponential mean stability of $\tilde{\mathcal{K}}(t)$ and $\boldsymbol{\delta}(t)$ can be proved, then the error dynamics given in Eq. (30) is exponentially stable in the first moment and bounded stable in higher order moments.

Consider the following non-negative function

$$\begin{aligned}
V(t) = &\int_{t_0}^{t} \left[\tilde{\mathbf{Y}}_s^T(\tau)L_{ss}^T L_{ss}\tilde{\mathbf{Y}}_s(\tau) + \tilde{\mathbf{Z}}_{m_s}^T(\tau)\tilde{\mathbf{Z}}_{m_s}(\tau) + \delta^T(\tau)\delta(\tau)\right] d\tau + \mathrm{Tr}\left\{G\Delta Q(t)G^T\right\} + \mathrm{Tr}\{\tilde{K_1}^T(t)\Gamma_1\tilde{K_1}(t)\} + \\
&\int_{t_0}^{t} \left[\mathrm{Tr}\left\{\tilde{K_1}(\tau)\delta(\tau)\delta^T(\tau)\tilde{K_1}^T(\tau) + \tilde{K_2}(\tau)\delta(\tau)\delta^T(\tau)\tilde{K_2}^T(\tau)\right\}\right] d\tau + \mathrm{Tr}\{\tilde{K_2}^T(t)\Gamma_2\tilde{K_2}(t) + \tilde{K}_{\boldsymbol{\mathcal{D}}}^T(t)\Gamma_{\boldsymbol{\mathcal{D}}}\tilde{K}_{\boldsymbol{\mathcal{D}}}(t)\}
\end{aligned} \tag{31}$$

where $\Delta Q(t) = Q^* - Q(t)$ and $Q^* \geq Q(t)$, $\forall t \geq t_0$. Note that Q^* would result in exponentially decaying estimation errors and L_{ss} is the steady-state Kalman gain corresponding to the initial process noise covariance,

$Q(t_0)$. Now $\mathfrak{L}V(t)$ can be calculated as

$$\mathfrak{L}V(t) = \tilde{\mathbf{y}}_s^T(t)L_{ss}^T L_{ss}\tilde{\mathbf{y}}_s(t) + \tilde{\mathbf{Z}}_{m_s}^T(t)\tilde{\mathbf{Z}}_{m_s}(t) - \text{Tr}\left\{G\dot{Q}(t)G^T + \delta(t)\delta^T(t) + \tilde{K}_1(t)\delta(t)\delta^T(t)\tilde{K}_1{}^T(t)\right\} + \\ \text{Tr}\left\{\tilde{K}_2(t)\delta(t)\delta^T(t)\tilde{K}_2{}^T(t) + 2\tilde{K}_1{}^T(t)\Gamma_1\dot{\tilde{K}}_1(t) + 2\tilde{K}_2{}^T(t)\Gamma_2\dot{\tilde{K}}_2(t) + 2\tilde{K}_{\mathcal{D}}^T(t)\Gamma_{\mathcal{D}}\dot{\tilde{K}}_{\mathcal{D}}(t)\right\} \tag{32}$$

Now if the following adaptive laws are selected

$$\dot{Q}(t) = \gamma G^T[L_{ss}\tilde{\mathbf{y}}_s(t)\tilde{\mathbf{y}}_s^T(t)L_{ss}^T + \tilde{\mathbf{Z}}_{m_s}(t)\tilde{\mathbf{Z}}_{m_s}^T(t)]G \tag{33}$$

$$\dot{\tilde{K}}_1(t) = -\frac{1}{2}\Gamma_1^{-1}\boldsymbol{\delta}^T(t)\boldsymbol{\delta}(t)\tilde{K}_1(t) \tag{34}$$

$$\dot{\tilde{K}}_2(t) = -\frac{1}{2}\Gamma_2^{-1}\boldsymbol{\delta}^T(t)\boldsymbol{\delta}(t)\tilde{K}_2(t) \tag{35}$$

$$\dot{\tilde{K}}_{\mathcal{D}}(t) = -\frac{1}{2}\Gamma_{\mathcal{D}}^{-1}\tilde{K}_{\mathcal{D}}^{-T}(t)\boldsymbol{\delta}(t)\boldsymbol{\delta}^T(t) \tag{36}$$

then we have $\mathfrak{L}V(t) \leq 0$. Thus $V(t)$ is a supermartingale and

$$\mathbb{P}\Big(\sup_{\infty > t \geq t_0} V(t) \geq \lambda\Big) \leq \frac{V(t_0)}{\lambda}$$

