Sensitivity Analysis for Constrained Relative Attitude Determination Involving Two Vehicle Formations

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In this paper a constrained relative attitude determination solution of a formation of two vehicles is considered and the effect of constraint violation on the estimation error is studied. The solution for the relative attitude between the two vehicles is obtained only using line-of-sight measurements between them and a common (unknown) object observed by both vehicles. The solution represents the minimum number of measurements required to determine the relative attitude and no ambiguities are present. The constraint used in the solution is a triangle constraint on the vector observations. This constraint represents an ideal situation, which may be violated in practice due to sensor misalignments and/or noisy measurements. A sensitivity analysis is performed in order to assess how out-of-plane observations affect the overall solution. In particular, an analytical expression for this sensitivity is derived. Simulations runs are also shown to verify the analytical expression.

I. Introduction

Most inertial navigation systems used for vehicles incorporate the Global Positioning System (GPS) along with inertial measurement units providing both inertial position and attitude. If relative information is required then these measurements must be converted to relative coordinates. Although GPS can be used to provide relative information using pseudolites, GPS and GPS-like signals are susceptible to interference and jamming, among other issues. Therefore developing GPS-less navigation systems is currently an active area of research.[1] Recent research concerning vision-based navigation for Uninhabited Air Vehicles (UAV)s indicates that relative navigation can be achieved using camera-based images. Line-of-sight (LOS) vectors between vehicles in formation can be used for relative navigation and in particular relative attitude determination. Reference 2 implements an extended Kalman filter to estimate the relative position and attitude of two air vehicles using multiple LOS measurements between them along with other onboard measurements from gyros and accelerometers. This approach has the advantage of not relying on external sensors but may require considerable onboard computations. Computing the relative attitude directly without filtering for the two-vehicle formation using LOS information between them can offer computational efficiency without reliance on filter convergence issues because point-by-point solutions are possible with deterministic methods.

The attitude is determined as the angular departure from some reference. Attitude sensors provide either arc lengths or dihedral angle observations that are known in a reference coordinate system. The angle measurements can be combined to determine entire directions. Oftentimes these directions are LOS observations to an observed object such as a star, the Sun, the Earth's magnetic field vector or landmarks. Since the attitude of an object is described by a 3×3 orthogonal rotation matrix with determinant $+1$, it has three independent parameters; two of which describe an axis and the third the rotation about this axis. Therefore at least two unit vector measurements are needed to determine the attitude. But since each unit vector contains two independent pieces of information, the attitude is over-determined in this case. Therefore

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it is convenient to divide attitude determination algorithms into two classes: 1) deterministic solutions where the minimal scalar measurements are used, and 2) over-deterministic where more than the minimal scalar measurement set is used to determine the attitude.

Many algorithms have been published to determine the attitude from two or multiple unit vectors, the most widely used of which are the TRIAD[3] and QUEST[4] algorithms. When more than the minimal set of vector observations is used to determine the attitude an optimal solution is obtained by minimizing an appropriate cost function, which was first introduced as the well known Wahba problem.[5] A purely deterministic solution for the attitude involves one direction and one angle or three angles but this case is shown to have a discrete ambiguity,[6] which needs further information to resolve. The advantages of a deterministic solution are 1) since the minimal scalar measurements are used there is no need to minimizing the cost function and 2) any deterministic algorithm will provide an optimal solution.

Using a set of LOS observations between vehicles in a three-vehicle formation has been shown to offer a deterministic solution in Ref. 2, which is not possible if each vehicle is considered separately. The observability of this relative attitude solution depends on both vehicle geometry and sensor location. It is well known that the rotation around a unit vector is unobservable when that unit vector is the only observation used for attitude determination. Reference 2 shows that having only one LOS set between each of the individual vehicles provides sufficient information to determine all relative attitudes in a three-vehicle system. An unobservable case arises when all vectors are in the same plane, e.g. they form a triangle. Reference 7 extends the previous result to a two-vehicle formation with a common observed object, which can be another vehicle or a landmark, by applying a parametric constraint to the attitude solution. This constraint is based on assuming that a triangle set of observations is given. In the work of Ref. 2 this issue causes problems in the solution, while in Ref. 7 this constraint is forced to be true and hence relieves the arising difficulties. This results in a deterministic solution for the relative attitude with no ambiguity and no observability issues.

The triangle scenario does reflect a realistic physical situation. For example, this occurs naturally when two UAVs have a common LOS between them and measure some common object other than each other, which forms a triangle of LOS observations. It is important to note that no information on the location of the object is required in the solution, only the fact that both vehicles *observe* the common object. This constitutes a significant departure from standard navigation or attitude approaches that use *known* objects or landmarks. The triangle constraint is used to determine a solution, however, due to sensor misalignments and/or noise in the measurements the actual LOS observations will not form a perfect triangle. In this paper this error will be studied by deriving an analytical expression of the error sensitivity for out-of-plane vectors.

The organization of this paper is as follows. First, the configuration for the constrained observation geometry and a description of the sensor model are given. Then, the constrained relative attitude solution is summarized. Next, a review of quaternions is provided. Then, a sensitivity expression to out-of-plane deflections is derived. Finally, simulation results are shown for a static formation.

