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Adaptive Stochastic Disturbance Accommodating Control

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This paper presents a Kalman filter based adaptive disturbance accommodating stochastic control scheme for linear uncertain systems to minimize the adverse effects of both model uncertainties and external disturbances. Instead of dealing with system uncertainties and external disturbances separately, the disturbance accommodating control scheme lumps the overall effects of these errors in a to-be-determined model-error vector, and then utilizes a Kalman filter in the feedback loop for simultaneously estimating the system states and the model-error vector from noisy measurements. Since the model-error dynamics is unknown, the process noise covariance associated with the model-error dynamics is used to empirically tune the Kalman filter to yield accurate estimates. A rigorous stochastic stability analysis reveals a lower bound requirement on the assumed system process noise covariance to ensure the stability of the controlled system when the nominal control action on the true plant is unstable. An adaptive law is synthesized for the selection of stabilizing system process noise covariance. Simulation results are presented where the proposed control scheme is implemented on a two degree-of-freedom helicopter.

Keywords: Disturbance accommodating control; Stochastic adaptive control; Kalman filter; Stochastic stability

1 Introduction

Uncertainty in dynamic systems may take numerous forms, but among them the most significant are noise/disturbance uncertainty and model/parameter uncertainty. External disturbances and system uncertainties can obscure the development of a viable control law. This paper presents the formulation and analysis of a stochastic robust control scheme known as the Disturbance Accommodating Control. The main objective of Disturbance Accommodating Control (DAC) is to make necessary corrections to the nominal control input to accommodate for external disturbances and system uncertainties. Instead of dealing with system uncertainties and disturbances separately, DAC lumps the overall effects of these errors in a to-be-determined term that is used to directly update a nominal control input.

Disturbance accommodating control was first proposed by Johnson in 1971 (Johnson 1971). Though the traditional DAC approach only considers disturbance functions which exhibit waveform patterns over short intervals of time (Johnson and Kelly 1981), a more general formulation of DAC can accommodate the simultaneous presence of both "noise" type disturbances and "waveform structured" disturbances (Johnson 1984, 1985). The disturbance accommodating observer approach has shown to be extremely effective for disturbance attenuation (Biglari and Mobasher 2000, Profeta et al. 1990, Kim and Oh 1998); however, the performance of the observer can significantly vary for different types of exogenous disturbances, which is due to observer gain sensitivity.

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This paper presents a robust control approach based on a significant extension of the conventional observer-based DAC concept, which compensates for both unknown model parameter uncertainties and external disturbances by estimating a model-error vector (throughout this paper the phrase "disturbance term" will be used to refer to this quantity) in real time. The estimated model-error vector is further used as a signal synthesis adaptive correction to the nominal control input to achieve maximum performance. This control approach utilizes a Kalman filter in the feedback loop for simultaneously estimating the system states and the disturbance term from measurements (Sorrells 1982, 1989, Davari and Chandramohan 2003). The estimated states are then used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input to minimize the effects of both system uncertainties and the external disturbance. There are several advantages of implementing the Kalman filter in the DAC approach: i) tuning of the estimator parameters, such as the process noise matrix, can be done easily unlike conventional DAC approaches in which the adaptation involves the entire feedback gain, ii) the estimated disturbance term is a natural byproduct of state estimation, and iii) the Kalman filter can be used to filter noisy measurements.

It is a well-known fact that the closed-loop performance and the stability of the Kalman filter-based DAC approach depends on the accuracy of the estimated disturbance term. Since the dynamics of the disturbance term is unknown, the process noise covariance associated with the disturbance term is used to empirically tune the Kalman filter to yield accurate estimates. Although the Kalman filter-based DAC approach has been successfully utilized for practical applications, there has not been any rigorous stochastic stability analysis to reveal the interdependency between the estimator process noise covariance and controlled system stability. The first main contribution of this paper is a detailed stability analysis, which examines the explicit dependency of the controlled system's closed-loop stability on the disturbance term process noise covariance and the measurement noise covariance. Since the system under consideration is stochastic in nature, the notion of stability is depicted in two separate fashions. The first method deals with moment stability and the second technique considers stability in a probabilistic sense.

Stochastic stability analysis on the Kalman filter-based DAC approach indicates that the effectiveness of the proposed control scheme depends on the estimator parameters such as the process noise covariance matrix. The stability analysis also indicates that the DAC scheme is most effective when the assumed process noise covariance satisfies a lower bound requirement which depends on the system uncertainties. In general, it is difficult to select a stabilizing process noise covariance for the broad type of uncertain systems considered here. One could always try to select an extremely large value of process noise covariance that might stabilize the system or even monotonically increase the process noise covariance matrix in an ad-hoc manner until the system stabilizes. However, it is important to keep in mind that selecting a large process noise covariance matrix would result in noisy control signal which could lead to problems such as chattering. The second main contribution of this paper is the formulation of a stochastic adaptive scheme for selecting the appropriate process noise covariance that would guarantee closed-loop stability of the controlled system.

The structure of this paper is as follows. A detailed formulation of the stochastic DAC approach for multi-input multi-output (MIMO) systems, followed by a stochastic stability analysis, is first given. Next, an adaptive scheme is developed for the selection of stabilizing the disturbance term process noise covariance. Simulation results are then presented where the proposed control scheme is implemented on a two degree-of-freedom helicopter.

2 Disturbance Accommodating Controller Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space, where Ω is the sample space, $\mathcal{F}$ is a σ-field, and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$. Additionally, the elements of Ω are denoted by ω and the members of $\mathcal{F}$ are called events. Now consider a linear time-invariant stochastic system of the following form:

$$\begin{aligned}\dot{\mathbf{X}}_1(t) &= \bar{A}_1\mathbf{X}_1(t) + \bar{A}_2\mathbf{X}_2(t), \quad \mathbf{X}_1(t_0) = \mathbf{X}_{1_0}\\ \dot{\mathbf{X}}_2(t) &= A_3\mathbf{X}_1(t) + A_4\mathbf{X}_2(t) + B\mathbf{u}(t) + \mathbf{W}(t), \quad \mathbf{X}_2(t_0) = \mathbf{X}_{2_0}\end{aligned} \tag{1}$$

Here, the stochastic state vector, $\left[\mathbf{X}_1^T(t)\ \mathbf{X}_2^T(t)\right]^T = \mathbf{X}(t) \triangleq \mathbf{X}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^n$, is an n-dimensional random variable for fixed t. The state vectors, $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ are of dimensions $\mathbf{X}_1(t) \triangleq \mathbf{X}_1(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^{n-r}$ and $\mathbf{X}_2(t) \triangleq \mathbf{X}_2(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^r$, respectively. The system given in (1) is in the typical kinematics-dynamics form, where the kinematics is assumed to be fully known, i.e., the state matrices $\bar{A}_1 \in \Re^{(n-r)\times(n-r)}$ and $\bar{A}_2 \in \Re^{(n-r)\times r}$ are precisely known. Uncertainty is only associated with the dynamics, i.e., the state and control distribution matrices, $A_3 \in \Re^{r\times(n-r)}, A_4 \in \Re^{r\times r}, B \in \Re^{r\times r}$, are assumed to be unknown. Also, the input matrix, B is assumed to be nonsingular. Finally, the stochastic external disturbance $\mathbf{W}(t) \triangleq \mathbf{W}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^r$ is modeled as a linear time-invariant system driven by a Gaussian white noise process, i.e.,

$$\dot{\mathbf{W}}(t) = \mathbb{L}\left(\mathbf{X}(t), \mathbf{W}(t)\right) + \boldsymbol{\mathcal{V}}(t), \quad \mathbf{W}(t_0) = \mathbf{0}_{r\times 1} \tag{2}$$

where $\mathbb{L}(\cdot)$ is an unknown linear operator and $\boldsymbol{\mathcal{V}}(t) \triangleq \boldsymbol{\mathcal{V}}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^r$, is assumed to be zero-mean Gaussian white noise process, i.e., $\boldsymbol{\mathcal{V}}(t) \sim \mathcal{N}\big(\mathbf{0}, \mathcal{Q}\delta(\tau)\big)$. It is important to note that the linear operator $\mathbb{L}(\cdot)$ and the covariance of the white noise process $\boldsymbol{\mathcal{V}}(t)$, are unknown. The measurement equation is given as

$$\mathbf{Y}(t) = C\mathbf{X}(t) + \mathbf{V}(t) \tag{3}$$

where $\mathbf{Y}(t) \triangleq \mathbf{Y}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^m$ is the measurement vector and $C \in \Re^{m\times n}$ denotes the known output matrix. The measurement noise, $\mathbf{V}(t) \triangleq \mathbf{V}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^m$, is assumed to be zero-mean Gaussian white noise with known covariance, i.e., $\mathbf{V}(t) \sim \mathcal{N}\big(\mathbf{0}, R\delta(\tau)\big)$.

The assumed (known) system matrices are given as A_{3_m}, A_{4_m}, and B_m. Now the external disturbance and the model uncertainties can be lumped into a disturbance term, $\boldsymbol{\mathcal{D}}(t) \in \Re^r$, through

$$\boldsymbol{\mathcal{D}}(t) = \Delta A_1\mathbf{X}_1(t) + \Delta A_2\mathbf{X}_2(t) + \Delta B\mathbf{u}(t) + \mathbf{W}(t) \tag{4}$$

where $\Delta A_1 = (A_3 - A_{3_m})$, $\Delta A_2 = (A_4 - A_{4_m})$ and $\Delta B = (B - B_m)$. Using this disturbance term the true model can be written in terms of the known system matrices as follows:

$$\begin{aligned}\dot{\mathbf{X}}_1(t) &= \bar{A}_1\mathbf{X}_1(t) + \bar{A}_2\mathbf{X}_2(t)\\ \dot{\mathbf{X}}_2(t) &= A_{3_m}\mathbf{X}_1(t) + A_{4_m}\mathbf{X}_2(t) + B_m\mathbf{u}(t) + \boldsymbol{\mathcal{D}}(t)\end{aligned} \tag{5}$$

The control law, $\mathbf{u}(t)$, consists of a nominal control and a control correction term to minimize the adverse effect of the disturbance term, $\boldsymbol{\mathcal{D}}(t)$, i.e.,

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) + \tilde{\mathbf{u}}(t) \tag{6}$$

Here $\bar{\mathbf{u}}(t)$ is the nominal control and $\tilde{\mathbf{u}}(t)$ is the control correction term. For the purpose of analysis, the control correction term is selected to ensure the complete cancelation of the disturbance term. Thus the disturbance accommodating control law can be written as

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) - B_m^{-1}\boldsymbol{\mathcal{D}}(t) \tag{7}$$

The disturbance term is not known, but an estimator can be implemented in the feedback loop to estimate the disturbance term online. Estimating the disturbance term requires knowledge of its dynamic model. Since the dynamics of the disturbance term is not precisely known, the disturbance term dynamics is modeled as

$$\dot{\boldsymbol{\mathcal{D}}}_m(t) = \mathrm{A}_{\mathcal{D}_m}\boldsymbol{\mathcal{D}}_m(t) + \boldsymbol{\mathcal{W}}(t), \quad \boldsymbol{\mathcal{D}}_m(t_0) = \mathbf{0} \tag{8}$$

where $\mathrm{A}_{\mathcal{D}_m}$ is Hurwitz and $\boldsymbol{\mathcal{W}}(t) \triangleq \boldsymbol{\mathcal{W}}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^r$ is zero-mean Gaussian white noise process, i.e., $\boldsymbol{\mathcal{W}}(t) \sim \mathcal{N}\big(\mathbf{0}, Q\delta(\tau)\big)$. Equation (8) is used solely in the estimator design to estimate the true disturbance term. After constructing the assumed augmented state vector as $\mathbf{Z}_m(t) = \big[\mathbf{X}_{1_m}^T(t)\ \mathbf{X}_{2_m}^T(t)\ \boldsymbol{\mathcal{D}}_m^T(t)\big]^T$, the assumed extended model of the system can be written as

$$\begin{bmatrix}\dot{\mathbf{X}}_{1_m}(t)\\ \dot{\mathbf{X}}_{2_m}(t)\\ \dot{\boldsymbol{\mathcal{D}}}_m(t)\end{bmatrix} = \begin{bmatrix}\bar{A}_1 & \bar{A}_2 & 0_{(n-r)\times r}\\ A_{3_m} & A_{4_m} & I_{r\times r}\\ 0_{r\times(n-r)} & 0_{r\times r} & \mathrm{A}_{\mathcal{D}_m}\end{bmatrix}\begin{bmatrix}\mathbf{X}_{1_m}(t)\\ \mathbf{X}_{2_m}(t)\\ \boldsymbol{\mathcal{D}}_m(t)\end{bmatrix} + \begin{bmatrix}0_{(n-r)\times r}\\ B_m\\ 0_{r\times r}\end{bmatrix}\mathbf{u}(t) + \begin{bmatrix}\mathbf{0}_{(n-r)\times 1}\\ \mathbf{0}_{r\times 1}\\ \boldsymbol{\mathcal{W}}(t)\end{bmatrix} \tag{9}$$