Therefore

$$\lim_{t\to\infty}\int_{t_0}^{t} \tilde{\mathbf{Y}}_s^T(\tau)L_{ss}^T L_{ss}\tilde{\mathbf{Y}}_s(\tau)d\tau < \infty \quad \text{a.s.} \Rightarrow \lim_{t\to\infty}\tilde{\mathbf{Y}}_s(t) = 0 \quad \text{a.s.,}$$

$$\lim_{t\to\infty}\int_{t_0}^{t} \tilde{\mathbf{Z}}_{m_s}^T(\tau)\tilde{\mathbf{Z}}_{m_s}(\tau)d\tau < \infty \quad \text{a.s.} \Rightarrow \lim_{t\to\infty}\tilde{\mathbf{Z}}_{m_s}(t) = 0 \quad \text{a.s. and}$$

$$\lim_{t\to\infty}\int_{t_0}^{t} \boldsymbol{\delta}^T(\tau)\boldsymbol{\delta}(\tau)d\tau < \infty \quad \Rightarrow \lim_{t\to\infty}\boldsymbol{\delta}(t) = 0$$

Also $\Delta Q(t)$, $\tilde{K}_1(t)$, $\tilde{K}_2(t)$, and $\tilde{K}_{\mathcal{D}}(t)$ are bounded. Now based on the results obtained from Theorem 1, one could conclude that the dynamics in Eq. (30) is exponentially stable in the first moment and bounded in all higher order moments.

IV. Numerical Simulations

A detailed investigation of the above mentioned adaptive algorithm through numerical simulations is given in this section. For simulation purposes, we consider a two degree of freedom helicopter that pivots about the pitch axis by angle θ and about the yaw axis by angle ψ. After linearizing about $\theta_0 = \psi_0 = \dot{\theta}_0 = \dot{\psi}_0 = 0$, the helicopter equations of motion can be written as

$$(J_{eq,p} + m_{heli}l_{cm}^2)\ddot{\theta}(t) = K_{pp}V_{m,p}(t) + K_{py}V_{m,y}(t) - B_p\dot{\theta}(t) + W_1(t) \tag{37a}$$

$$(J_{eq,y} + m_{heli}l_{cm}^2)\ddot{\psi}(t) = K_{yy}V_{m,y}(t) + K_{yp}V_{m,p}(t) - B_y\dot{\psi}(t) + W_2(t) \tag{37b}$$

A detailed description of system parameters and assumed values are given in Table 1. The input to the system are the input voltages of the pitch and yaw motors, $V_{m,p}$ and $V_{m,y}$, respectively. Let $\mathbf{X}_1(t) = \begin{bmatrix}\theta(t) & \psi(t)\end{bmatrix}^T$, $\mathbf{X}_2(t) = \begin{bmatrix}\dot{\theta}(t) & \dot{\psi}(t)\end{bmatrix}^T$, and $\mathbf{W}(t) = \begin{bmatrix}W_1(t) & W_2(t)\end{bmatrix}^T$. For simulation purposes, the external disturbance $\mathbf{W}(t)$ is selected to be

$$\begin{aligned}\dot{W}_1(t) &= -W_1(t) + 2W_2(t) + \mathcal{V}_1(t)\\ \dot{W}_2(t) &= W_1(t) - 3W_2(t) + \mathcal{V}_2(t)\end{aligned} \tag{38}$$