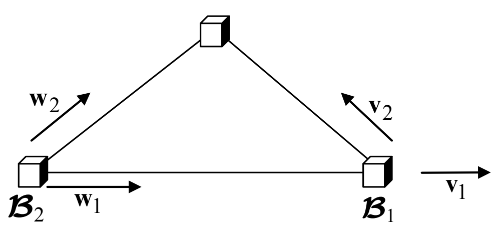

Figure 1. Observation Geometry

II. Configuration and Sensor Model

Figure 1 shows the configuration and observations used for the solution of the relative attitude from frame $\mathcal{B}_1$ to frame $\mathcal{B}_2$. The vector $\mathbf{w}_1$ is the LOS observation from $\mathcal{B}_2$ to $\mathcal{B}_1$ expressed in $\mathcal{B}_2$ coordinates. The vector $\mathbf{v}_1$ is the LOS observation from $\mathcal{B}_2$ to $\mathcal{B}_1$ expressed in $\mathcal{B}_1$ coordinates; note, in practice the negative of this vector is measured. The vector $\mathbf{w}_2$ is the LOS observation from $\mathcal{B}_2$ to the common object expressed in $\mathcal{B}_2$ coordinates. Finally, the vector $\mathbf{v}_2$ is the LOS observation from $\mathcal{B}_1$ to the common object expressed in $\mathcal{B}_1$ coordinates.

Line-of-sight observations between multiple vehicles can be obtained using standard light-beam and focal-plane-detector technology. One such system is the vision-based navigation (VISNAV) system,[8] which consists of a position sending diode (PSD) as the focal plane that captures incident light from a beacon omitted from a neighboring vehicle from which a LOS vector can be determined. The light source is such that the system can achieve selective vision. This sensor have the advantage of having a small size and a very wide field-of-view (FOV).[9] The measurement can be expressed as coordinates in the PSD focal plane, denoted by α and β. The focal plane coordinates can be written in a 2×1 vector $\mathbf{m} \equiv [\alpha \ \ \beta]^T$ and the measurement model follows

$$\tilde{\mathbf{m}} = \mathbf{m} + \mathbf{w} \tag{1}$$

A typical noise model used to describe the uncertainty in the focal-plane coordinate observations is given as

$$\mathbf{w} \sim \mathcal{N}\left(\mathbf{0}, R^{\text{FOCAL}}\right) \tag{2a}$$

$$R^{\text{FOCAL}} = \frac{\sigma^2}{1 + d\left(\alpha^2 + \beta^2\right)} \begin{bmatrix} \left(1 + d\alpha^2\right)^2 & \left(d\alpha\beta\right)^2 \\ \left(d\alpha\beta\right)^2 & \left(1 + d\beta^2\right)^2 \end{bmatrix} \tag{2b}$$

where σ^2 is the variance of the measurement errors associated with α and β, and d is on the order of 1. The covariance for the focal plane measurements is a function of the true values and this covariance realistically increases as the distance from the boresight increases. The measurement error associated with the focal plane measurements results in error in the measured LOS vector. A general sensor LOS observation can be expressed in unit vector form given by

$$\mathbf{b} = \frac{1}{\sqrt{f + \alpha^2 + \beta^2}} \begin{bmatrix} \alpha \\ \beta \\ f \end{bmatrix} \tag{3}$$

where f denotes the focal length. The LOS observation has two independent parameters α and β. Therefore in the presence of random noise in these parameters the LOS vector still must maintain a unit norm. Although the LOS measurement noise must lie on the unit sphere we can approximate the measurement noise as additive noise, given by

$$\tilde{\mathbf{b}} = \mathbf{b} + \boldsymbol{\upsilon} \tag{4}$$

with

$$\boldsymbol{\upsilon} \sim \mathcal{N}\left(\mathbf{0}, \Omega\right) \tag{5}$$

where $\boldsymbol{\upsilon}$ is assumed to be a Gaussian random vector with zero mean and covariance Ω. Shuster[4] has shown that the probability density for unit vector measurements lies on a sphere and can accurately be approximated by a density on a plane tangent to the vector for a small FOV sensors. This approximation is known as the QUEST measurement model,[4] which characterizes the LOS noise process resulting from the focal plane model as

$$\Omega \equiv E\left\{\boldsymbol{\upsilon}\boldsymbol{\upsilon}^T\right\} = \sigma^2\left(I_{3\times 3} - \mathbf{b}\,\mathbf{b}^T\right) \tag{6}$$

It is clear that this is only valid for a small FOV in which a tangent plane closely approximates the surface of a unit sphere. For wide FOV sensors, a more accurate measurement covariance is shown in Ref. 10. This formulation employs a first-order Taylor series approximation about the focal-plane axes. The partial derivative operator is used to linearly expand the focal-plane covariance in Eq. (2), given by

$$J = \frac{\partial \mathbf{b}}{\partial \mathbf{m}} = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{1 + \alpha^2 + \beta^2} \mathbf{b}\,\mathbf{m}^T \tag{7}$$

Then the wide-FOV covariance model is given by

$$\Omega = J\, R^{\text{FOCAL}} J^T \tag{8}$$

If a small FOV model is valid, then Eq. (8) can still be used, but is nearly identical to Eq. (6). For both equations, Ω is a 3×3 covariance matrix for a unit vector measurement with two independent parameters and therefore must be singular. A nonsingular covariance matrix for the LOS measurements can be obtained by a rank-one update to Ω:

$$\Omega_{\text{new}} = \Omega + \frac{1}{2}\text{trace}\left(\Omega\right)\mathbf{b}\mathbf{b}^T \tag{9}$$

which can be used without loss in generality to develop attitude-error covariance expressions.[2] Equation (8) represents the covariance for the LOS measurements in their respective body frame. The four measurements, using notation defined by Figure 1 instead of $\mathbf{b}$, and their respective covariances are summarized by

$$\tilde{\mathbf{w}}_1 = \mathbf{w}_1 + \boldsymbol{\upsilon}_{w1}, \quad \boldsymbol{\upsilon}_{w1} \sim \mathcal{N}\left(\mathbf{0}, R_{w_1}\right) \tag{10a}$$

$$\tilde{\mathbf{w}}_2 = \mathbf{w}_2 + \boldsymbol{\upsilon}_{w2}, \quad \boldsymbol{\upsilon}_{w2} \sim \mathcal{N}\left(\mathbf{0}, R_{w_2}\right) \tag{10b}$$

$$\tilde{\mathbf{v}}_1 = \mathbf{v}_1 + \boldsymbol{\upsilon}_{v1}, \quad \boldsymbol{\upsilon}_{v1} \sim \mathcal{N}\left(\mathbf{0}, R_{v_1}\right) \tag{10c}$$

$$\tilde{\mathbf{v}}_2 = \mathbf{v}_2 + \boldsymbol{\upsilon}_{v2}, \quad \boldsymbol{\upsilon}_{v2} \sim \mathcal{N}\left(\mathbf{0}, R_{v_2}\right) \tag{10d}$$

Since in practice each vehicle will have their own set of LOS measurement devices, then the measurements in Eq. (10a) can be assumed to be uncorrelated.