The zero elements in the disturbance term dynamics are assumed for sake of simplicity, however the control formulation given here is also valid if nonzero elements are assumed. Equation (9) can be written in terms of the appended state vector, $\mathbf{Z}_m$, as

$$\dot{\mathbf{Z}}_m(t) = F_m\mathbf{Z}_m(t) + D_m\mathbf{u}(t) + G\boldsymbol{\mathcal{W}}(t), \quad \mathbf{Z}_m(t_0) = [\mathbf{X}_{1_0}^T\ \ \mathbf{X}_{2_0}^T\ \ \mathbf{0}^T]^T \tag{10}$$

where

$$F_m = \begin{bmatrix}\bar{A}_1 & \bar{A}_2 & 0_{(n-r)\times r}\\ A_{3_m} & A_{4_m} & I_{r\times r}\\ 0_{r\times(n-r)} & 0_{r\times r} & \mathrm{A}_{\mathcal{D}_m}\end{bmatrix}, \quad D_m = \begin{bmatrix}0_{(n-r)\times r}\\ B_m\\ 0_{r\times r}\end{bmatrix}, \quad G = \begin{bmatrix}0_{n\times r}\\ I_{r\times r}\end{bmatrix}$$

Note that the uncertainty is only associated with the dynamics of the disturbance term. Let $\mathbf{Z}(t) = \big[\mathbf{X}_1^T(t)\ \mathbf{X}_2^T(t)\ \boldsymbol{\mathcal{D}}^T(t)\big]^T$ and $H = [C\ \ 0_{m\times r}]$. Now the measured output equation can be written as

$$\mathbf{Y}(t) = H\mathbf{Z}(t) + \mathbf{V}(t) \tag{11}$$

Though the disturbance term is unknown, an estimator such as a Kalman filter can be implemented in the feedback loop to estimate the unmeasured system states and the disturbance term from the noisy measurements. Let $\hat{\mathbf{Z}}(t) = \Big[\hat{\mathbf{X}}_1^T(t)\ \hat{\mathbf{X}}_2^T(t)\ \hat{\boldsymbol{\mathcal{D}}}^T(t)\Big]^T$, now the estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m\mathbf{u}(t) + K(t)[\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)], \quad \hat{\mathbf{Z}}(t_0) = \mathbf{Z}_m(t_0) \tag{12}$$

where $K(t)$ is the Kalman gain and $\hat{\mathbf{Y}}(t) = H\hat{\mathbf{Z}}(t)$. The Kalman gain can be calculated as $K(t) = P(t)H^TR^{-1}$, where $P(t)$ is obtained by solving the continuous-time matrix differential Riccati equation (Crassidis and Junkins 2004):

$$\dot{P}(t) = F_mP(t) + P(t){F_m}^T - P(t)H^TR^{-1}HP(t) + GQG^T, \quad P(t_0) \tag{13}$$

Let $\mathbf{Z}(t) = \big[\mathbf{X}_1^T(t)\ \mathbf{X}_2^T(t)\ \boldsymbol{\mathcal{D}}^T(t)\big]^T$, now the estimator dynamics can be rewritten as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m\mathbf{u}(t) + K(t)H[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)] + K(t)\mathbf{V}(t) \tag{14}$$

The estimator uses the assumed system model given in (10) for the propagation stage and the actual measurements for the update stage, i.e., $\hat{\mathbf{Z}}(t) = E[\mathbf{Z}_m(t)|\{\mathbf{Y}(t_0)\ldots\mathbf{Y}(t)\}]$. Due to system uncertainties, the estimator in (14) is sub-optimal and the estimates $\hat{\mathbf{Z}}(t)$ may be biased.

Remark 1 : Accuracy of the estimates depends on Q which indicates how well the disturbance term dynamics is modeled via (8). A large Q indicates that (8) is a poor model of the true disturbance term dynamics and a small Q indicates that (8) is an accurate model of the true disturbance term dynamics. Note that selecting a small Q, while having a poor model, would results in inaccurate estimates.

The total control law, $\mathbf{u}(t)$, consists of a nominal control and necessary corrections to the nominal control to compensate for the disturbance term as shown in (7). The nominal control is assumed to be a state feedback control, where the feedback gain, $K_m \triangleq \begin{bmatrix} K_{m_1} & K_{m_2} \end{bmatrix}$, is selected so that $(\mathcal{A}_m - \mathcal{B}_m K_m)$ is Hurwitz, where

$$\mathcal{A}_m = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ A_{3_m} & A_{4_m} \end{bmatrix} \quad \text{and} \quad \mathcal{B}_m = \begin{bmatrix} 0_{(n-r)\times r} \\ B_m \end{bmatrix}$$

While the nominal controller guarantees the desired performance of the assumed model, the second term, $-\boldsymbol{\mathcal{D}}(t)$, in (7) ensures the complete cancelation of the disturbance term which is compensating for the external disturbance and model uncertainties. Now the disturbance accommodating control law can be written in terms of the estimated system states and the estimated disturbance term as

$$\mathbf{u}(t) = -\begin{bmatrix} K_m & B_m^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}}(t) \\ \hat{\boldsymbol{\mathcal{D}}}(t) \end{bmatrix} = S\,\hat{\mathbf{Z}}(t) \tag{15}$$

where $S \triangleq -\begin{bmatrix} K_m & B_m^{-1} \end{bmatrix}$. Notice that B_m is assumed to be a nonsingular matrix. A summary of the proposed control scheme is given in Table 1.

Table 1. Summary of Disturbance Accommodating Control Process

Plant	$\dot{\mathbf{X}}_1(t) = \bar{A}_1\mathbf{X}_1(t) + \bar{A}_2\mathbf{X}_2(t)$ $\dot{\mathbf{X}}_2(t) = A_{3_m}\mathbf{X}_1(t) + A_{4_m}\mathbf{X}_2(t) + B_m\mathbf{u}(t) + \boldsymbol{\mathcal{D}}(t)$ $\mathbf{Y}(t) = C\mathbf{X}(t) + \mathbf{V}(t)$
Initialize	$\hat{\mathbf{Z}}(t_0),\ P(t_0)$
Estimator Gain	$\dot{P}(t) = F_m P(t) + P(t){F_m}^T - P(t)H^T R^{-1} H P(t) + GQG^T$ $K(t) = P(t)H^T R^{-1}$
Estimate	$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m\mathbf{u}(t) + K(t)\big[\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)\big]$
Control Synthesis	$\mathbf{u}(t) = -\begin{bmatrix} K_m & B_m^{-1} \end{bmatrix}\hat{\mathbf{Z}}(t)$

Remark 2 : It is important to note that if $Q \approx 0$, then $\boldsymbol{\mathcal{D}}_m(t) \approx \boldsymbol{\mathcal{D}}_m(t_0) = \mathbf{0}$, and the total control law given in (15) becomes just the nominal control. If the nominal control, $\bar{\mathbf{u}}(t)$, on the true plant would result in an unstable system, i.e., the matrix $(\mathcal{A} - \mathcal{B}K_m)$ is unstable, where

$$\mathcal{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0_{(n-r)\times r} \\ B \end{bmatrix}$$

then selecting a small Q would also result in an unstable system. On the other hand, selecting a large Q value would compel the estimator to completely rely upon the measurement signal and therefore the noise associated with the measurement signal is directly transmitted into the estimates. This could result in a noisy control signal which could lead to problems such as chattering. Also note that as R, the measurement noise covariance, increases, the estimator gain decreases and thus the estimator fails to update the propagated disturbance term based on measurements. Thus, for a highly uncertain system, if the nominal control action on the true plant would result in an unstable system, then selecting a small Q or a large R would also result in an unstable closed-loop system.

If the estimator in (14) is able to obtain accurate estimates of the system states and the disturbance term, then the control law in (15) guarantees the desired closed-loop performance. The accuracy of the estimated system states and the disturbance term depends on the estimator parameters such as the process noise covariance, Q, and the measurement noise covariance, R. Thus the performance of the DAC approach presented here depends on the estimator design parameters. A schematic representation of the proposed controller is given in figure. 1.

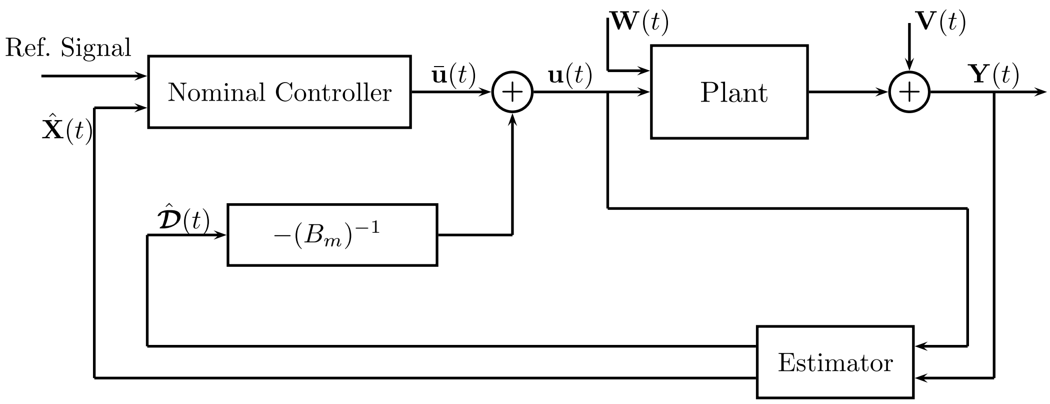


Figure 1. DAC Block Diagram

3 Stochastic Stability Analysis

Without loss of generality, the following assumption can be made about the external disturbance model.

Assumption 3.1 **:** The linear operator, $\mathbb{L}(\cdot)$, in the external disturbance term model in (2) is assumed to be

$$\mathbb{L}\left(\mathbf{X}(t), \mathbf{W}(t)\right) = A_{\mathbf{w}_1}\mathbf{X}_1(t) + A_{\mathbf{w}_2}\mathbf{X}_2(t) + A_{\mathbf{w}_3}\mathbf{W}(t) \tag{16}$$

where $A_{\mathbf{w}_1}$, $A_{\mathbf{w}_2}$, and $A_{\mathbf{w}_3}$ are unknown matrices.