Table 1. Two Degree-of-Freedom Helicopter Model Parameters

System Parameter	Description	Assumed Values	True Values	Unit
B_p	Equivalent viscous damping about pitch axis	0.8000	0.6000	N/V
B_y	Equivalent viscous damping about yaw axis	0.3180	-0.1590	N/V
$J_{eq,p}$	Total moment of inertia about yaw pivot	0.0384	0.0384	$Kg{\cdot}m^2$
$J_{eq,y}$	Total moment of inertia about pitch pivot	0.0432	0.0432	$Kg{\cdot}m^2$
K_{pp}	Trust torque constant acting on pitch axis from pitch motor/propeller	0.2040	0.2040	$N{\cdot}m/V$
K_{py}	Trust torque constant acting on pitch axis from yaw motor/propeller	0.0068	0.0068	$N{\cdot}m/V$
K_{yp}	Trust torque constant acting on yaw axis from pitch motor/propeller	0.0219	0.0219	$N{\cdot}m/V$
K_{yy}	Trust torque constant acting on yaw axis from yaw motor/propeller	0.0720	0.0720	$N{\cdot}m/V$
m_{heli}	Total mass of the helicopter	1.3872	1.3872	Kg
l_{cm}	Location of center-of-mass	0.1860	0.1860	m

and

$$\begin{bmatrix}\mathcal{V}_1(t)\\ \mathcal{V}_2(t)\end{bmatrix} = \boldsymbol{\mathcal{V}}(t) \sim \mathcal{N}\big(\mathbf{0}, 1\times 10^{-3} I_{2\times 2}\delta(\tau)\big)$$

Now the state-space representation of the above system is

$$\begin{aligned}\dot{\mathbf{X}}_1(t) &= \mathbf{X}_2(t)\\ \dot{\mathbf{X}}_2(t) &= A_2\mathbf{X}_2(t) + \mathbf{u}(t) + \mathbf{W}(t)\end{aligned} \tag{39}$$

where

$$A_2 = \begin{bmatrix} a_1 & 0\\ 0 & a_2\end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} b_1 V_{m,p}(t) + b_2 V_{m,y}(t)\\ b_3 V_{m,p}(t) + b_4 V_{m,y}(t)\end{bmatrix}$$

and the system parameters are given as

$$a_1 = \frac{-B_p}{(J_{eq,p} + m_{heli} l_{cm}^2)} \qquad a_2 = \frac{-B_y}{(J_{eq,y} + m_{heli} l_{cm}^2)}$$

$$b_1 = \frac{K_{pp}}{(J_{eq,p} + m_{heli} l_{cm}^2)} \qquad b_2 = \frac{K_{py}}{(J_{eq,p} + m_{heli} l_{cm}^2)}$$

$$b_3 = \frac{K_{yp}}{(J_{eq,y} + m_{heli} l_{cm}^2)} \qquad b_4 = \frac{K_{yy}}{(J_{eq,y} + m_{heli} l_{cm}^2)}$$

The state-space representation of the assumed system model is

$$\begin{aligned}\dot{\mathbf{x}}_{1_m}(t) &= \mathbf{x}_{2_m}(t)\\ \dot{\mathbf{x}}_{2_m}(t) &= A_{2_m}\mathbf{x}_{2_m}(t) + \mathbf{u}(t)\end{aligned}$$

where

$$A_{2_m} = \begin{bmatrix} a_{1_m} & 0\\ 0 & a_{2_m}\end{bmatrix}$$

The measured output equations are given as

$$\mathbf{Y}(t) = C\mathbf{X}(t) + \mathbf{V}(t)$$

where $\mathbf{X}(t) = \begin{bmatrix}\mathbf{X}_1^T(t) & \mathbf{X}_2^T(t)\end{bmatrix}^T$ and $C = \begin{bmatrix}I_{2\times 2} & 0_{2\times 2}\end{bmatrix}$. Notice that the disturbance term, $\boldsymbol{\mathcal{D}}(t) = [\mathcal{D}_{\dot{\theta}}(t) \quad \mathcal{D}_{\dot{\psi}}(t)]^T$, can be written as

$$\mathcal{D}_{\dot{\theta}}(t) = \triangle a_1 \dot{\theta}(t) + W_1(t)$$
$$\mathcal{D}_{\dot{\psi}}(t) = \triangle a_2 \dot{\psi}(t) + W_2(t)$$

The assumed disturbance term dynamics is modeled as

$$\dot{\mathcal{D}}_{\dot{\theta}_m}(t) = -\mathcal{D}_{\dot{\theta}_m}(t) + \mathcal{W}_1(t)$$
$$\dot{\mathcal{D}}_{\dot{\psi}_m}(t) = -3\mathcal{D}_{\dot{\psi}_m}(t) + \mathcal{W}_2(t)$$