III. Constrained Solution

This section summarizes the constrained attitude solution. More details can be found in Ref. 7. Considering the measurements shown in Figure 1, to determine the full attitude between the $\mathcal{B}_2$ and $\mathcal{B}_1$ frames the attitude matrix must satisfy the following measurement equations:

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{11a}$$

$$d = \mathbf{w}_2^T A\mathbf{v}_2 \tag{11b}$$

We assume that $|d| \leq 1$; otherwise a solution will not exist. Here it is assumed that the LOS vectors $\mathbf{v}_1$ and $\mathbf{w}_1$ are parallel. Also note that from Figure 1 no observation information is required from the third object to either $\mathcal{B}_1$ or $\mathcal{B}_2$. Hence, no information such as position is required for this object to determine the relative attitude. A solution for the attitude satisfying Eq. (11) is discussed in Ref. 6 and will be utilized to form a solution for the constrained problem discussed here. The solution for the rotation matrix that satisfies Eq. (11) can be found by first finding a rotation matrix that satisfies that first equation and then finding the angle that one must rotate about the reference direction to align the two remaining vectors such that their dot product is equivalent to that measured in the remaining frame in the formation. The first rotation can be found by rotating about any direction by any angle, where $B = R\left(\mathbf{n}_1, \theta\right)$ is a general rotation about some axis rotation that satisfies Eq. (11a). The choice of the initial rotation axis is arbitrary, here the vector between the two reference direction vectors is used and the rotation is as follows:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times3} \tag{12}$$

where $\mathbf{n}_1 = (\mathbf{w}_1 + \mathbf{v}_1)$ and $\theta = \pi$. This rotation matrix will align the LOS vectors between frames, but the frames could still have some rotation about this vector, so therefore the angle about this axis must be determined to solve the second equation. To do so the vector $\mathbf{w}^*$ is first defined, which is the vector produced after applying the rotation B on the vector $\mathbf{v}_2$. This will allow us to determine the second rotation needed to map $\mathbf{v}_2$ properly to the $\mathcal{B}_2$ frame with $\mathbf{w}^* = B\,\mathbf{v}_2$. Since the rotation axis is the $\mathbf{w}_1$ vector, this vector will be invariant under this transformation and the solution to the full attitude can be written as $A = R\left(\mathbf{n}_2, \theta\right)B$.

Consider solving for the rotation angle using the planar constraint, the constraint can be written as the following:

$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times]R\left(\mathbf{n}_2, \theta\right)\mathbf{w}^* \tag{13}$$

Substituting the second rotation matrix into Eq. (13), and with $\mathbf{n}_2 = \mathbf{w}_1$, leads to

$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times][\mathbf{w}_1\mathbf{w}_1^T - \cos(\theta)[\mathbf{w}_1\times]^2\mathbf{w}^* - \sin(\theta)[\mathbf{w}_1\times]\mathbf{w}^*] \tag{14}$$

Expanding out this expression we can write

$$\left(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*\right)\cos(\theta) = \left(\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*\right)\sin(\theta) \tag{15}$$

Notice that if we divide Eq. (15) by -1 the equation would be unchanged but the solution for the angle θ would differ by π. Therefore, using the planar constraint the solution for the angle θ can be written as $\theta = \beta + \phi$, where

$$\beta = \operatorname{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) \tag{16}$$

and $\phi = 0$ or π. An ambiguity exists when using this approach but it is important to note that one of the possible solutions for this approach is equivalent to the triangle constraint case.

Finally the solution for the attitude is given by $A = R(\mathbf{w}_1, \theta)\, B$. The solution is now summarized:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times 3} \tag{17a}$$

$$R(\mathbf{w}_1, \theta) = I_{3\times 3}\cos(\theta) + (1 - \cos(\theta))\mathbf{w}_1\mathbf{w}_1^T - \sin(\theta)[\mathbf{w}_1\times] \tag{17b}$$

$$\theta = \operatorname{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \pi \tag{17c}$$

$$A = R(\mathbf{w}_1, \theta)\, B \tag{17d}$$

This result shows that for any formation of two vehicles a deterministic solution will exist using one direction and one angle. Due to the fact that our case is truly deterministic there is no need to minimize a cost function and the solution will always be the maximum likelihood one. It is very important to note that without the resolution of the attitude ambiguity any covariance development might not have any meaning since although the covariance might take a small value if the wrong possible attitude is used then the error might be fairly large and not bounded by the attitude covariance.