Based on equation (4), the true disturbance term dynamics can now be written as

$$\dot{\boldsymbol{\mathcal{D}}}(t) = \Delta A_1 \dot{\mathbf{X}}_1(t) + \Delta A_2 \dot{\mathbf{X}}_2(t) + \Delta B \dot{\mathbf{u}}(t) + \mathbb{L}\left(\mathbf{X}(t), \mathbf{W}(t)\right) + \boldsymbol{\mathcal{V}}(t) \tag{17}$$

Substituting the control law (15) the above equation can be written as

$$\begin{aligned}\dot{\boldsymbol{\mathcal{D}}}(t) =& \Delta A_1\dot{\mathbf{X}}_1(t) + \Delta A_2\dot{\mathbf{X}}_2(t) + \Delta BS\dot{\hat{\mathbf{Z}}}(t) + \mathbb{L}\left(\mathbf{X}(t), \mathbf{W}(t)\right) + \boldsymbol{\mathcal{V}}(t)\\ =& \Delta A_1\left\{\bar{A}_1\mathbf{X}_1(t) + \bar{A}_2\mathbf{X}_2(t)\right\} + \Delta A_2\left\{A_{3_m}\mathbf{X}_1(t) + A_{4_m}\mathbf{X}_2(t) + B_m S\hat{\mathbf{Z}}(t) + \boldsymbol{\mathcal{D}}(t)\right\} +\\ & \Delta BS\left\{\left[F_m + D_m S - K(t)H\right]\hat{\mathbf{Z}}(t) + K(t)H\mathbf{Z}(t) + K(t)\mathbf{V}(t)\right\} +\\ & A_{\mathbf{w}_1}\mathbf{X}_1(t) + A_{\mathbf{w}_2}\mathbf{X}_2(t) + A_{\mathbf{w}_3}\mathbf{W}(t) + \boldsymbol{\mathcal{V}}(t)\end{aligned}$$

Assume the output matrix can be partitioned as $C \triangleq \left[C_1\ C_2\right]$, and H can be written as $H \triangleq \left[C_1\ C_2\ 0_{m\times r}\right]$. Thus $K(t)H\mathbf{Z}(t) = K(t)C_1\mathbf{X}_1(t) + K(t)C_2\mathbf{X}_2(t)$. Also note that $\mathbf{W}(t)$ can be written as $\mathbf{W}(t) = \boldsymbol{\mathcal{D}}(t) - \Delta A_1\mathbf{X}_1(t) - \Delta A_2\mathbf{X}_2(t) - \Delta B\mathbf{u}(t)$. Now the true disturbance term dynamics can be written as

$$\begin{aligned}\dot{\boldsymbol{\mathcal{D}}}(t) =& \left\{\Delta A_1\bar{A}_1 + \Delta A_2 A_{3_m} + \Delta BSK(t)C_1 + A_{\mathbf{w}_1} - A_{\mathbf{w}_3}\Delta A_1\right\}\mathbf{X}_1(t)+\\ & \left\{\Delta A_1\bar{A}_2 + \Delta A_2 A_{4_m} + \Delta BSK(t)C_2 + A_{\mathbf{w}_2} - A_{\mathbf{w}_3}\Delta A_2\right\}\mathbf{X}_2(t)+\\ & \left\{\Delta BS\left[F_m + D_m S - K(t)H\right] + \Delta A_2 B_m S - A_{\mathbf{w}_3}\Delta BS\right\}\hat{\mathbf{Z}}(t)+\\ & \left\{\Delta A_2 + A_{\mathbf{w}_3}\right\}\boldsymbol{\mathcal{D}}(t) + \Delta BSK(t)\mathbf{V}(t) + \boldsymbol{\mathcal{V}}(t)\end{aligned}$$

Let $\Delta BSK(t)\mathbf{V}(t) + \boldsymbol{\mathcal{V}}(t) = \boldsymbol{\mathcal{W}}_a(t)$, thus $\boldsymbol{\mathcal{W}}_a(t)$ is also a zero-mean stochastic process with

$$E\left[\boldsymbol{\mathcal{W}}_a(t)\boldsymbol{\mathcal{W}}_a^T(t+\tau)\right] = \left\{\Delta BSK(t)RK^T(t)S^T\Delta B^T + \mathcal{Q}\right\}\delta(\tau) = Q_a(t)\delta(\tau)$$

Notice $\dot{\boldsymbol{\mathcal{D}}}(t)$ is a linear function of the true extended system state, $\mathbf{Z}(t)$, the estimated states, $\hat{\mathbf{Z}}(t)$, and the noise term, $\boldsymbol{\mathcal{W}}_a(t)$. Thus the system in (1) is rewritten as the following extended dynamically equivalent system:

$$\begin{bmatrix}\dot{\mathbf{X}}_1(t)\\ \dot{\mathbf{X}}_2(t)\\ \dot{\boldsymbol{\mathcal{D}}}(t)\end{bmatrix} = \begin{bmatrix}\bar{A}_1 & \bar{A}_2 & 0_{(n-r)\times r}\\ A_{3_m} & A_{4_m} & I_{r\times r}\\ A_{\mathcal{D}_1}(t) & A_{\mathcal{D}_2}(t) & A_{\mathcal{D}_3}(t)\end{bmatrix}\begin{bmatrix}\mathbf{X}_1(t)\\ \mathbf{X}_2(t)\\ \boldsymbol{\mathcal{D}}(t)\end{bmatrix} + \begin{bmatrix}0_{(n-r)\times(n+r)}\\ B_m S\\ B_{\mathcal{D}}(t)\end{bmatrix}\hat{\mathbf{Z}}(t) + \begin{bmatrix}\mathbf{0}_{(n-r)\times 1}\\ \mathbf{0}_{r\times 1}\\ \boldsymbol{\mathcal{W}}_a(t)\end{bmatrix} \tag{18}$$

where

$$A_{\mathcal{D}_1}(t) = \left\{\Delta A_1\bar{A}_1 + \Delta A_2 A_{3_m} + \Delta BSK(t)C_1 + A_{\mathbf{w}_1} - A_{\mathbf{w}_3}\Delta A_1\right\}$$

$$A_{\mathcal{D}_2}(t) = \left\{\Delta A_1\bar{A}_2 + \Delta A_2 A_{4_m} + \Delta BSK(t)C_2 + A_{\mathbf{w}_2} - A_{\mathbf{w}_3}\Delta A_2\right\}$$

$$A_{\mathcal{D}_3}(t) = \left\{\Delta A_2 + A_{\mathbf{w}_3}\right\}$$

and

$$B_{\mathcal{D}}(t) = \left\{\Delta BS\left[F_m + D_m S - K(t)H\right] + \Delta A_2 B_m S - A_{\mathbf{w}_3}\Delta BS\right\}$$

Equation (18) can be written in concise form as

$$\dot{\mathbf{Z}}(t) = F(t)\mathbf{Z}(t) + D(t)\hat{\mathbf{Z}}(t) + G\boldsymbol{\mathcal{W}}_a(t) \tag{19}$$

where

$$F(t) = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & 0_{(n-r)\times r} \\ A_{3_m} & A_{4_m} & I_{r\times r} \\ A_{\mathcal{D}_1}(t) & A_{\mathcal{D}_2}(t) & A_{\mathcal{D}_3}(t) \end{bmatrix} \quad \text{and} \quad D(t) = \begin{bmatrix} 0_{(n-r)\times(n+r)} \\ B_m S \\ B_{\mathcal{D}}(t) \end{bmatrix}$$

After substituting the control law, the estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m S\hat{\mathbf{Z}}(t) + K(t)H[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)] + K(t)\mathbf{V}(t)$$

Let $\tilde{\mathbf{Z}}(t) = \mathbf{Z}(t) - \hat{\mathbf{Z}}(t)$ be the estimation error, then the error dynamics can be written as

$$\dot{\tilde{\mathbf{Z}}}(t) = [F_m - K(t)H + \Delta F(t)]\tilde{\mathbf{Z}}(t) + [\Delta F(t) + \Delta D(t)]\hat{\mathbf{Z}}(t) + G\boldsymbol{\mathcal{W}}_a(t) - K(t)\mathbf{V}(t) \tag{20}$$

where $\triangle F(t) = F(t) - F_m$ and $\triangle D(t) = D(t) - D_m S$. Combining the error dynamics and the estimator dynamics yields

$$\begin{bmatrix} \dot{\tilde{\mathbf{Z}}}(t) \\ \dot{\hat{\mathbf{Z}}}(t) \end{bmatrix} = \begin{bmatrix} (F_m - K(t)H + \Delta F(t)) & (\Delta F(t) + \Delta D(t)) \\ K(t)H & (F_m + D_m S) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Z}}(t) \\ \hat{\mathbf{Z}}(t) \end{bmatrix} + \begin{bmatrix} G & -K(t) \\ 0_{(n+r)\times r} & K(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{W}}_a(t) \\ \mathbf{V}(t) \end{bmatrix} \tag{21}$$

or in a more compact form as

$$\dot{\boldsymbol{\mathcal{Z}}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t) \tag{22}$$

where

$$\Upsilon(t) = \begin{bmatrix} (F_m - K(t)H + \Delta F(t)) & (\Delta F(t) + \Delta D(t)) \\ K(t)H & (F_m + D_m S) \end{bmatrix}, \quad \Gamma(t) = \begin{bmatrix} G & -K(t) \\ 0_{(n+r)\times r} & K(t) \end{bmatrix}$$

$$\boldsymbol{\mathcal{Z}}(t) = \begin{bmatrix} \tilde{\mathbf{Z}}(t) \\ \hat{\mathbf{Z}}(t) \end{bmatrix}, \quad \boldsymbol{\mathcal{G}}(t) = \begin{bmatrix} \boldsymbol{\mathcal{W}}_a(t) \\ \mathbf{V}(t) \end{bmatrix}$$

Although the Kalman filter-based DAC approach has been successfully utilized for practical applications, there has not been any rigorous stochastic stability analysis to reveal the interdependency between the estimator process noise covariance and controlled system stability. Since the system under consideration is stochastic in nature, the notion of stability is depicted in two separate fashions. The first method deals with moment stability; for the Gaussian stochastic processes presented here, the first two moments are considered. The second technique considers stability in a probabilistic sense.

3.1 *First Moment Stability*

In this section a detailed stability analysis which examines the explicit dependency of the controlled system's first moment stability or the mean stability on the estimator parameters, such as the disturbance term process noise covariance Q, and the measurement noise covariance R, is given. First, a few definitions regarding the closed-loop system's mean stability are given. These definitions and notations are first introduced for a system without any parameter uncertainties and are used throughout the rest of this manuscript.

3.1.1 System without Uncertainties

Here a system without any parameter uncertainties is considered, i.e., $F(t) = F_m$, $D(t) = D_m S$, and $\boldsymbol{\mathcal{W}}_a(t) = \boldsymbol{\mathcal{W}}(t)$. If there is no model error, then the estimator is unbiased, i.e., $E\left[\bar{\tilde{\mathbf{Z}}}(t)\right] \equiv \boldsymbol{\mu}_{\bar{\tilde{\mathbf{Z}}}}(t) = \mathbf{0}$. Note that the overline is used to indicate the states of the system when there is no model uncertainties. Now (21) may be written as

$$\begin{bmatrix} \dot{\bar{\tilde{\mathbf{Z}}}}(t) \\ \dot{\bar{\hat{\mathbf{Z}}}}(t) \end{bmatrix} = \begin{bmatrix} F_m - K(t)H & 0_{(n+r)\times(n+r)} \\ K(t)H & F_m + D_m S \end{bmatrix} \begin{bmatrix} \bar{\tilde{\mathbf{Z}}}(t) \\ \bar{\hat{\mathbf{Z}}}(t) \end{bmatrix} + \begin{bmatrix} G & -K(t) \\ 0_{(n+r)\times r} & K(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{W}}(t) \\ \mathbf{V}(t) \end{bmatrix}$$

where $\bar{\tilde{\mathbf{Z}}}(t)$ and $\bar{\hat{\mathbf{Z}}}(t)$ denote the estimation error and estimated states when there is no model error, respectively. Let $\bar{\boldsymbol{\mathcal{Z}}}(t) = \left[\bar{\tilde{\mathbf{Z}}}^T(t)\ \bar{\hat{\mathbf{Z}}}^T(t)\right]^T$ and $\bar{\boldsymbol{\mathcal{G}}}(t) = \left[\boldsymbol{\mathcal{W}}^T(t)\ \mathbf{V}^T(t)\right]^T$, now the above equation can be written in a more compact form as

$$\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\bar{\boldsymbol{\mathcal{Z}}}(t) + \Gamma(t)\bar{\boldsymbol{\mathcal{G}}}(t) \tag{23}$$

where

$$\bar{\Upsilon}(t) = \begin{bmatrix} F_m - K(t)H & 0_{(n+r)\times(n+r)} \\ K(t)H & F_m + D_m S \end{bmatrix}$$

Notice that $\bar{\boldsymbol{\mathcal{G}}}(t)$ is a zero-mean Gaussian white noise process with

$$E[\bar{\boldsymbol{\mathcal{G}}}(t)\bar{\boldsymbol{\mathcal{G}}}^T(t-\tau)] = \begin{bmatrix} Q & 0_{r\times m} \\ 0_{m\times r} & R \end{bmatrix} \delta(\tau) = \bar{\Lambda}\delta(\tau)$$

Since the first moment stability is of concern here, the first moment dynamics or the mean dynamics is written as

$$E[\dot{\bar{\boldsymbol{\mathcal{Z}}}}(t)] = \dot{\boldsymbol{\mu}}_{\bar{\boldsymbol{\mathcal{Z}}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t) \tag{24}$$

Definition 3.2: Given $M \geq 1$ and $\beta \in \mathbb{R}$, the system in (23) is said to be (M, β)-stable in the mean if

$$|\bar{\Phi}(t, t_0)\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t_0)| \leq M e^{\beta(t-t_0)}|\boldsymbol{\mu}_{\bar{\boldsymbol{\mathcal{Z}}}}(t_0)|, \quad \forall t \geq t_0 \tag{25}$$

where $\bar{\Phi}(t, t_0)$ is the evolution operator generated by $\bar{\Upsilon}(t)$ and $|\cdot|$ indicates the Euclidean norm, i.e.,

$$|\mathbf{m}| = \sqrt{m_1^2 + m_2^2 + \ldots}$$

Since most applications involve the case where $\beta \leq 0$, (M, β)-stability guarantees both a specific decay rate of the mean response (given by β) and a specific bound on the transient behavior of the mean (given by M).