Let the extended assumed state vector be, $\mathbf{Z}_m(t) = \begin{bmatrix}\mathbf{X}_m^T(t) & \mathcal{D}_{\dot{\theta}_m}(t) & \mathcal{D}_{\dot{\psi}_m}(t)\end{bmatrix}^T$. Now the assumed extended state-space equation can be written as

$$\dot{\mathbf{Z}}_m(t) = F_m \mathbf{Z}_m(t) + B_m \mathbf{u}(t) + G\boldsymbol{\mathcal{W}}(t)$$

where $F_m = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_{1_m} & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{2_m} & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$, $B_m = \begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \\ 0_{2\times 2} \end{bmatrix}$, $G = \begin{bmatrix} 0_{4\times 2} \\ I_{2\times 2} \end{bmatrix}$, and $\boldsymbol{\mathcal{W}} = \begin{bmatrix} \mathcal{W}_1(t) \\ \mathcal{W}_2(t) \end{bmatrix}$. The estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m \hat{\mathbf{Z}}(t) + B_m \mathbf{u}(t) + L(t)[C\mathbf{X}(t) - H\hat{\mathbf{Z}}(t)] + L(t)\mathbf{V}(t) \tag{40}$$

where $H = [C \quad 0_{2\times 2}]$. The nominal controller is a linear quadratic regulator which minimizes the cost function

$$J = \frac{1}{2} E\left[\int_0^\infty \big((\mathbf{X}(t) - \mathbf{x}_d)^T \mathcal{Q}_{\mathbf{X}} (\mathbf{X}(t) - \mathbf{x}_d) + \mathbf{u}^T(t) \mathcal{R}_{\mathbf{u}} \mathbf{u}(t)\big) dt\right] \tag{41}$$

where $\mathbf{x}_d^T = [\theta_d \quad \psi_d \quad 0 \quad 0]$, θ_d and ψ_d are some desired final values of θ and ψ, respectively, and $\mathcal{Q}_{\mathbf{X}}$ and $\mathcal{R}_{\mathbf{u}}$ are two symmetric positive definite matrices. The nominal control that minimizes the above cost function is

$$\bar{\mathbf{u}}(t) = -\begin{bmatrix} K_{1_m} & K_{2_m} \end{bmatrix} \left\{ \hat{\mathbf{X}}(t) - \mathbf{x}_d \right\}$$

where K_m is the feedback gain that minimizes the cost Eq. (41). Now the total control law can be written in terms of the estimated states and the estimated disturbance term as

$$\mathbf{u}(t) = -\begin{bmatrix} K_{1_m} & K_{2_m} & I_{2\times 2} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}}(t) - \mathbf{x}_d \\ \hat{\mathcal{D}}_{\dot{\theta}}(t) \\ \hat{\mathcal{D}}_{\dot{\psi}}(t) \end{bmatrix} = S\hat{\mathbf{Z}}(t) + K_m \mathbf{x}_d$$

Table 2 shows the nominal controller and estimator matrices. Since the measurement noise covariance, R, can be obtained from sensor calibration, the process noise matrix, Q, is treated as a tuning parameter. Based on the weighting matrices given in Table 2, the feedback gains are calculated to be

$$K_{1_m} = \begin{bmatrix} 22.2649 & 2.0680 \\ -2.0680 & 22.2649 \end{bmatrix} \quad \text{and} \quad K_{2_m} = \begin{bmatrix} 7.6295 & 0.9156 \\ -0.3360 & 8.8047 \end{bmatrix}$$

For simulation purposes the initial states are selected to be $[\theta_0 \quad \psi_0 \quad \dot{\theta}_0 \quad \dot{\psi}_0]^T = [-45^\circ \quad 0 \quad 0 \quad 0]^T$ and the desired states θ_d and ψ_d are selected to be 50° and 30°, respectively.