The solution in Eq. (17) can be rewritten without the use of any transcendental functions. The following relationships can be derived:

$$\cos(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{18a}$$

$$\sin(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{18b}$$

This leads to $\cos(\theta) = -b/c$ and $\sin(\theta) = -a/c$ with

$$a = \mathbf{w}_2^T[\mathbf{w}_1\times]\left([\mathbf{w}_1\times] + [\mathbf{v}_1\times]\right)[\mathbf{v}_1\times]\mathbf{v}_2 \tag{19a}$$

$$b = \mathbf{w}_2^T[\mathbf{w}_1\times]\left([\mathbf{w}_1\times][\mathbf{v}_1\times] - I_{3\times 3}\right)[\mathbf{v}_1\times]\mathbf{v}_2 \tag{19b}$$

$$c = (1 + \mathbf{v}_1^T\mathbf{w}_1)\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\| \tag{19c}$$

Note that $c = \sqrt{a^2 + b^2}$. Then the matrix R is given by

$$R = -\frac{b}{c}\, I_{3\times 3} + \left(1 + \frac{b}{c}\right)\mathbf{w}_1\mathbf{w}_1^T + \frac{a}{c}\,[\mathbf{w}_1\times] \tag{20}$$

Noting that $\mathbf{w}_1\mathbf{w}_1^T B = \mathbf{w}_1\mathbf{v}_1^T$ then the solution in Eq. (17d) can be rewritten as

$$\begin{aligned} A = {} & \frac{b}{c}\left(I_{3\times 3} - \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} + \mathbf{w}_1\mathbf{v}_1^T\right) \\ & + \frac{a}{c}\,[\mathbf{w}_1\times]\left(\frac{\mathbf{v}_1\mathbf{w}_1^T + \mathbf{v}_1\mathbf{v}_1^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times 3}\right) + \mathbf{w}_1\mathbf{v}_1^T \end{aligned} \tag{21}$$

Note in practice the measured quantities from the previous section are used in place of the observed quantities shown in Eq. (17), and Eqs. (19) and (21).

The covariance matrix for an attitude estimate is defined as the covariance of a small angle rotation taking the true attitude to the estimated attitude. Typically the small Euler angles are used to parameterize the attitude error-matrix. Reference 7 derives the attitude error-covariance for the constrained solution by using the attitude matrix with respect to the small angle errors. The attitude error-covariance is given by

$$P = \left(\begin{bmatrix} -[A_{\text{true}}\mathbf{v}_2\times] \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix}\begin{bmatrix} R_{\Delta_1} & R_{\Delta_1\Delta_2} \\ R_{\Delta_1\Delta_2}^T & R_{\Delta_2} \end{bmatrix}^{-1}\begin{bmatrix} -[A_{\text{true}}\mathbf{v}_2\times] \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix}^T\right)^{-1} \tag{22}$$

where

$$R_{\Delta_1} = R_{w_1} + A_{\text{true}} R_{v_1} A_{\text{true}}^T \tag{23a}$$

$$\begin{aligned} R_{\Delta_2} &= \mathbf{w}_2^T [A_{\text{true}}\mathbf{v}_2 \times] R_{w_1} [A_{\text{true}}\mathbf{v}_2 \times] \mathbf{w}_2 + (A_{\text{true}}\mathbf{v}_2)^T [\mathbf{w}_1 \times] R_{w_2} [\mathbf{w}_1 \times] (A_{\text{true}}\mathbf{v}_2) \\ &+ \mathbf{w}_2^T [\mathbf{w}_1 \times] A_{\text{true}} R_{v_2} A_{\text{true}}^T [\mathbf{w}_1 \times] \mathbf{w}_2 \end{aligned} \tag{23b}$$

$$R_{\Delta_1 \Delta_2} = -R_{w_1} [A_{\text{true}}\mathbf{v}_2 \times] \mathbf{w}_2 \tag{23c}$$

This expression is a function of the true attitude, A_{true}, but the true attitude can effectively be replaced with the estimated attitude to within first order.

IV. Attitude Matrix Error Representation

The attitude matrix is parameterized using the quaternion, which is based on the Euler axis/angle parameterization of the attitude matrix. The quaternion is defined as

$$\mathbf{q} = \begin{bmatrix} \boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{24}$$

where $\boldsymbol{\varrho} = \mathbf{e} \sin(\theta/2)$ and $\mathbf{q}_4 = \cos(\theta/2)$; $\mathbf{e}$ is the Euler rotation axis and θ is Euler rotation angle. The quaternion must also satisfy the unit constraint $\mathbf{q}^T\mathbf{q} = 1$. The attitude matrix can be written in terms of the quaternion parameterization:

$$A = \Xi^T(\mathbf{q})\,\Psi(\mathbf{q}) \tag{25}$$

where

$$\Xi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{26a}$$

$$\Psi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{26b}$$

Also the inverse quaternion is defined by $\mathbf{q}^{-1} = [-\boldsymbol{\varrho}^T \;\; q_4]^T$ so that $A(\mathbf{q}^{-1}) = A^T(\mathbf{q})$.

Successive rotations can be represented using quaternion multiplication made in the same order as the attitude matrix multiplication:[11]

$$A(\mathbf{q}')A(\mathbf{q}) = A(\mathbf{q} \otimes \mathbf{q}') \tag{27}$$

The bilinear composition of the quaternion is $\mathbf{q} \otimes \mathbf{q}'$ and is defined by

$$\mathbf{q} \otimes \mathbf{q}' = \begin{bmatrix} -q'_4 \boldsymbol{\varrho} + q_4 \boldsymbol{\varrho}' - \boldsymbol{\varrho}[\boldsymbol{\varrho}'\times] \\ -q'_4 q_4 + \boldsymbol{\varrho}'^T \boldsymbol{\varrho} \end{bmatrix} \tag{28}$$

In this paper a small angle-error approach is used to determine the attitude error induced by an out-of-plane vector. The attitude error, denoted by δA, is given by a multiplication of two attitude matrices, A_1 and A_2, with $\delta A = A_1 A_2^T$. For small attitude errors δA can be approximated using a first-order expansion so that

$$A_1 = e^{-[\boldsymbol{\delta\alpha}\times]} A_2 \approx (I_{3\times 3} - [\boldsymbol{\delta\alpha}\times])\, A_2 \tag{29}$$

where $\boldsymbol{\delta\alpha} = [\delta\alpha_1 \;\; \delta\alpha_2 \;\; \delta\alpha_3]^T$ represents the small roll, pitch and yaw error rotations. Then using Eq. (27) the error quaternion can be expressed as

$$\boldsymbol{\delta}\mathbf{q} = \mathbf{q}_1 \otimes \mathbf{q}_2^{-1} \tag{30}$$

where the error quaternion can be related to small Euler rotation by $\boldsymbol{\delta}\mathbf{q} = \begin{bmatrix} \boldsymbol{\delta\alpha}^T/2 & 1 \end{bmatrix}^T$.