Definition 3.3: If the stochastic system in (23) is (M, β)-stable in the mean, then the transient bound of the system mean response for the exponential rate β is defined to be

$$M_\beta = \inf\left\{M \in \mathbb{R}; \forall t \geq t_0 : \parallel \bar{\Phi}(t, t_0) \parallel \leq M e^{\beta(t-t_0)}\right\} \tag{26}$$

Here $\| \cdot \|$ indicates the matrix two-norm, i.e.,

$$\| M \| = \sigma_{\max}(M)$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value.

As shown in the Theorem given below, the (M, β)-stability and the transient bound of the system's mean response are related to a continuous time Lyapunov matrix differential equation.

Theorem 3.4: *Assume there exists a bounded, continuously differentiable positive definite matrix function $\bar{\mathcal{P}}(t)$ satisfying the Lyapunov matrix differential equation*

$$\dot{\bar{\mathcal{P}}}(t) = \bar{\Upsilon}(t)\bar{\mathcal{P}}(t) + \bar{\mathcal{P}}(t)\bar{\Upsilon}^T(t) + \Gamma(t)\bar{\Lambda}\Gamma^T(t), \quad \bar{\mathcal{P}}(t_0) \tag{27}$$

then the system in (23) is (M, β)-stable in the mean and the transient bound M_β of the system mean response can be obtained as

$$M_\beta^2 \leq \sup_{t \geq t_0} \sigma_{\max}(\bar{\mathcal{P}}(t))/\sigma_{\min}(\bar{\mathcal{P}}(t_0)) \tag{28}$$

where $\sigma_{\min}(\cdot)$ denotes the minimum singular value.

Proof Since $\Gamma(t)\bar{\Lambda}\Gamma^T(t) \geq 0 \; \forall t \geq t_0$, the (M, β)-stability in the mean follows directly from the existence of bounded positive definite solution, $\bar{\mathcal{P}}(t)$, satisfying equation (27). Now the solution to (27) can be written as

$$\bar{\mathcal{P}}(t) = \bar{\Phi}(t, t_0)\bar{\mathcal{P}}(t_0)\bar{\Phi}^T(t, t_0) + \int_{t_0}^{t} \bar{\Phi}(t, \tau)\Gamma(\tau)\bar{\Lambda}\Gamma^T(\tau)\bar{\Phi}^T(t, \tau)d\tau$$

Notice that $\forall t \geq t_0$, $\bar{\mathcal{P}}(t) \geq \bar{\Phi}(t, t_0)\bar{\mathcal{P}}(t_0)\bar{\Phi}^T(t, t_0) \geq \sigma_{\min}(\bar{\mathcal{P}}(t_0))\bar{\Phi}(t, t_0)\bar{\Phi}^T(t, t_0)$, i.e.,

$$\sigma_{\max}(\bar{\mathcal{P}}(t)) \geq \| \bar{\Phi}(t, t_0)\bar{\mathcal{P}}(t_0)\bar{\Phi}^T(t, t_0) \| \geq \sigma_{\min}(\bar{\mathcal{P}}(t_0)) \| \bar{\Phi}(t, t_0) \|^2, \quad t \geq t_0$$

Now (28) follows from

$$\sigma_{\max}(\bar{\mathcal{P}}(t))/\sigma_{\min}(\bar{\mathcal{P}}(t_0)) \geq \| \bar{\Phi}(t, t_0) \|^2, \quad t \geq t_0$$

□

Remark 3: Assume $\bar{\mathcal{P}}(t_0)$ is selected as $\bar{\mathcal{P}}(t_0) = E[\bar{\boldsymbol{\mathcal{Z}}}(t_0)\bar{\boldsymbol{\mathcal{Z}}}^T(t_0)]$, then the positive definite solution, $\bar{\mathcal{P}}(t)$, satisfying equation (27) denotes the correlation matrix, i.e.,

$$\bar{\mathcal{P}}(t) = E[\bar{\boldsymbol{\mathcal{Z}}}(t)\bar{\boldsymbol{\mathcal{Z}}}^T(t)]$$

Thus the transient bound of the system mean response can be obtained in terms of the bounded correlation matrix.

Note that $\Gamma(t)\bar{\Lambda}\Gamma^T(t)$ in (27) can be factored as shown below:

$$\begin{aligned}\Gamma(t)\bar{\Lambda}\Gamma^T(t) &= \begin{bmatrix} (GQG^T + KRK^T) & -KRK^T \\ -KRK^T & KRK^T \end{bmatrix} = \begin{bmatrix} GQG^T & -0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} KRK^T & -KRK^T \\ -KRK^T & KRK^T \end{bmatrix} \\ &= \begin{bmatrix} G \\ 0 \end{bmatrix} Q \begin{bmatrix} G^T & 0 \end{bmatrix} + \begin{bmatrix} PH^T \\ -PH^T \end{bmatrix} R^{-1} \begin{bmatrix} HP & -HP \end{bmatrix} = LQL^T + N(t)R^{-1}N^T(t)\end{aligned}$$

where

$$L = \begin{bmatrix} G \\ 0 \end{bmatrix} \quad \text{and} \quad N(t) = \begin{bmatrix} P(t)H^T \\ -P(t)H^T \end{bmatrix}$$

Thus (27) can be written as

$$\dot{\bar{\mathcal{P}}}(t) = \bar{\Upsilon}(t)\bar{\mathcal{P}}(t) + \bar{\mathcal{P}}(t)\bar{\Upsilon}^T(t) + LQL^T + N(t)R^{-1}N^T(t) \tag{29}$$

3.1.2 Uncertain System

In this subsection, the first moment stability of the perturbed system given in (22) is considered, i.e.,

$$\dot{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t) \tag{30}$$

where

$$\Delta\Upsilon(t) = \begin{bmatrix} \Delta F(t) & (\Delta F(t) + \Delta D(t)) \\ 0 & 0 \end{bmatrix}$$

The correlation matrix $\mathcal{P}(t) = E\left[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)\right]$ satisfies the following matrix Lyapunov differential equation:

$$\dot{\mathcal{P}}(t) = \left(\bar{\Upsilon}(t) + \Delta\Upsilon(t)\right)\mathcal{P}(t) + \mathcal{P}(t)\left(\bar{\Upsilon}(t) + \Delta\Upsilon(t)\right)^T + \Gamma(t)\Lambda(t)\Gamma^T(t) \tag{31}$$

where

$$\Lambda(t)\delta(\tau) = E[\boldsymbol{\mathcal{G}}(t)\boldsymbol{\mathcal{G}}^T(t-\tau)] = \begin{bmatrix} Q_a(t) & 0_{r\times m} \\ 0_{m\times r} & R \end{bmatrix}\delta(\tau)$$

Assuming the nominal control action on the true plant would result in an unstable system, stability of extended uncertain system given in (30) depends on the disturbance term process noise covariance, Q, and the measurement noise covariance, R. The Theorem given below indicates that the stability of the extended uncertain system given in (30) is guaranteed if the selected Q and R satisfies a lower and an upper bound, respectively.

Theorem 3.5: *The uncertain system in* (30) *is* (M, β)*-stable in the mean if*

$$\left\{\sigma_{\min}(Q) + \sigma_{\min}(R^{-1}) \parallel N(t)N^T(t) \parallel -1\right\} \parallel \breve{\mathcal{P}}(t) \parallel^{-2} > \ 2 \parallel \Delta\Upsilon(t) \parallel^2, \quad t \geq t_0 \tag{32}$$

where $\breve{\mathcal{P}}(t)$ *satisfies the matrix differential equation*

$$\dot{\breve{\mathcal{P}}}(t) = \bar{\Upsilon}(t)\breve{\mathcal{P}}(t) + \breve{\mathcal{P}}(t)\bar{\Upsilon}^T(t) + \parallel Q \parallel I + \parallel N(t)R^{-1}N(t)^T \parallel I - \breve{\mathcal{P}}(t)\Delta\Upsilon^T(t)\Delta\Upsilon(t)\breve{\mathcal{P}}(t) \tag{33}$$

Proof For the linear time-varying system given in (30), uniform asymptotic stability in the mean implies (M, β)-stability in the mean. In order to show the uniform asymptotic stability of the mean, consider the mean dynamics of the system in (30):

$$\dot{\boldsymbol{\mu}}_{\boldsymbol{\mathcal{Z}}}(t) = \bar{\Upsilon}(t)\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) + \Delta\Upsilon(t)\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) \tag{34}$$

where $E[\boldsymbol{\mathcal{Z}}(t)] = \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)$. Construct the following Lyapunov candidate function:

$$V[\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)] = \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T(t)\breve{\mathcal{P}}^{-1}(t)\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) \tag{35}$$

Note that the solution, $\breve{\mathcal{P}}(t)$, of (33) is required to be a bounded positive definite matrix as long as $\Delta\Upsilon$ is norm-bounded (Abou-Kandil et al. 2003). Thus $\breve{\mathcal{P}}^{-1}(t)$ exists and $V[\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)] > 0$ for all $\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t) \neq \mathbf{0}$. Since $\breve{\mathcal{P}}(t)\breve{\mathcal{P}}^{-1}(t) = I$, the time derivative of $\breve{\mathcal{P}}(t)\breve{\mathcal{P}}^{-1}(t)$ is 0:

$$\frac{d}{dt}\left[\breve{\mathcal{P}}(t)\breve{\mathcal{P}}^{-1}(t)\right] = \dot{\breve{\mathcal{P}}}(t)\breve{\mathcal{P}}^{-1}(t) + \breve{\mathcal{P}}(t)\dot{\breve{\mathcal{P}}}^{-1}(t) = 0$$

Solving the above equation for $\dot{\breve{\mathcal{P}}}^{-1}(t)$ and substituting (33) gives

$$\begin{aligned}\dot{\breve{\mathcal{P}}}^{-1}(t) &= -\breve{\mathcal{P}}^{-1}(t)\dot{\breve{\mathcal{P}}}(t)\breve{\mathcal{P}}^{-1}(t)\\ &= -\breve{\mathcal{P}}^{-1}(t)\bar{\Upsilon}(t) - \bar{\Upsilon}^T(t)\breve{\mathcal{P}}^{-1}(t) - \{\parallel Q \parallel + \parallel N(t)R^{-1}N(t)^T \parallel\}\breve{\mathcal{P}}^{-1}(t)\breve{\mathcal{P}}^{-1}(t)\\ &\qquad + \Delta\Upsilon^T(t)\Delta\Upsilon(t)\end{aligned}$$