Table 2. Nominal Controller/Estimator Matrices

LQR Weighting Matrices	Covariance Matrices
$\mathcal{R}_{\mathbf{u}} = I_{2\times 2}$	$Q(t_0) = \begin{bmatrix} 1\times 10^{-3} & 0 \\ 0 & 1\times 10^{-2} \end{bmatrix}, Q(t) = \begin{bmatrix} q_1(t) & q_2(t) \\ q_3(t) & q_3(t) \end{bmatrix},$
$\mathcal{Q}_{\mathbf{x}} = \begin{bmatrix} 500\times I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 100\times I_{2\times 2} \end{bmatrix}$	$R = 1\times 10^{-5}\times I_{2\times 2},\ P(t_0) = \begin{bmatrix} 1\times 10^{-3}\times I_{4\times 4} & 0_{4\times 2} \\ 0_{2\times 4} & I_{2\times 2} \end{bmatrix}$

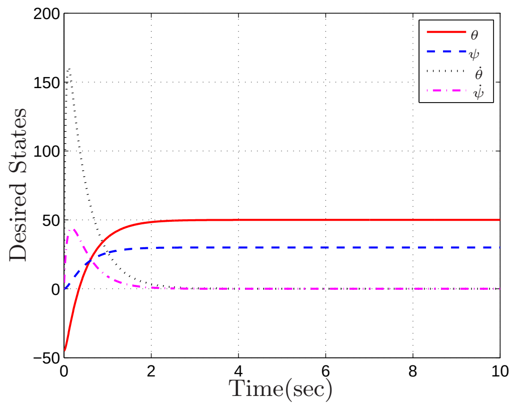
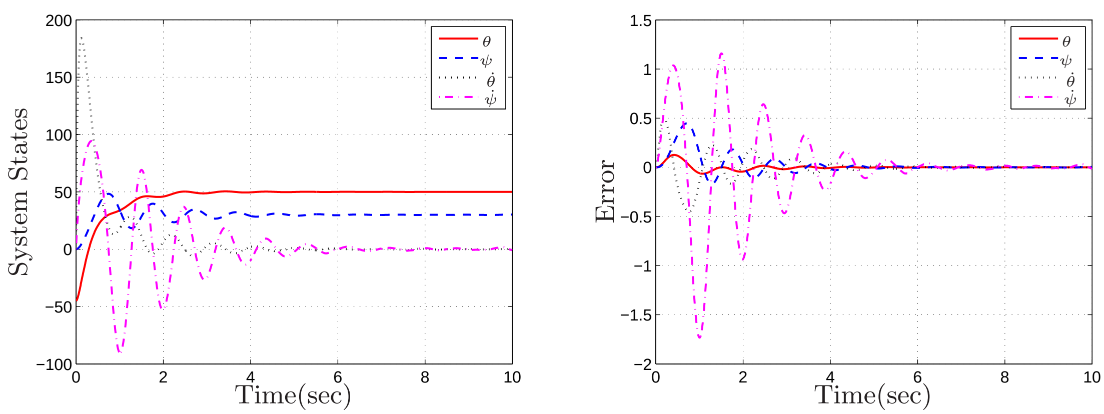


Figure 1. Desired System Response

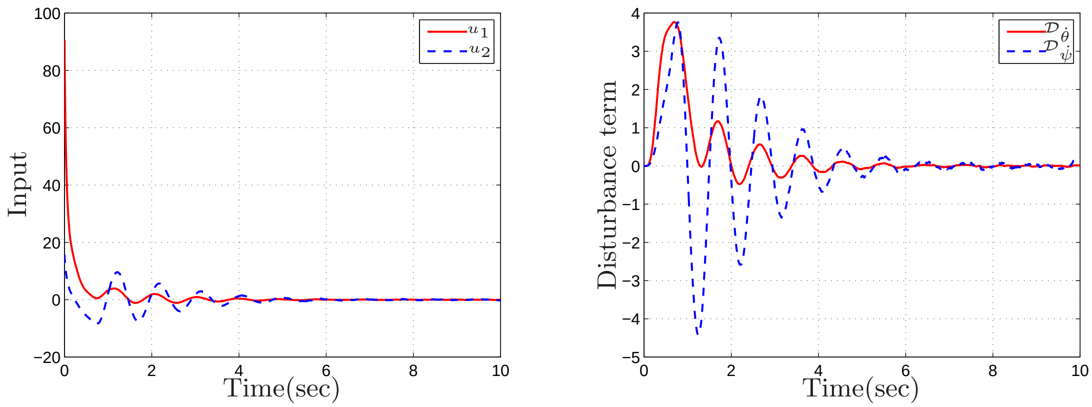
The desired response given in Fig. 1 is the system response to nominal control when there is no model error and external disturbance. For the first simulation results presented here, we consider the system response to the adaptive disturbance accommodating control when there is no saturation. Figure 2(a) shows the actual system response obtained for the first simulation when there is no actuator saturation. Figure 2(b) shows the error corresponding to the difference between the desired states given in Fig. 1 and the true states given in Fig. 2(a) for the first simulation scenario. Figure 3(a) shows the input corresponding to the first simulation scenario and Fig. 3(b) shows the estimated disturbance term obtained for the first simulation. Finally, given in Fig. 4 is the adaptive process noise covariance obtained for the first simulation.