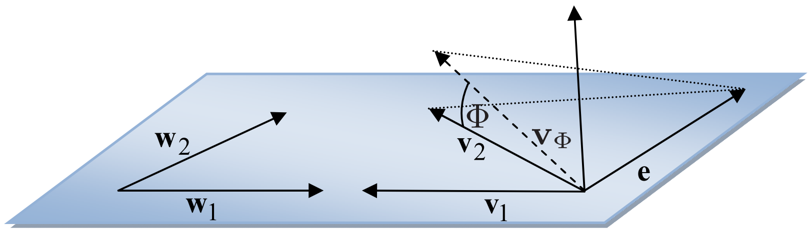

Figure 2. Out-of-Plane Geometry

V. Sensitivity to Out-of-Plane Deflection

In this section an expression for the sensitivity of the attitude error is derived for the case that a perfect triangle configuration is not given. The triangle assumption can only be violated by the case where one of the observation vectors is out of the plane containing the other two observations. Since the $\mathbf{w}_1$ and $\mathbf{v}_1$ vectors are common LOS observations expressed in different coordinates, then by definition they must be in the same direction and therefore these vectors can't be out of the plane. The $\mathbf{v}_2$ and $\mathbf{w}_2$ vectors are the only two vectors that can be out-of-plane. Since one of these vectors has to be used with the $\mathbf{w}_1$ and $\mathbf{v}_1$ direction to define a plane, then only one observation vector needs to be chosen to be out of the plane.

Consider rotating the $\mathbf{v}_2$ vector out of the plane by an angle Φ. Then the resulting out-of-plane vector can be defined as

$$\mathbf{v}_\Phi = R(\Phi, \mathbf{e})\mathbf{v}_2 \tag{31}$$

where $\mathbf{e}$ is the axis of rotation; see Figure 2. Then it follows that

$$\mathbf{e} = -\frac{[\mathbf{v}_2\times]^2\mathbf{v}_1}{\|\mathbf{v}_2 \times \mathbf{v}_1\|} \tag{32}$$

Note that $\|[\mathbf{v}_2\times]^2\mathbf{v}_1\| = \|\mathbf{v}_2\times\mathbf{v}_1\|$. The out-of-plane vector can be written using the definition of the attitude matrix:[11]

$$\mathbf{v}_\Phi = \left[I_{3\times 3}\cos(\Phi) + (1-\cos(\Phi))\mathbf{e}\mathbf{e}^T + \sin(\Phi)[\mathbf{e}\times]\right]\mathbf{v}_2 \tag{33}$$

Noting that $\mathbf{e}^T\mathbf{v}_2 = 0$ simplifies Eq. (33) to give

$$\mathbf{v}_\Phi = \left(I_{3\times 3}\cos(\Phi) + \sin(\Phi)[\mathbf{v}_1\times]/\|\mathbf{v}_2\times\mathbf{v}_1\|\right)\mathbf{v}_2 \tag{34}$$

The goal is to obtain an expression that relates the increase in the attitude error due to the out-of-plane deflection. This can be accomplished using the solution in Eq. (17d) to obtain an expression for the sensitivity of the error to the out-of-plane deflection. The attitude error matrix can be written as

$$\delta A_\Phi = A_\Phi A^T \tag{35}$$

where the matrix A is the attitude matrix formed using the observation set $\{\mathbf{w}_1, \mathbf{v}_1, \mathbf{w}_2, \mathbf{v}_2\}$ and the matrix A_Φ is the attitude matrix formed using the observation set $\{\mathbf{w}_1, \mathbf{v}_1, \mathbf{w}_2, \mathbf{v}_\Phi\}$. As explained previously the only out-of-plane vector is given by rotating $\mathbf{v}_2$ onto $\mathbf{v}_\Phi$. The attitude solution for A is accomplished by two successive rotations, the first rotation is given by the matrix B and the second rotation is given by the matrix R, where $R \equiv R(\mathbf{w}_1, \theta)$ is used for convenience. The solution for the estimated attitude is written as $A = R\,B$. The matrix B aligns the $\mathbf{v}_1$ and $\mathbf{w}_1$ directions and therefore this matrix is independent of $\mathbf{v}_\Phi$. The second rotation is a simple Euler axis/angle rotation about the $\mathbf{w}_1$ vector by the angle of θ. Its equivalent quaternion is given by

$$\mathbf{q}_R = \begin{bmatrix} \sin(\theta/2)\mathbf{w}_1 \\ \cos(\theta/2) \end{bmatrix} \tag{36}$$

This shows that all of the out-of-error is expressed solely by θ, as seen by Eq. (17c). The attitude matrix associated with $\mathbf{q}_R$ is denoted by R. The quantity θ_Φ is used to represent the error in θ due to the out-of-plane deflection. The quaternion associated with θ_Φ is given by

$$\mathbf{q}_{R_\Phi} = \begin{bmatrix} \sin(\theta_\Phi/2)\mathbf{w}_1 \\ \cos(\theta_\Phi/2) \end{bmatrix} \tag{37}$$

and its associated attitude matrix is R_Φ. Then $A_\Phi = R_\Phi\, B$, so that

$$\delta A_\Phi = R_\Phi\, B(R\,B)^T = R_\Phi B\, B^T\, R^T = R_\Phi\, R^T \tag{38}$$