Now the time derivative of (35) can be written as

$$\begin{aligned}\dot{V}[\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}(t)] =& \dot{\boldsymbol{\mu}}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\dot{\breve{\mathcal{P}}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-1}\dot{\boldsymbol{\mu}}_{\boldsymbol{\mathcal{Z}}}\\ =& [\bar{\Upsilon}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}]^T\breve{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-1}\bar{\Upsilon}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\bar{\Upsilon}^T\breve{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} -\\ & \{\parallel Q \parallel + \parallel NR^{-1}N^T \parallel\}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-2}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\Delta\Upsilon^T\Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-1}[\bar{\Upsilon}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}]\\ =& \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\Delta\Upsilon^T\breve{\mathcal{P}}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-1}\Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}} - \{\parallel Q \parallel + \parallel NR^{-1}N^T \parallel\}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\breve{\mathcal{P}}^{-2}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}\\ & + \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\Delta\Upsilon^T\Delta\Upsilon\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}\\ =& \boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}^T\left\{\Delta\Upsilon^T\breve{\mathcal{P}}^{-1} + \breve{\mathcal{P}}^{-1}\Delta\Upsilon - \{\parallel Q \parallel + \parallel NR^{-1}N^T \parallel\}\breve{\mathcal{P}}^{-2} + \Delta\Upsilon^T\Delta\Upsilon\right\}\boldsymbol{\mu}_{\boldsymbol{\mathcal{Z}}}\end{aligned}$$

Asymptotic stability in the first moment is guaranteed if

$$\left\{\Delta\Upsilon^T\breve{\mathcal{P}}^{-1} + \breve{\mathcal{P}}^{-1}\Delta\Upsilon - \{\parallel Q \parallel + \parallel NR^{-1}N^T \parallel\}\breve{\mathcal{P}}^{-2} + \Delta\Upsilon^T\Delta\Upsilon\right\} < 0$$

Note

$$\left[\Delta\Upsilon^T - \breve{\mathcal{P}}^{-1}\right]\left[\Delta\Upsilon^T - \breve{\mathcal{P}}^{-1}\right]^T \geq 0 \Rightarrow \Delta\Upsilon^T\Delta\Upsilon + \breve{\mathcal{P}}^{-2} \geq \Delta\Upsilon^T\breve{\mathcal{P}}^{-1} + \breve{\mathcal{P}}^{-1}\Delta\Upsilon$$

Thus the above condition for asymptotic stability is satisfied as soon as

$$\left\{2\Delta\Upsilon^T\Delta\Upsilon + \breve{\mathcal{P}}^{-2} - \{\parallel Q \parallel + \parallel NR^{-1}N^T \parallel\}\breve{\mathcal{P}}^{-2}\right\} < 0$$

or

$$\left\{2\breve{\mathcal{P}}\Delta\Upsilon^T\Delta\Upsilon\breve{\mathcal{P}} + I - \{\parallel Q \parallel + \parallel NR^{-1}N^T \parallel\}I\right\} < 0$$

Using the inequalities

$$\parallel \breve{\mathcal{P}} \parallel^2 \parallel \Delta\Upsilon \parallel^2 I \geq \breve{\mathcal{P}}\Delta\Upsilon^T\Delta\Upsilon\breve{\mathcal{P}},\;\; \sigma_{\min}(Q) \leq \parallel Q \parallel,\;\; \text{and}\;\; \sigma_{\min}(R^{-1}) \parallel NN^T \parallel \leq \parallel NR^{-1}N^T \parallel$$

yields

$$2 \parallel \breve{\mathcal{P}} \parallel^2 \parallel \Delta\Upsilon \parallel^2 < \sigma_{\min}(Q) + \sigma_{\min}(R^{-1}) \parallel NN^T \parallel -1$$

Hence the uniform asymptotic stability in the first moment is guaranteed if

$$2 \parallel \Delta\Upsilon(t) \parallel^2 < \left\{\sigma_{\min}(Q) + \sigma_{\min}(R^{-1}) \parallel N(t)N^T(t) \parallel -1\right\} \parallel \breve{\mathcal{P}}(t) \parallel^{-2}, \quad t \geq t_0$$

□

Remark 4: The uncertain system in (30) is (M, β)-stable in the mean if the selected Q and R satisfy the inequality in (32). Thus for a highly uncertain system, if the nominal control action on the true plant would result in an unstable system, then selecting a small Q or a large R would also result in an unstable closed-loop system.

3.2 *Mean Square Stability*

In this subsection the controlled system's stability in the second moment or the mean square stability is considered. It is shown here that the (M, β)-stability in the mean implies mean square stability. More details on mean square stability can be found in Kushner (1967) and Soong (1973).

Definition 3.6: A stochastic system of the following form $\dot{\boldsymbol{\mathcal{Z}}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t) + \Gamma(t)\boldsymbol{\mathcal{G}}(t)$ is mean square stable if

$$\lim_{t\to\infty} E[\boldsymbol{\mathcal{Z}}^T(t)\boldsymbol{\mathcal{Z}}(t)] < \mathrm{M} \tag{36}$$

where M is a constant square matrix whose elements are finite.

Note that $E[\boldsymbol{\mathcal{Z}}^T(t)\boldsymbol{\mathcal{Z}}(t)] = \mathrm{Tr}\,\{\mathcal{P}(t)\}$, i.e.,

$$\frac{d}{dt}E[\boldsymbol{\mathcal{Z}}(t)\boldsymbol{\mathcal{Z}}^T(t)] = \dot{\mathcal{P}}(t) = \Upsilon(t)\mathcal{P}(t) + \mathcal{P}(t)\Upsilon^T(t) + \Gamma(t)\Lambda(t)\Gamma^T(t)$$

and the solution to the above equation can be written as

$$\mathcal{P}(t) = \int_{-\infty}^{t} \Phi(t,\tau)\Gamma(\tau)\Lambda(\tau)\Gamma^T(\tau)\Phi^T(t,\tau)d\tau$$

The (M, β)-stable in the mean implies the system matrix, $\Upsilon(t) = \bar{\Upsilon}(t) + \Delta\Upsilon(t)$, generates an exponentially stable evolution operator, and therefore $\mathcal{P}(t)$ has a bounded solution (Abou-Kandil et al. 2003). Therefore, for the system given in (30), (M, β)-stability in the mean implies mean square stability.

3.3 *Almost Sure Asymptotic Stability*

The solution to the stochastic system given in (30) cannot be based on the ordinary mean square calculus because the integral involved in the solution depends on $\boldsymbol{\mathcal{G}}(t)$, which is of unbounded variation (Soong and Grigoriu 1993). For the treatment of this class of problems, the stochastic differential equation may be rewritten in Itô form as

$$d\boldsymbol{\mathcal{Z}}(t) = \left[\bar{\Upsilon}(t)\boldsymbol{\mathcal{Z}}(t) + \Delta\Upsilon(t)\boldsymbol{\mathcal{Z}}(t)\right]dt + \Gamma(t)\Lambda^{1/2}(t)d\mathcal{B}(t)$$

or simply as

$$d\boldsymbol{\mathcal{Z}}(t) = \Upsilon(t)\boldsymbol{\mathcal{Z}}(t)dt + \Gamma(t)\Lambda^{1/2}(t)d\mathcal{B}(t) \tag{37}$$

where $d\mathcal{B}(t)$ is an increment of Brownian motion process with zero-mean, Gaussian distribution and

$$E[d\mathcal{B}(t)d\mathcal{B}^T(t)] = Idt \tag{38}$$

The solution $\boldsymbol{\mathcal{Z}}(t)$ of (37) is a semimartingale process that is also a Markov process (Grigoriu 2002). Details on the almost sure ($a.s.$) stability for the stochastic system in (37) is presented in this section.

Definition 3.7: The linear stochastic system given in (37) is asymptotically stable with probability 1, or almost surely asymptotically stable, if

$$\mathbb{P}\big(\boldsymbol{\mathcal{Z}}(t) \to \mathbf{0} \quad \text{as} \quad t \to \infty\big) = 1 \tag{39}$$

(M, β)-stability in the mean response implies that $\Upsilon(t)$ generates an asymptotically stable evolution for the linear system in (37), but it does not imply almost sure asymptotic stability due to the persistently acting disturbance. In fact, given $\Upsilon(t)$ generates an asymptotically stable evolution, the necessary and sufficent condition for almost sure asymptotic stability is

$$\lim_{t\to\infty} \parallel \Gamma(t) \parallel^2 \log(t) = 0 \tag{40}$$

A detailed proof of this argument can be found in Appleby (2002). Equation (40) constitutes the sufficent condition for the almost sure asymptotic stability of a linear stochastic system given (M, β)-stability in the mean.

4 Stabilizing Q and Transient Bound on Uncertain System

The Lyapunov analysis given in Theorem 3.5 indicates a lower bound requirement on the system process noise covariance, Q, and an upper bound requirement on system measurement noise covariance, R, in order for the controlled system to be (M, β)-stable in the mean. Since the measurement noise covariance can be obtained from sensor calibration, the process noise matrix Q is usually treated as a tuning parameter. This would compel one to select an extremely large Q so that the stability is always guaranteed. Selecting a large Q value would force the estimator to completely rely upon the measurement signal and therefore the noise associated with the measurement signal is directly transmitted into the estimates. This could result in a noisy control signal which could lead to problems such as chattering. This section shows a systematic approach to select a stabilizing Q using the overbounding method of Petersen and Hollot (Petersen and Hollot 1986, Douglas and Athans 1994).

Assume the structure of the uncertainty $\Delta\Upsilon(t)$ is given as

$$\Delta\Upsilon(t) = \sum_{i=1}^{l} r_i(t)\Upsilon_i \tag{41}$$

where Υ_i is assumed to be a rank-one matrix of the form $\Upsilon_i = \mathbf{t}_i\mathbf{e}_i^T$. In the above description,

$r_i(t)$ is the i^{th} component of the vector $\mathbf{r}(t) \in \mathbb{R}^l$ and is upper bounded by

$$\bar{r} \geq \sup_{t \geq t_0} |r_i(t)|, \quad \forall\, i \in \{1,\, 2,\, \ldots,\, l\} \tag{42}$$

Define matrices T and E as

$$T = \sum_{i=1}^{l} \mathbf{t}_i \mathbf{t}_i^T \quad \text{and} \quad E = \sum_{i=1}^{l} \mathbf{e}_i \mathbf{e}_i^T \tag{43}$$

Lemma 4.1: *If the uncertain matrix $\Delta\Upsilon(t)$ has the structure given in (41), then the following matrix inequality is valid for all matrices $\mathcal{P}^*(t)$ of appropriate dimensions:*

$$\mathcal{P}^{*T}(t)\Delta\Upsilon^T(t) + \Delta\Upsilon(t)\mathcal{P}^*(t) \leq \bar{r}^2 T + \mathcal{P}^{*T}(t)E\mathcal{P}^*(t), \quad \forall t \geq t_0 \tag{44}$$

where $\bar{r}$, T, and E are from (42) and (43).