(a) States (b) Error

Figure 2. Unsaturated Adaptive Disturbance Accommodating Control Results: Actual States and Error

For the second simulation results presented here, we consider the system response to the adaptive dis-



(a) Input

(b) Disturbance term

Figure 3. Unsaturated Adaptive Disturbance Accommodating Control Results: Control Input and Estimated Disturbance term

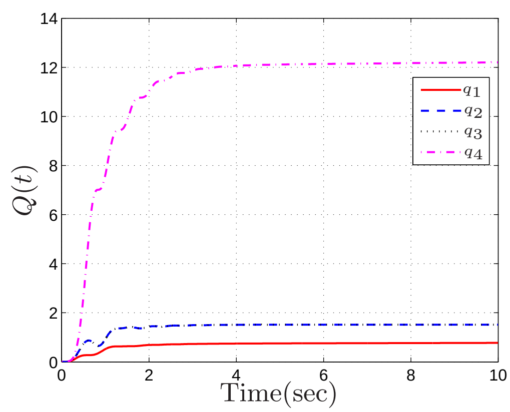


Figure 4. Unsaturated Adaptive Disturbance Accommodating Control Results: Adaptive Q(t)

turbance accommodating control with saturation, i.e.,

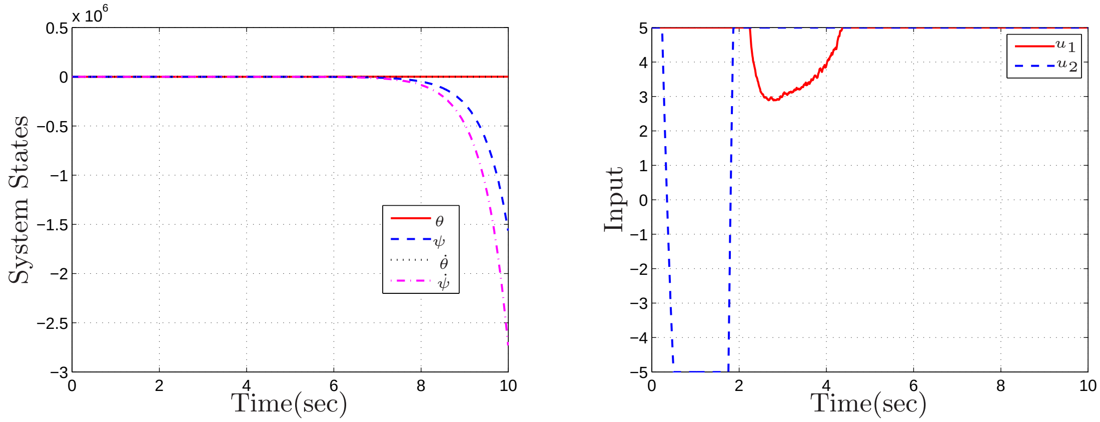
$$u_{a_1}(t) = \begin{cases} 5 \text{ if } u_1(t) > 5 \\ u_1(t) \text{ if } -5 \le u_1(t) \le 5 \\ -5 \text{ if } -5 > u_1(t) \end{cases} \quad \& \quad u_{a_2}(t) = \begin{cases} 5 \text{ if } u_2(t) > 5 \\ u_2(t) \text{ if } -5 \le u_2(t) \le 5 \\ -5 \text{ if } -5 > u_2(t) \end{cases}$$

Figure 5(a) shows the actual system response obtained for the second simulation when there is control saturation. Figure 5(b) shows the input corresponding to the second simulation scenario. Results shown in Fig. 5 indicate that the controlled system is unstable when there is actuator saturation.