So the error is only a function of R_Φ and R. Using quaternion multiplication Eq. (38) can be rewritten as

$$\boldsymbol{\delta}\mathbf{q}_{R_\Phi} = \mathbf{q}_{R_\Phi} \otimes \mathbf{q}_R^{-1} \tag{39}$$

where the error quaternion can be related to small angle errors by $\boldsymbol{\delta}\mathbf{q}_{R_\Phi} = [\boldsymbol{\delta\alpha}_\Phi^T/2\ \ 1]^T$. By carrying out the quaternion multiplication the vector component of the error quaternion can be shown to be given by

$$\boldsymbol{\delta\alpha}_\Phi = 2\left[\sin(\theta_\Phi/2)\cos(\theta/2) - \cos(\theta_\Phi/2)\sin(\theta/2)\right]\mathbf{w}_1 \tag{40}$$

By noting that $\sin((\theta_\Phi - \theta)/2) = \sin(\theta_\Phi/2)\cos(\theta/2) - \cos(\theta_\Phi/2)\sin(\theta/2)$ and assuming $(\theta_\Phi - \theta)$ is small, Eq. (40) becomes

$$\boldsymbol{\delta\alpha}_\Phi = (\theta_\Phi - \theta)\mathbf{w}_1 \tag{41}$$

Note that $\mathbf{w}_1$ is independent of Φ and only θ_Φ depends on Φ, where $\mathbf{w}_1$ defines the direction of $\boldsymbol{\delta\alpha}_\Phi$. The magnitude of the small angle vector gives the angle of rotation about $\mathbf{w}_1$, taking A_Φ to A. Calculating the derivative of the magnitude of $\boldsymbol{\delta\alpha}_\Phi$ with respect to Φ quantifies the sensitivity of the solution to the out-of-plane deflection angle. Since $\mathbf{w}_1$ is assumed to be a unit vector the magnitude of the small error angle can be written as $\Theta = \boldsymbol{\delta\alpha}_\Phi^T\mathbf{w}_1$. To consider the sensitivity of the solution to out-of-plane deflection the sensitivity in Θ is considered. Hence the following defined quantity is used to study the sensitivity is $\Theta \equiv (\theta_\Phi - \theta)$. The sensitivity of the solution to out-of-plane deflection is given by

$$\left.\frac{d\Theta}{d\Phi}\right|_{\Phi=0} = \left.\frac{d\theta_\Phi}{d\mathbf{v}_\Phi^T}\frac{d\mathbf{v}_\Phi}{d\Phi}\right|_{\Phi=0} \tag{42}$$

Expressions for the derivatives in Eq. (42) are needed. Using $\mathbf{w}^* = B\mathbf{v}_2$, θ can be written as

$$\theta = \operatorname{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]B\mathbf{v}_2, \mathbf{w}_2^T[\mathbf{w}_1\times]^2 B\mathbf{v}_2) + \pi \tag{43}$$

To simplify the derivation of the sensitivity expression Eq. (43) can be rearranged using $\mathbf{w}_1 = B\mathbf{v}_1$ and $\mathbf{v}_2^* = B^T\mathbf{w}_2$. Then by defining $\mathcal{Y} \equiv \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]\mathbf{v}_2$ and $\mathcal{X} \equiv \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2\mathbf{v}_2$ the angle θ is given by

$$\theta \equiv \operatorname{atan2}(\mathcal{Y}, \mathcal{X}) + \pi \tag{44}$$

To compute the sensitivity in Eq. (42) first θ_Φ is calculated using $\{\mathbf{w}_1, \mathbf{v}_1, \mathbf{w}_2, \mathbf{v}_\Phi\}$ and then this expression is differentiated with respect to $\mathbf{v}_\Phi$. The expression for θ_Φ is given by

$$\theta_\Phi = \operatorname{atan2}(\mathcal{Y}_\Phi, \mathcal{X}_\Phi) + \pi \tag{45}$$

where the terms in Eq. (45) are defined by $\mathcal{Y}_\Phi = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]\mathbf{v}_\Phi$ and $\mathcal{X}_\Phi = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2\mathbf{v}_\Phi$. The expression for the sensitivity $\frac{d\theta_\Phi}{d\mathbf{v}_\Phi^T}$ can now be calculated from Eq. (45). It follows that

$$\frac{d\theta_\Phi}{d\mathbf{v}_\Phi^T} = \frac{1}{\mathcal{X}_\Phi^2 + \mathcal{Y}_\Phi^2}\left(\mathcal{X}_\Phi\frac{\partial\mathcal{Y}_\Phi}{d\mathbf{v}_\Phi^T} - \mathcal{Y}_\Phi\frac{\partial\mathcal{X}_\Phi}{d\mathbf{v}_\Phi^T}\right) \tag{46}$$

Then the expression for $\frac{d\theta_\Phi}{d\mathbf{v}_\Phi^T}$ is evaluated at $\Phi = 0$, resulting in $\mathbf{v}_\Phi = \mathbf{v}_2$ and the derivative terms can be written as $\frac{\partial\mathcal{Y}_\Phi}{d\mathbf{v}_\Phi^T} = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]$ and $\frac{\partial\mathcal{X}_\Phi}{d\mathbf{v}_\Phi^T} = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2$. Using these expressions in Eq. (46) the first sensitivity term can be written as

$$\left.\frac{d\theta_\Phi}{d\mathbf{v}_\Phi^T}\right|_{\mathbf{v}_\Phi=\mathbf{v}_2} = \frac{1}{\mathcal{X}^2 + \mathcal{Y}^2}\left(\mathcal{X}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times] - \mathcal{Y}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2\right) \tag{47}$$

The expression for $\frac{d\,\mathbf{v}_\Phi}{d\,\Phi}$ can be determined from Eq. (34):

$$\frac{d\,\mathbf{v}_\Phi}{d\,\Phi} = \left(-I_{3\times3}\sin(\Phi) + \cos(\Phi)[\mathbf{v}_1\times]/\|\mathbf{v}_2\times\mathbf{v}_1\|\right)\mathbf{v}_2 \tag{48}$$