Proof Substituting $\Upsilon_i = \mathbf{t}_i \mathbf{e}_i^T$ into (41) yields,

$$\Delta\Upsilon(t)\mathcal{P}^*(t) + \mathcal{P}^{*T}(t)\Delta\Upsilon^T(t) = \sum_{i=1}^{l} \left\{ r_i(t)\mathbf{t}_i\mathbf{e}_i^T\mathcal{P}^*(t) + r_i(t)\mathcal{P}^{*T}(t)\mathbf{e}_i\mathbf{t}_i^T \right\}$$

Notice

$$\left[r_i(t)\mathbf{t}_i - \mathcal{P}^{*T}(t)\mathbf{e}_i \right] \left[r_i(t)\mathbf{t}_i - \mathcal{P}^{*T}(t)\mathbf{e}_i \right]^T \geq 0$$

Thus

$$r_i^2(t)\mathbf{t}_i\mathbf{t}_i^T + \mathcal{P}^{*T}(t)\mathbf{e}_i\mathbf{e}_i^T\mathcal{P}^*(t) \geq r_i(t)\mathbf{t}_i\mathbf{e}_i^T\mathcal{P}^*(t) + r_i(t)\mathcal{P}^{*T}(t)\mathbf{e}_i\mathbf{t}_i^T$$

and

$$\sum_{i=1}^{l} \left\{ r_i^2(t)\mathbf{t}_i\mathbf{t}_i^T + \mathcal{P}^{*T}(t)\mathbf{e}_i\mathbf{e}_i^T\mathcal{P}^*(t) \right\} \geq \sum_{i=1}^{l} \left\{ r_i(t)\mathbf{t}_i\mathbf{e}_i^T\mathcal{P}^*(t) + r_i(t)\mathcal{P}^{*T}(t)\mathbf{e}_i\mathbf{t}_i^T \right\}$$

Now substituting for T and E yields

$$\bar{r}^2 T + \mathcal{P}^{*T}(t)E\mathcal{P}^*(t) \geq \Delta\Upsilon(t)\mathcal{P}^*(t) + \mathcal{P}^{*T}(t)\Delta\Upsilon^T(t)$$

□

A computationally feasible procedure for the calculation of a stabilizing Q is given next.

Theorem 4.2: *Assume the uncertain matrix $\Delta\Upsilon(t)$ has the structure given in (41) and the process noise covariance, Q^*, is selected so that the following matrix differential Riccati equation has a bounded positive definite matrix solution, $\bar{\mathcal{P}}^*(t)$:*

$$\dot{\bar{\mathcal{P}}}^*(t) = \bar{\Upsilon}(t)\bar{\mathcal{P}}^*(t) + \bar{\mathcal{P}}^*(t)\bar{\Upsilon}^T(t) - \bar{\gamma}\bar{\mathcal{P}}^*(t)\bar{\mathcal{P}}^*(t) + \mathcal{R}\left(Q^*, R^{-1}\right) \tag{45}$$

and

$$\mathcal{R}\left(Q^*, R^{-1}\right) \geq \bar{\gamma}\bar{\mathcal{P}}^*(t)\bar{\mathcal{P}}^*(t) + \bar{r}^2 T + \mathcal{P}^{*T}(t)E\mathcal{P}^*(t), \quad \forall t \geq t_0 \tag{46}$$

where $\bar{\gamma}$ is a positive constant and $\mathcal{R}\left(Q^*, R^{-1}\right)$ denotes a positive definite matrix function. Then, the uncertain system in (30) *is* (M, β)*-stable in the mean and*

$$M_\beta^2 \leq \sup_{t \geq t_0} \sigma_{\max}\left(\bar{\mathcal{P}}^*(t)\right) / \sigma_{\min}\left(\bar{\mathcal{P}}^*(t_0)\right) \tag{47}$$

where M_β represents the transient bound of the uncertain system's mean response.

Proof Since $\bar{\Upsilon}(t)$ is assumed to generate an exponentially stable evolution operator, there exists a bounded positive definite matrix, $\bar{\mathcal{P}}^*(t)$, that satisfies equation (45). Note that (45) can be written as

$$\begin{aligned}\dot{\bar{\mathcal{P}}}^*(t) = \left[\bar{\Upsilon}(t) + \Delta\Upsilon(t)\right]\bar{\mathcal{P}}^*(t) + \bar{\mathcal{P}}^*(t)\left[\bar{\Upsilon}(t) + \Delta\Upsilon(t)\right]^T - \bar{\gamma}\bar{\mathcal{P}}^*(t)\bar{\mathcal{P}}^*(t) + \mathcal{R}\left(Q^*, R^{-1}\right) \\ - \Delta\Upsilon(t)\bar{\mathcal{P}}^*(t) - \bar{\mathcal{P}}^*(t)\Delta\Upsilon^T(t)\end{aligned}$$

The solution to above equation is

$$\begin{aligned}\bar{\mathcal{P}}^*(t) = \Phi(t,t_0)\bar{\mathcal{P}}^*(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t} \Phi(t,\tau)\Big\{\mathcal{R}\left(Q^*, R^{-1}\right) - \bar{\gamma}\bar{\mathcal{P}}^*(\tau)\bar{\mathcal{P}}^*(\tau) - \\ \Delta\Upsilon(\tau)\bar{\mathcal{P}}^*(\tau) - \bar{\mathcal{P}}^*(\tau)\Delta\Upsilon^T(\tau)\Big\}\Phi^T(t,\tau)d\tau\end{aligned}$$

where $\Phi(t, t_0)$ is the evolution operator generated by $\Upsilon(t) = \bar{\Upsilon}(t) + \Delta\Upsilon(t)$. Based on Lemma 4.1 and the matrix inequality equation (46)

$$\mathcal{R}\left(Q^*, R^{-1}\right) - \bar{\gamma}\bar{\mathcal{P}}^*(t)\bar{\mathcal{P}}^*(t) - \Delta\Upsilon(t)\bar{\mathcal{P}}^*(t) - \bar{\mathcal{P}}^*(t)\Delta\Upsilon^T(t) \geq 0$$

Thus

$$\bar{\mathcal{P}}^*(t) \geq \Phi(t,t_0)\bar{\mathcal{P}}^*(t_0)\Phi^T(t,t_0) \geq \sigma_{\min}\left(\bar{\mathcal{P}}^*(t_0)\right)\Phi(t,t_0)\Phi^T(t,t_0)$$

Now (47) follows from

$$\sigma_{\max}\left(\bar{\mathcal{P}}^*(t)\right) / \sigma_{\min}\left(\bar{\mathcal{P}}^*(t_0)\right) \geq \parallel \Phi(t,t_0) \parallel^2$$

Therefore, $\Phi(t, t_0)$ generates an exponentially stable evolution. □

Assuming the system uncertainties can be written in the form given in (41), a stabilizing process noise covariance, Q^*, can be calculated. Notice that bounds on the system uncertainties used here may be highly conservative and therefore it may result in an extremely large value of Q. As mentioned earlier, selecting a large Q results in a noisy control signal and it could lead to problems such as chattering. Also note that obtaining the upper bound $\bar{r}$ is rather difficult since the system uncertainties, $\Delta F(t)$ and $\Delta D(t)$, may depend on the estimator gain, $K(t)$. Thus increasing the process noise covariance would also increase the upper bound on the uncertainty, i.e., $\bar{r}$. Finally, the reader should realize that the dependency of system uncertainties on the estimator gain is eliminated if the control distribution matrix is precisely known, i.e., $\Delta B = 0$. For more details, please refer to (18).

5 Adaptive Scheme

After substituting the disturbance accommodating control law, $\mathbf{u}(t) = S\hat{\mathbf{Z}}(t)$, the plant dynamics in (1) can written as

$$\dot{\mathbf{X}}(t) = \mathcal{A}\mathbf{X}(t) + \mathcal{B}\left[S\hat{\mathbf{Z}}(t) + B^{-1}\mathbf{W}(t)\right] \tag{48}$$

and the estimator dynamics in (14) can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = [F_m + D_m S]\,\hat{\mathbf{Z}}(t) + K(t)\widetilde{\mathbf{Y}}(t) \tag{49}$$

where $\widetilde{\mathbf{Y}}(t) = [\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)]$. Let $\mathbf{X}_{\text{ext}}(t) = \left[\mathbf{X}_1^T(t)\ \mathbf{X}_2^T(t)\ \mathbf{W}^T(t)\right]^T$, now based on assumption 3.1, the controlled plant in (48) can be written as

$$\dot{\mathbf{X}}_{\text{ext}}(t) = \mathcal{A}_{\text{ext}}\mathbf{X}_{\text{ext}}(t) + \mathcal{B}_{\text{ext}}S\hat{\mathbf{Z}}(t) + G\boldsymbol{\mathcal{V}}(t) \tag{50}$$

where

$$\mathcal{A}_{\text{ext}} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & 0_{(n-r)\times r} \\ A_3 & A_4 & I_{r\times r} \\ A_{\mathbf{w}_1} & A_{\mathbf{w}_2} & A_{\mathbf{w}_3} \end{bmatrix} \quad \text{and} \quad \mathcal{B}_{\text{ext}} = \begin{bmatrix} 0_{(n-r)\times r} \\ B \\ 0_{r\times r} \end{bmatrix}$$

The following assumptions are now made.

Assumption 5.1**:** The pair $(\mathcal{A}_{\text{ext}}, \mathcal{B}_{\text{ext}})$ is controllable and the pair $(\mathcal{A}_{\text{ext}}, H)$ is observable.

Assumption 5.2**:** There exist an $r \times m$ matrix Π such that $\Pi^T\Pi \geq I_{m\times m}$, i.e., $m \leq r$. If $m > r$, then the r-outputs considered here are selected such that the corresponding $(\mathcal{A}_{\text{ext}}, H)$ is observable.

Assumption 5.3**:** There exists an $m \times m$ matrix $\bar{R} > 0$ such that $\forall t \geq t_0$, we have

$$E\left[\widetilde{Y}(t)\widetilde{Y}^T(t)\right] \geq \bar{R}$$

Based on assumptions 5.1, 5.2, and 5.3, an adaptive scheme for selecting the stabilizing process noise covariance matrix can be developed as shown next.

Theorem 5.4: *Given assumptions 5.1, 5.2, and 5.3, the controlled system is mean square stable,* $E[\mathbf{X}(t)] \in L_2 \cap L_\infty$ *and* $\mathbf{X}(t)$ *is asymptotically stable in the first moment, i.e.,*

$$\lim_{t\to\infty} E\left[\mathbf{X}(t)\right] = \mathbf{0}$$

if the process noise covariance is updated online using the adaptive law

$$dQ(t) = \left\{A_Q Q(t) + Q(t)A_Q^T + \gamma\Pi\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\Pi^T\right\}dt \tag{51}$$

where A_Q *is an* $r\times r$ *negative definite matrix such that* $0 < -2\text{Tr}\left\{A_Q\right\} \leq 1$ *and* γ *is the adaptive gain.*

Proof of this theorem is based on the following lemmas.

Lemma 5.5: *Consider the following linear stochastic system*

$$\dot{\mathbf{Z}}(t) = A\mathbf{Z}(t) + \mathbf{U}(t)$$

If the matrix A generates an exponentially stable evolution operator $\Phi_A(t-t_0)$ and $\mathbf{U}(t) \in L_2$, i.e.,

$$E\left[\int_{t_0}^{\infty} |\mathbf{U}(\tau)|^2 d\tau\right]^{1/2} < \infty$$

where $|\cdot|$ represents the Euclidean norm, then $\mathbf{Z}(t) \in L_2 \cap L_\infty$ and

$$\lim_{t\to\infty} E\left[\mathbf{Z}(t)\right] = \mathbf{0}$$

Proof The solution $\mathbf{Z}(t)$ can be written as

$$\mathbf{Z}(t) = \Phi_A(t-t_0)\mathbf{Z}(t_0) + \int_{t_0}^{t} \Phi_A(t-\tau)\mathbf{U}(\tau)d\tau$$

Since $\Phi_A(t-t_0)$ is exponentially stable

$$\| \Phi_A(t-t_0) \| \leq \lambda_0 e^{-a(t-t_0)} \leq \lambda_0, \quad \forall t \geq t_0$$

where $\|\cdot\|$ represents any induced matrix norm and λ_0 and a are two positive constants. Thus

$$\begin{aligned}
|\mathbf{Z}(t)| &\leq \| \Phi_A(t-t_0) \| \, |\mathbf{Z}(t_0)| + \int_{t_0}^{t} \| \Phi_A(t-\tau) \| \, |\mathbf{U}(\tau)| d\tau \\
&\leq \lambda_0 e^{-a(t-t_0)} |\mathbf{Z}(t_0)| + \int_{t_0}^{t} \lambda_0 e^{-a(t-\tau)} |\mathbf{U}(\tau)| d\tau \\
&\leq \lambda_0 e^{-a(t-t_0)} |\mathbf{Z}(t_0)| + \int_{t_0}^{t} \lambda_0 e^{-(a-a_0/2)(t-\tau)} e^{-a_0/2(t-\tau)} |\mathbf{U}(\tau)| d\tau
\end{aligned}$$