For the final simulation, the saturated scenario given above is repeated with the adaptive laws given in Eqs. (33), (34), (35), and (36). The adaptive gains are selected to be $\Gamma_1 = \Gamma_2 = \Gamma_{\mathcal{D}} = 1 \times 10^2 I_{2\times 2}$. Figure 6 shows the actual system response and the error obtained for the third simulation. Figure 7(a) shows the input corresponding to the third simulation scenario and Fig. 7(b) shows the estimated disturbance term obtained for the third simulation. Given in Fig. 8 is the adaptive process noise covariance obtained for the third simulation. The adaptive feedback gain obtained for the third simulation is given in Fig. 9.

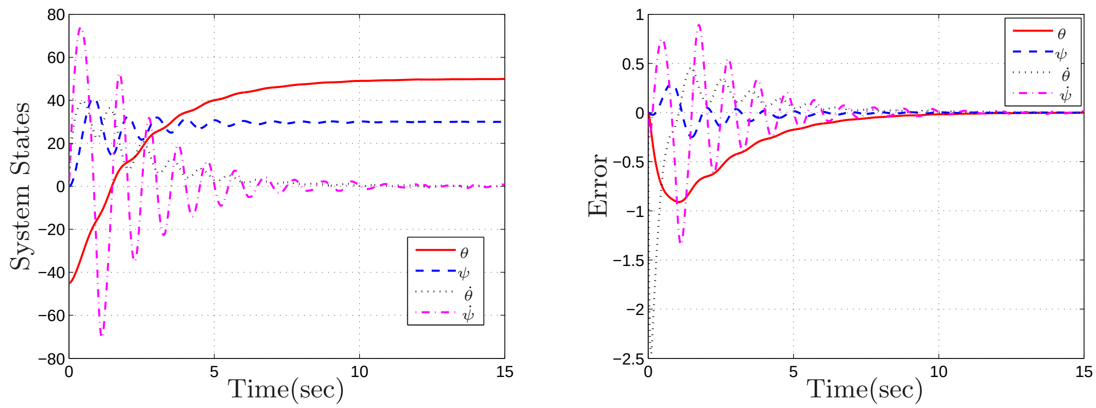
V. Conclusion

Observer based adaptive stochastic disturbance accommodating control scheme utilizes a Kalman estimator to determine the necessary corrections to the nominal control input and thus minimizes the adverse



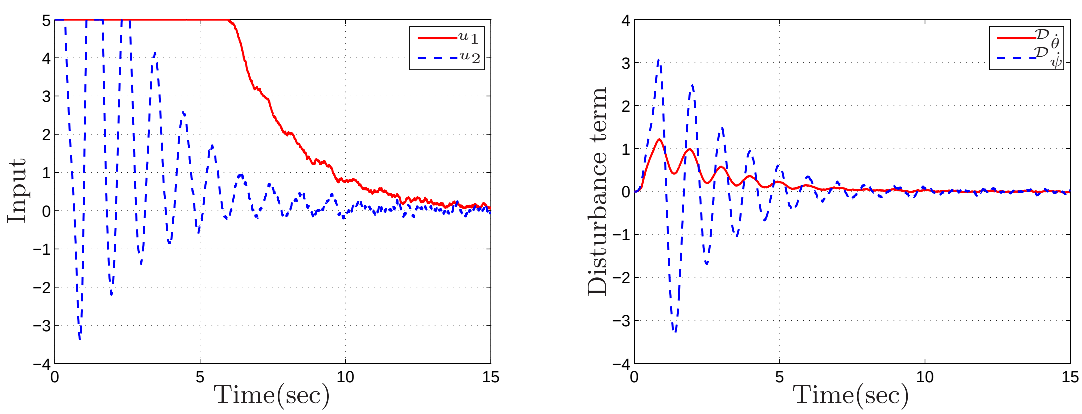
(a) States (b) Input

Figure 5. Saturated Adaptive Disturbance Accommodating Control Results: Actual States and Control Input



(a) States (b) Error

Figure 6. Saturated Adaptive Disturbance Accommodating Control with Adaptive Feedback Gain: Actual States and Error