By setting $\Phi = 0$ in Eq. (48), the expression for $\frac{d\mathbf{v}_\Phi}{d\Phi}\Big|_{\Phi=0}$ can be determined to be

$$\left.\frac{d\mathbf{v}_\Phi}{d\Phi}\right|_{\Phi=0} = \frac{[\mathbf{v}_1\times]\mathbf{v}_2}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{49}$$

Then combining Eq. (47) and Eq. (49) the sensitivity of the solution to out-of-plane deflection defined in Eq. (42) can be expressed as

$$\left.\frac{d\theta_\Phi}{d\Phi}\right|_{\Phi=0} = \frac{1}{\mathcal{X}^2+\mathcal{Y}^2}\left(\mathcal{X}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times] - \mathcal{Y}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2\right)\frac{[\mathbf{v}_1\times]\mathbf{v}_2}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{50}$$

Using the identity $[\mathbf{v}_1\times]^3 = -[\mathbf{v}_1\times]$ and the definitions of $\mathcal{Y}$ and $\mathcal{X}$ then Eq. (50) becomes

$$\left.\frac{d\theta_\Phi}{d\Phi}\right|_{\Phi=0} = \frac{1}{\mathcal{X}^2+\mathcal{Y}^2}\frac{(\mathcal{X}^2+\mathcal{Y}^2)}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{51}$$

Then finally by simplifying Eq. (51) the final expression for the sensitivity of the solution to out-of-plane deflection is given by

$$\left.\frac{d\theta_\Phi}{d\Phi}\right|_{\Phi=0} = \frac{1}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{52}$$

It is expected that the out-of-plane deflection is small under most operating conditions and therefore Eq. (52) gives a good approximation for the sensitivity of the relative attitude solution due to constraint validation.

VI. Simulations

The simulations use a static formation of three vehicles, with each vehicle having two focal plane detectors (FPD)s and two of the three vehicles having light source devices. As mentioned previously the third vehicle does not require a light source because the triangle constraint is used in the solution. The relative attitude mapping between each vehicle's body frame is determined from LOS measurements. The formation configuration uses the following true LOS vectors:

$$\mathbf{w}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}\cos(135°)\\0\\-\sin(135°)\end{bmatrix} \tag{53}$$

The last vector is chosen so that a triangle configuration is assured for the true vectors. The remaining LOS truth vectors are determined from those listed in Eq. (11), without noise added, using the appropriate attitude transformation. For this configuration the true relative attitude is given by

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & -1 & 0\end{bmatrix} \tag{54}$$

For the simulation the LOS vectors are converted into focal-plane coordinates and random noise is added to the true values having covariances described in §II, with $\sigma = 17\times10^{-6}$ rad. Since each FPD has its own boresight axis, and the measurement covariance in Eq. (2) is described with respect to the boresight, individual sensor frames must be defined to generate the FPD measurements. The measurement error-covariance for each FPD is determined with respect to the corresponding sensor frames and must be rotated to the vehicle's body frame as well. The letter S is used to denote sensor frame. The orthogonal transformations for their respective sensor frames, denoted by the subscript, used to orientate the FPD to

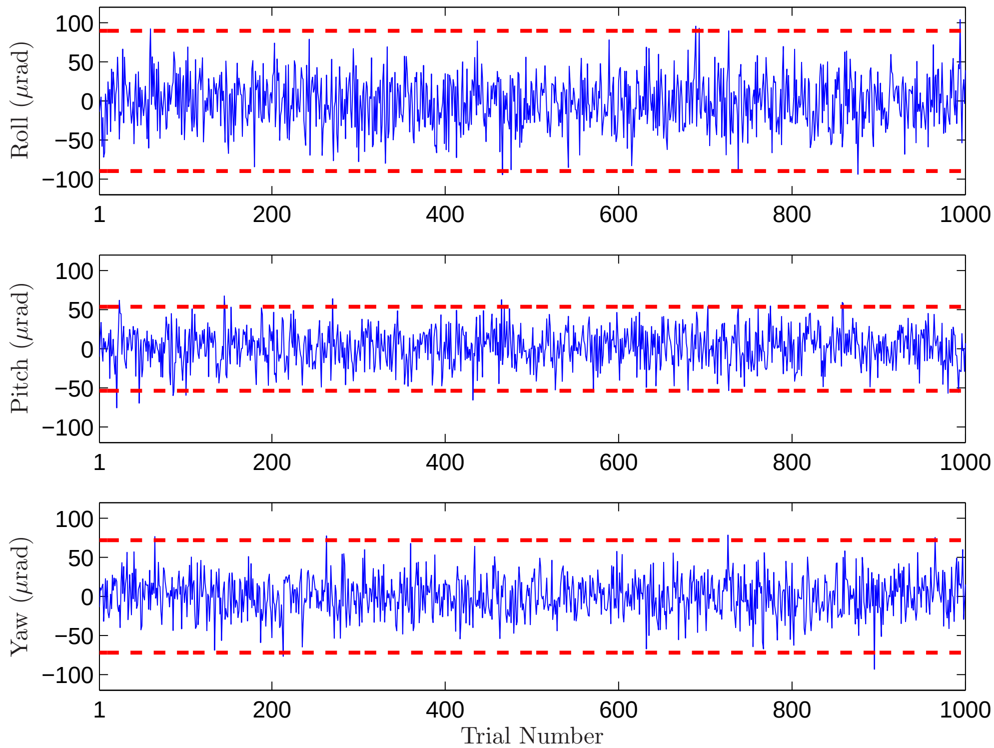

Figure 3. Relative Attitude Estimate Errors

the specific vehicle, denoted by the superscript, are given by

$$A_{s_{\mathcal{B}_1}}^{\mathbf{v}_1} = \begin{bmatrix} -0.8373 & -0.2962 & 0.4596 \\ -0.2962 & -0.4609 & 0.8366 \\ 0.4596 & -0.8366 & 0.2981 \end{bmatrix}, \quad A_{s_{\mathcal{B}_1}}^{\mathbf{v}_2} = \begin{bmatrix} -0.8069 & 0.4487 & 0.3843 \\ 0.4487 & -0.0423 & 0.8927 \\ 0.3843 & 0.8927 & -0.2355 \end{bmatrix} \tag{55a}$$