The last inequality is obtained by expressing $e^{-a(t-\tau)}$ as $e^{-(a-a_0/2)(t-\tau)}e^{-a_0/2(t-\tau)}$, where $a_0 < 2a$ is a positive constant. Applying the Schwartz inequality yields

$$|\mathbf{Z}(t)| \leq \lambda_0 e^{-a(t-t_0)} |\mathbf{Z}(t_0)| + \lambda_0 \left(\int_{t_0}^{t} e^{-(2a-a_0)(t-\tau)} d\tau\right)^{1/2} \left(\int_{t_0}^{t} e^{-a_0(t-\tau)} |\mathbf{U}(\tau)|^2 d\tau\right)^{1/2}$$

Thus

$$|\mathbf{Z}(t)| \leq \lambda_0 e^{-a(t-t_0)} |\mathbf{Z}(t_0)| + \frac{\lambda_0}{\sqrt{(2a-a_0)}} \left(\int_{t_0}^{t} e^{-a_0(t-\tau)} |\mathbf{U}(\tau)|^2 d\tau\right)^{1/2}$$

and

$$E\left[|\mathbf{Z}(t)|\right] \leq \lambda_0 e^{-a(t-t_0)} E\left[|\mathbf{Z}(t_0)|\right] + \frac{\lambda_0}{\sqrt{(2a-a_0)}} E\left[\left(\int_{t_0}^{t} e^{-a_0(t-\tau)} |\mathbf{U}(\tau)|^2 d\tau\right)^{1/2}\right]$$

Therefore

$$\lim_{t\to\infty} E\left[\mathbf{Z}(t)\right] = \mathbf{0}$$

Also note that $E[|\mathbf{Z}(t)|]$ is bounded by

$$E[|\mathbf{Z}(t)|] \leq \lambda_0 e^{-a(t-t_0)} E[|\mathbf{Z}(t_0)|] + \frac{\lambda_0}{\sqrt{(2a-a_0)}} E\left[\left(\int_{t_0}^{t} |\mathbf{U}(\tau)|^2 d\tau\right)^{1/2}\right]$$

Since $\mathbf{U}(t) \in L_2$

$$E\left[\left(\int_{t_0}^{\infty} |\mathbf{U}(\tau)|^2 d\tau\right)^{1/2}\right] \leq E\left[\int_{t_0}^{\infty} |\mathbf{U}(\tau)|^2 d\tau\right]^{1/2} < \infty$$

Note that $\mathbf{Z}(t) \in L_2$ since

$$\int_{t_0}^{\infty}\left(\int_{t_0}^{t} e^{-a_0(t-\tau)}|\mathbf{U}(\tau)|^2 d\tau\right) dt \leq \int_{t_0}^{\infty} |\mathbf{U}(\tau)|^2 \left(\int_{t_0}^{\infty} e^{-a_0(t-\tau)} dt\right) d\tau \leq \frac{1}{a_0}\int_{t_0}^{\infty} |\mathbf{U}(\tau)|^2 d\tau$$

Finally note that

$$\mathbf{Z}(t) \in L_2 \Rightarrow E\left[\mathbf{Z}(t)\right] \in L_2$$

□

Lemma 5.6: *Consider the following linear stochastic system*

$$\dot{\mathbf{Z}}(t) = A\mathbf{Z}(t) + \mathbf{U}(t)$$
$$\mathbf{Y}(t) = C\mathbf{Z}(t)$$

If (A, C) is observable, $\mathbf{Y}(t) \in L_2$ and $\mathbf{U}(t) \in L_2$, then $\mathbf{Z}(t) \in L_2 \cap L_\infty$ and

$$\lim_{t\to\infty} E\left[\mathbf{Z}(t)\right] = \mathbf{0}$$

Proof If (A, C) is observable, then there exist a matrix K such that $A_o = A - KC$ is exponentially stable. Now $\dot{\mathbf{Z}}(t)$ can be written as

$$\dot{\mathbf{Z}}(t) = A_o\mathbf{Z}(t) + \mathbf{U}(t) + K\mathbf{Y}(t)$$

Thus from Lemma. 5.5 one could conclude that $\mathbf{Z}(t) \in L_2 \cap L_\infty$ and

$$\lim_{t\to\infty} E\left[\mathbf{Z}(t)\right] = \mathbf{0}$$

□

The stability analysis given in section 3 reveals that selecting a sufficiently large process noise covariance would guarantee asymptotic stability of the controlled system's mean response. Thus the adaptive law given in Theorem 5.4 increases the process noise covariance to ensure that $E\left[\widetilde{\mathbf{Y}}(t)\right] \in L_2$. Now based on the above lemmas the proof of Theorem 5.4 can be easily obtained as shown next.

Proof

Let $\mathcal{F}_t^{\widetilde{\mathbf{Y}}}$ denotes a filtration generated by $\widetilde{\mathbf{Y}}(t)$, i.e.

$$E\left[\widetilde{\mathbf{Y}}(s) \mid \mathcal{F}_t^{\widetilde{\mathbf{Y}}}\right] = \widetilde{\mathbf{Y}}(s) \qquad s \leq t$$

Now consider the following nonnegative function:

$$V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) = \int_{t_0}^{t} E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right] d\tau + E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right] + \text{Tr}\left\{Q^* - Q(t)\right\} + \text{Tr}\left\{P^*_{\max} - P(t)\right\}$$

where $Q^* \geq Q(t)\ \ \forall t \geq t_0$ is a stabilizing process noise covariance and $P^*_{\max}$ is selected such that $P^*_{\max} \geq P^*(t)$, $\forall t \geq t_0$, where $P^*(t)$ may be obtained by solving the continuous-time matrix differential Riccati equation:

$$\dot{P}^*(t) = F_m P^*(t) + P^*(t) {F_m}^T - P^*(t) H^T R^{-1} H P^*(t) + G Q^* G^T, \quad P^*(t_0) = P_0$$

Note that for any $Q(t) \leq Q^*$, $P(t) \leq P^*_{\max}$, where $P(t)$ satisfies

$$\dot{P}(t) = F_m P(t) + P(t) {F_m}^T - P(t) H^T R^{-1} H P(t) + G Q(t) G^T, \quad P(t_0) = P_0 \tag{52}$$

More details on this can be found in the *comparison results* given in chapter 4 of Abou-Kandil et al. (2003). The matrix $\mathcal{X}$ is a positive definite matrix of appropriate dimensions and it is selected so that it satisfies the following matrix inequality

$$\mathcal{X}\left[F_m + D_m S\right] + \left[F_m + D_m S\right]^T \mathcal{X} + \mathcal{X}\mathcal{X} + \mathcal{M} \leq 0$$

where $\mathcal{M} > 0$. It is important to note that the expectation given in the above nonnegative function is conditioned on the filtration at the lower time limit. For example, consider a time instant s such that $t_0 \leq s \leq t$, now $V(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)$ can be written as

$$V(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) = \int_{s}^{t} E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] d\tau + E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] + \text{Tr}\left\{Q^* - Q(t)\right\} + \text{Tr}\left\{P^*_{\max} - P(t)\right\}$$

Now $dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)$ can be calculated as

$$dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) = E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] dt + E\left[d\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] + E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[d\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] - \text{Tr}\left\{dQ(t)\right\} - \text{Tr}\left\{dP(t)\right\}$$

Note that

$$E\left[d\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] = \left[F_m + D_m S\right] E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] dt + E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_{s}^{\widetilde{\mathbf{Y}}}\right] dt$$

Thus

$$dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) = E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\Big\{dQ(t)\Big\} - \mathrm{Tr}\Big\{dP(t)\Big\} +$$

$$E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T [F_m + D_m S]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt +$$

$$E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} [F_m + D_m S] E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt$$

Notice that for any two vectors $\mathbf{a}$ and $\mathbf{b}$ of same dimensions, the following inequality holds

$$\mathbf{a}^T\mathbf{a} + \mathbf{b}^T\mathbf{b} \geq \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{a}$$

i.e.,

$$E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X}\mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] + E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] \geq$$

$$E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] + E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]$$

Therefore

$$dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \leq E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\Big\{dQ(t)\Big\} - \mathrm{Tr}\Big\{dP(t)\Big\} +$$

$$E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \Big\{\mathcal{X}[F_m + D_m S] + [F_m + D_m S]^T \mathcal{X} + \mathcal{X}\mathcal{X}\Big\} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt +$$

$$E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt$$

Now employing the Cauchy-Schwarz's inequality gives

$$dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \leq E\left[\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + E\left[\widetilde{\mathbf{Y}}^T(t)K^T(t)K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\Big\{dQ(t)\Big\}$$

$$+ E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \Big\{\mathcal{X}[F_m + D_m S] + [F_m + D_m S]^T \mathcal{X} + \mathcal{X}\mathcal{X}\Big\} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\Big\{dP(t)\Big\}$$

Substituting (51) and (52) yields

$$\begin{aligned} dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \leq\ & E\left[\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + E\left[\widetilde{\mathbf{Y}}^T(t)K^T(t)K(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt \\ & - E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\left\{A_Q Q(t) + Q(t)A_Q^T + \gamma\Pi\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\Pi^T\right\} dt \\ & - \mathrm{Tr}\left\{F_m P(t) + P(t){F_m}^T - P(t)H^T R^{-1} H P(t) + G Q(t) G^T\right\} dt \end{aligned}$$

Note that

$$-\mathrm{Tr}\left\{A_Q Q(t) + Q(t)A_Q^T\right\} = -2\mathrm{Tr}\left\{A_Q Q(t)\right\} \leq -2\mathrm{Tr}\left\{A_Q\right\}\mathrm{Tr}\left\{Q(t)\right\} \leq \mathrm{Tr}\left\{Q(t)\right\}$$

The first inequality is valid because $-A_Q$ is positive definite and the process noise covariance $Q(t)$ is positive semi-definite (Yang 2000). The last inequality holds since $0 < -2\mathrm{Tr}\left\{A_Q\right\} \leq 1$.

Also note that due to the nature of matrix G, we have

$$\mathrm{Tr}\left\{GQ(t)G^T\right\} = \mathrm{Tr}\left\{Q(t)\right\}$$

Thus

$$\begin{aligned} dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) &\leq \mathrm{Tr}\left\{E\left[\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\right\}dt + \mathrm{Tr}\left\{E\left[K^T(t)K(t)\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\right\}dt \\ &\quad - E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt - \gamma\mathrm{Tr}\left\{\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\Pi^T\Pi\right\}dt \\ &\quad + \mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t)\right\}dt - 2\mathrm{Tr}\left\{F_mP(t)\right\}dt \\ &\leq E\left[\left(1+ \parallel K(t)K^T(t) \parallel\right)\mathrm{Tr}\left\{\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\right\} \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt - E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \\ &\quad - \gamma\mathrm{Tr}\left\{\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\right\}dt + \mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t)\right\}dt - 2\mathrm{Tr}\left\{F_mP(t)\right\}dt \end{aligned}$$

The second inequality holds since

$$|\widetilde{\mathbf{Y}}(t)|^2 + |K(t)\widetilde{\mathbf{Y}}(t)|^2 \leq \left(1+ \parallel K(t)K^T(t) \parallel\right)|\widetilde{\mathbf{Y}}(t)|^2$$

Therefore $E\left[dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]$ can be written as

$$\begin{aligned} E\left[dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] &\leq E\left[\left(1+ \parallel K(t)K^T(t) \parallel\right)|\widetilde{\mathbf{Y}}(t)|^2 \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \\ &\quad - E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt - \gamma E\left[|\widetilde{\mathbf{Y}}(t)|^2 \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \\ &\quad + E\left[\mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t) - 2F_mP(t)\right\} \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \end{aligned}$$

Combining the similar terms yields

$$\begin{aligned} E\left[dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] &\leq E\left[\left(1+ \parallel K(t)K^T(t) \parallel -\gamma\right)|\widetilde{\mathbf{Y}}(t)|^2 \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \\ - E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt &+ E\left[\mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t) - 2F_mP(t)\right\} \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \end{aligned}$$