(a) Input (b) Disturbance term

Figure 7. Saturated Adaptive Disturbance Accommodating Control with Adaptive Feedback Gain: Control Input and Estimated Disturbance term

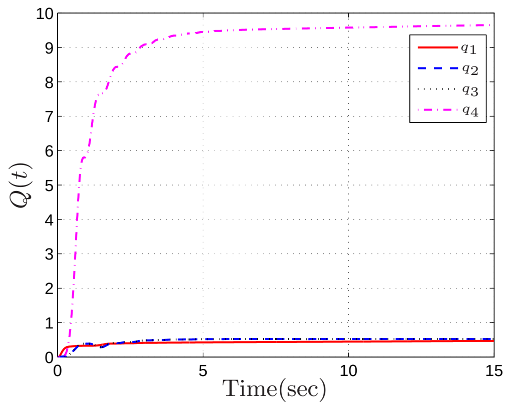
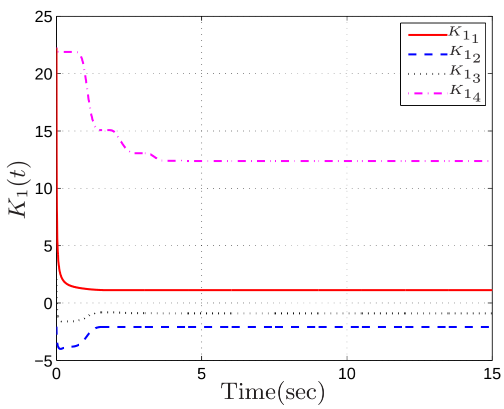
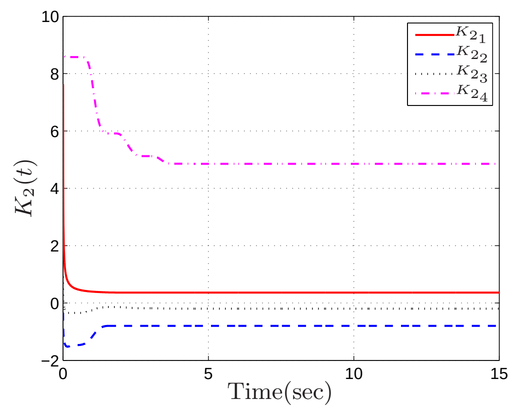


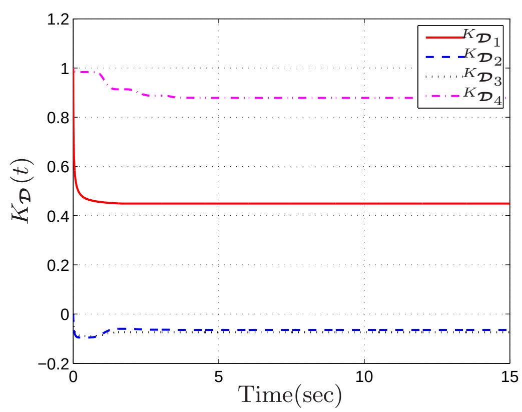
Figure 8. Saturated Adaptive Disturbance Accommodating Control with Adaptive Feedback Gain: Adaptive Q(t)



(a)



(b)



(c)

Figure 9. Adaptive Feedback Gain

effects of both model uncertainties and external disturbances on the controlled system. One of the major disadvantages of this stochastic adaptive controller is that the total control input, which includes the nominal control as well as the control corrections to the nominal control to accommodate for the disturbances, could exceed the actuator saturation limits. When the desired input exceeds the actuator limits, the controller fails to accommodate for the disturbances and this could drive the system unstable. The saturation phenomenon of the disturbance accommodating control system is more violent than that of the integral control system because of the positive feedback of the disturbance compensation. This paper presents the formulation a stochastic adaptive approach to update the control gains and the process noise covariance online so that the stability of closed-loop system under actuator saturation is guaranteed. The control approach presented here assumes that the actuator saturations are known and the saturated actuator still has enough control authority to stabilize the system. The simulation results presented here indicate that updating the control gains and the process noise covariance stabilized the controlled system while the nominal adaptive disturbance accommodating controller destabilized the saturated system.

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