$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_1} = \begin{bmatrix} -0.8889 & 0.0644 & 0.4535 \\ 0.0644 & -0.9626 & 0.2630 \\ 0.4535 & 0.2630 & 0.8515 \end{bmatrix}, \quad A_{s_{\mathcal{B}_2}}^{\mathbf{w}_2} = \begin{bmatrix} 0.4579 & -0.0169 & 0.8888 \\ -0.0169 & -0.9998 & -0.0103 \\ 0.8888 & 0.0103 & -0.4581 \end{bmatrix} \tag{55b}$$

The configuration is considered for 1,000 Monte Carlo trials. Measurements are generated in the sensor frame and rotated to the body frame to be combined with the other measurements to determine the full relative attitudes. The wide-FOV measurement model for the FPD LOS covariance is used. Relative attitude angle errors are displayed in Figure 3. Good performance characteristics are given using the constrained solution. This figure shows that the derived attitude-error covariance does indeed bound these errors in a 3σ sense, which is computed to be

$$P_{\boldsymbol{\delta\alpha\delta\alpha}} = 1 \times 10^{-9} \begin{bmatrix} 0.9790 & -0.1166 & -0.0889 \\ -0.1166 & 0.4056 & 0.0889 \\ -0.0889 & 0.0889 & 0.5308 \end{bmatrix} \tag{56}$$

This configuration is also considered for 100 Monte Carlo trials for various out-of-plane deflection angles. The angle is varied from -0.05 deg to 0.05 deg using 0.01 degree intervals. Measurements are generated in the sensor frame and rotated to the body frame to be combined with the other measurements to determine the full relative attitudes. The wide-FOV measurement model for the FPD LOS covariance is used. The Monte Carlo relative attitude angle errors are calculated for each trial and are plotted for all the considered out-of-plane deflection angles. The covariance of the angle error is calculated for out-of-plane deflection angles given the 100 Monte Carlo runs. The numerical variance runs are plotted with the angle errors for

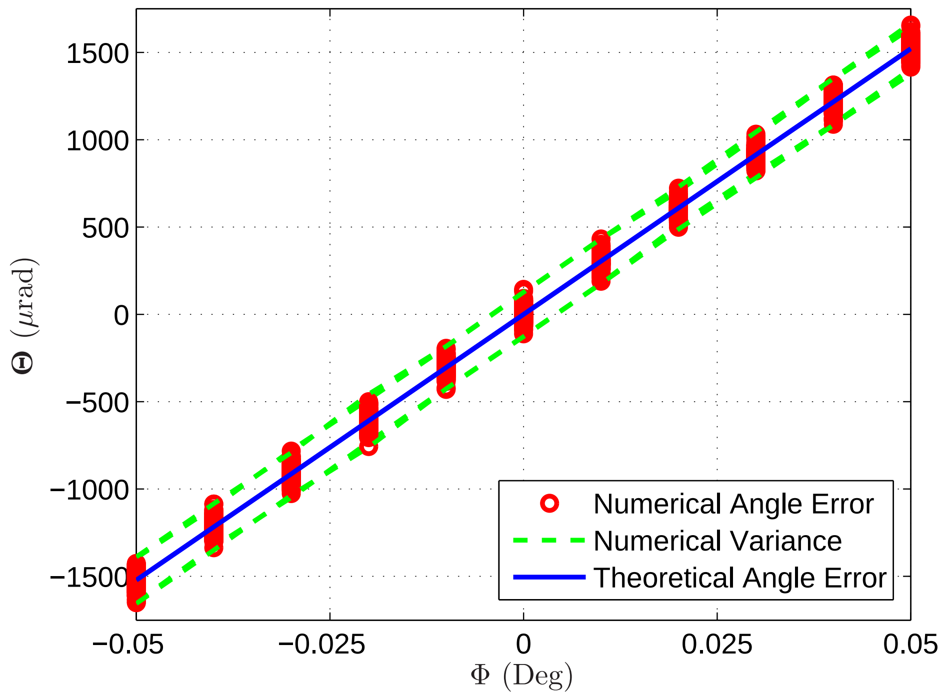

Figure 4. Plot of the Sensitivity to Out-of-Plane Deflections

all deflection angles. Results are shown in Figure 4. The theoretical error is calculated using Eq. (52) and the linear approximation about $\Phi = 0$, given by $\Theta = \frac{d\Theta}{d\Phi}\big|_{\Phi=0} \Phi$. Good agreement between computed errors through the Monte Carlo runs and the theoretical predictions is shown. Also the numerical variance does not vary with out-of-plane deflection; moreover, out-of-plane deflection biases the solution only and does not increase its variation about the mean.

VII. Conclusions

In this paper a sensitivity expression was derived for a relative attitude determination approach for two vehicles using a triangle constraint in the observations. The triangle constraint is useful because it requires two less observations than a deterministic relative approach without the constraint. In actual practice, the triangle scenario reflects a realistic physical situation; however, out-of-plane deflections can occur due to misalignments and/or noise. This paper studied how out-of-plane observations, which violate the constraint, affect the constrained solution. The case study shown in this paper showed that the derived expression matches simulated results. In essence the constraint violation leads to a bias in the attitude solution. The analytically derived expression for the sensitivity is useful to quantify whether or not a particular system using the constrained triangle solution causes issues in comparison to the required accuracy of the estimated solution.

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