Let $\gamma = \gamma_1 + \gamma_2$ where γ_1 is selected such that

$$\gamma_1 \geq 1+ \parallel K(t)K^T(t) \parallel \tag{53}$$

Thus

$$\begin{aligned} E\left[dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] &\leq -\gamma_2\mathrm{Tr}\left\{E\left[\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\right\}dt - \\ E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt &+ E\left[\mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t) - 2F_mP(t)\right\} \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \end{aligned}$$

Now based on assumption 5.3, we have

$$\begin{aligned} E\left[dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] &\leq -E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{Z}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt + \\ &\quad E\left[\mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t) - 2F_mP(t)\right\} - \gamma_2\mathrm{Tr}\left\{\bar{R}\right\} \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt \end{aligned}$$

Finally note that $E\left[dV(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq 0$ if

$$\gamma_2 \geq \mathrm{Tr}\left\{P(t)H^T R^{-1} H P(t) - 2F_m P(t)\right\}\mathrm{Tr}\left\{\bar{R}\right\}^{-1} \tag{54}$$

Assuming $\widetilde{\mathbf{Y}}(t_0)$ is precisely known yields,

$$E\left[\widetilde{\mathbf{Y}}(t)\mid\mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right] = E\left[\widetilde{\mathbf{Y}}(t)\right]$$

Thus selecting γ_1 and γ_2 according to Eqs. (53) and (54) yields

$$E\left[V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\right] - V(t_0,t_0,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) = \int_{t_0}^{t} E\left[dV(t_0,\tau,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\right]\leq 0$$

Therefore

$$E\left[V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\right]\leq \hat{\mathbf{z}}^T(t_0)\mathcal{X}\hat{\mathbf{z}}(t_0) + \mathrm{Tr}\left\{Q^* - Q(t_0)\right\} + \mathrm{Tr}\left\{P^*_{\max} - P(t_0)\right\}$$

Also note

$$E\left[V(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] - V(t_0,s,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q) = \int_{s}^{t} E\left[dV(s,\tau,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq 0$$

Thus

$$E\left[V(s,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq V(t_0,s,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)$$

Now the properties of $V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)$ may be summarized as

(*i*) $V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\geq 0$
(*ii*) $E\left[V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\right]<\infty$
(*iii*) $E\left[V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq V(t_0,s,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q),\qquad s\leq t$
(*iv*) $V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)$ is adapted to $\mathcal{F}_t^{\widetilde{\mathbf{Y}}}$

These properties imply that $V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)$ is a nonnegative $\mathcal{F}_s^{\widetilde{\mathbf{Y}}}$-supermartingale (Kushner 1967, Liptser and Shiryayev 1989) and the nonnegative supermartingale probability inequality yields (Doob 1953)

$$\mathbb{P}\Big(\sup_{t\geq t_0} V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)\geq\lambda\Big)\leq\frac{\hat{\mathbf{z}}^T(t_0)\mathcal{X}\hat{\mathbf{z}}(t_0) + \mathrm{Tr}\left\{Q^* - Q(t_0)\right\} + \mathrm{Tr}\left\{P^*_{\max} - P(t_0)\right\}}{\lambda}$$

where $\lambda>0$ is any positive constant. Thus selecting sufficiently large λ yields

$$\mathbb{P}\Big(\sup_{t\geq t_0} V(t_0,t,\hat{\mathbf{Z}},\widetilde{\mathbf{Y}},P,Q)<\infty\Big) = 1$$

That is, $V(t_0, t, \hat{\mathbf{Z}}, \widetilde{\mathbf{Y}}, P, Q)$ is almost surely bounded. Note $V(t_0, t, \hat{\mathbf{Z}}, \widetilde{\mathbf{Y}}, P, Q)$ is defined as

$$V(t_0, t, \hat{\mathbf{Z}}, \widetilde{\mathbf{Y}}, P, Q) = \int_{t_0}^{t} E\left[\widetilde{\mathbf{Y}}(\tau)\right]^T E\left[\widetilde{\mathbf{Y}}(\tau)\right] d\tau + E\left[\hat{\mathbf{Z}}(t)\right]^T \mathcal{X} E\left[\hat{\mathbf{Z}}(t)\right] + \mathrm{Tr}\Big\{Q^* - Q(t)\Big\} + \mathrm{Tr}\Big\{P^*_{\max} - P(t)\Big\}$$

Therefore

$$V(t_0, t, \hat{\mathbf{Z}}, \widetilde{\mathbf{Y}}, P, Q) \in L_\infty \quad a.s. \quad \Longrightarrow E\left[\widetilde{\mathbf{Y}}(t)\right] \in L_2,\ Q(t) \in L_\infty \text{ and } \ P(t) \in L_\infty \quad a.s.$$

Since $P(t)$ is $a.s.$ bounded, the estimator gain, $K(t) = P(t)H^T R^{-1}$, is also $a.s.$ bounded. Thus there exist a k^* such that

$$\mathbb{P}\left(\sup_{t \geq t_0} \parallel K(t) \parallel > k^*\right) = 0$$

The estimator dynamics is given as

$$\dot{\hat{\mathbf{Z}}}(t) = [F_m + D_m S]\, \hat{\mathbf{Z}}(t) + K(t)\widetilde{\mathbf{Y}}(t)$$

Since $[F_m + D_m S]$ generates an exponentially stable evolution operator, and since $E[\widetilde{\mathbf{Y}}(t)] \in L_2$, based on Lemma 5.5, it can be shown that $E[\hat{\mathbf{Z}}(t)] \in L_2 \cap L_\infty$, and

$$\lim_{t \to \infty} E\left[\hat{\mathbf{Z}}(t)\right] = \mathbf{0}$$

Since $E[\widetilde{\mathbf{Y}}(t)] \in L_2$,

$$E[\hat{\mathbf{Z}}(t)] \in L_2 \Longrightarrow E[\mathbf{Y}(t)] \in L_2$$

Now given the observability assumption, based on Lemma 5.6, it can be shown that $E[\mathbf{X}(t)] \in L_2 \cap L_\infty$, and

$$\lim_{t \to \infty} E\left[\mathbf{X}(t)\right] = \mathbf{0}$$

Finally note that the controlled closed-loop system can be written as

$$\begin{bmatrix} \dot{\mathbf{X}}_{\mathrm{ext}}(t) \\ \dot{\hat{\mathbf{Z}}}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{\mathrm{ext}} & \mathcal{B}_{\mathrm{ext}} S \\ K(t)H & \{F_m + D_m S - K(t)H\} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathrm{ext}}(t) \\ \hat{\mathbf{Z}}(t) \end{bmatrix} + \begin{bmatrix} G\boldsymbol{\mathcal{V}}(t) \\ K(t)\mathbf{V}(t) \end{bmatrix} \tag{55}$$

Note that the closed-loop state matrix

$$F_{\mathrm{CL}}(t) = \begin{bmatrix} \mathcal{A}_{\mathrm{ext}} & \mathcal{B}_{\mathrm{ext}} S \\ K(t)H & \{F_m + D_m S - K(t)H\} \end{bmatrix}$$

is bounded. Also, the asymptotic stability of $E[\mathbf{X}_{\mathrm{ext}}(t)]$ and $E[\hat{\mathbf{Z}}(t)]$ implies that the matrix, $F_{\mathrm{CL}}(t)$, generates an asymptotically stable evolution operator, $\Phi_{\mathrm{CL}}(t, t_0)$, i.e.,

$$\lim_{t \to \infty} \|\Phi_{\mathrm{CL}}(t, t_0)\| = 0$$

Equation (55) can be written in Itô form as

$$d\mathbf{X}_{\rm CL}(t) = F_{\rm CL}(t)\mathbf{X}_{\rm CL}(t)dt + \Gamma_{\rm CL}(t)d\boldsymbol{\mathcal{B}}_{\rm CL}(t) \tag{56}$$

where

$$\Gamma_{\rm CL}(t) = \begin{bmatrix} G & 0 \\ 0 & K(t) \end{bmatrix} \quad \text{and} \quad E\left[d\boldsymbol{\mathcal{B}}_{\rm CL}(t)d\boldsymbol{\mathcal{B}}_{\rm CL}^T(t)\right] = \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & R \end{bmatrix} dt = Q_{\rm CL}dt$$

Remark 5: It is important to note that if one wishes to express (55) in Stratonovich form, the results given here holds since we are considering linear stochastic differential equations with state free diffusion term and the solution obtained from the Stratonovich integral equation converges *a.s.* and uniformly to that obtained from the Itô integral equation. For more details please refer to the Wong-Zakai theorem (Grigoriu 2002).

Now using Itô formula $d\left(\mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t)\right)$ can be written as

$$\begin{aligned} d\left(\mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t)\right) &= \mathbf{X}_{\rm CL}(t)d\left(\mathbf{X}_{\rm CL}(t)\right)^T + d\left(\mathbf{X}_{\rm CL}(t)\right)\mathbf{X}_{\rm CL}^T(t) + \Gamma_{\rm CL}(t)Q_{\rm CL}\Gamma_{\rm CL}(t)dt \\ &= \Big\{\mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t)F_{\rm CL}^T(t) + F_{\rm CL}(t)\mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t) + \Gamma_{\rm CL}(t)Q_{\rm CL}\Gamma_{\rm CL}(t)\Big\}dt \\ &\quad + \Gamma_{\rm CL}(t)d\boldsymbol{\mathcal{B}}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t) + \mathbf{X}_{\rm CL}(t)d\boldsymbol{\mathcal{B}}_{\rm CL}^T(t)\Gamma_{\rm CL}^T(t) \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t) &= \Phi_{\rm CL}(t,t_0)\mathbf{X}_{\rm CL}(t_0)\mathbf{X}_{\rm CL}^T(t_0)\Phi_{\rm CL}^T(t,t_0) + \int_{t_0}^t \Big[\mathbf{X}_{\rm CL}(\tau)\left[\Gamma_{\rm CL}(\tau)d\boldsymbol{\mathcal{B}}_{\rm CL}(\tau)\right]^T\Big]^T \\ &\quad + \int_{t_0}^t \mathbf{X}_{\rm CL}(\tau)\left[\Gamma_{\rm CL}(\tau)d\boldsymbol{\mathcal{B}}_{\rm CL}(\tau)\right]^T + \int_{t_0}^t \Phi_{\rm CL}(t,\tau)\Gamma_{\rm CL}(\tau)Q_{\rm CL}\Gamma_{\rm CL}^T(\tau)\Phi_{\rm CL}^T(t,\tau)d\tau \end{aligned}$$

Therefore

$$\begin{aligned} E\Big[\mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t)\Big] &= E\Big[\Phi_{\rm CL}(t,t_0)\mathbf{X}_{\rm CL}(t_0)\mathbf{X}_{\rm CL}^T(t_0)\Phi_{\rm CL}^T(t,t_0) \\ &\quad + \int_{t_0}^t \Phi_{\rm CL}(t,\tau)\Gamma_{\rm CL}(\tau)Q_{\rm CL}\Gamma_{\rm CL}^T(\tau)\Phi_{\rm CL}^T(t,\tau)d\tau\Big] \end{aligned}$$

Since $\Phi_{\rm CL}(t,t_0)$ is an asymptotically stable evolution operator and $\Gamma_{\rm CL}(t)$ is bounded, it can be easily shown that the closed-loop system is mean square stable, i.e.,

$$\lim_{t\to\infty} E[\mathbf{X}_{\rm CL}(t)\mathbf{X}_{\rm CL}^T(t)] < \mathrm{M}$$

where M is a constant square matrix whose elements are finite (Soong 1973). □

Even though the initial process noise covariance, $Q(t_0)$, may not be the stabilizing Q, the adaptive law given in (51) can be used to update the process noise covariance online so that the controlled system is asymptotically stable. A schematic representation of the proposed adaptive controller is given in figure 2.

6 Simulation Results

For simulation purposes, consider a two degree-of-freedom helicopter that pivots about the pitch axis by angle θ and about the yaw axis by angle ψ. As shown in figure 3, pitch is defined positive

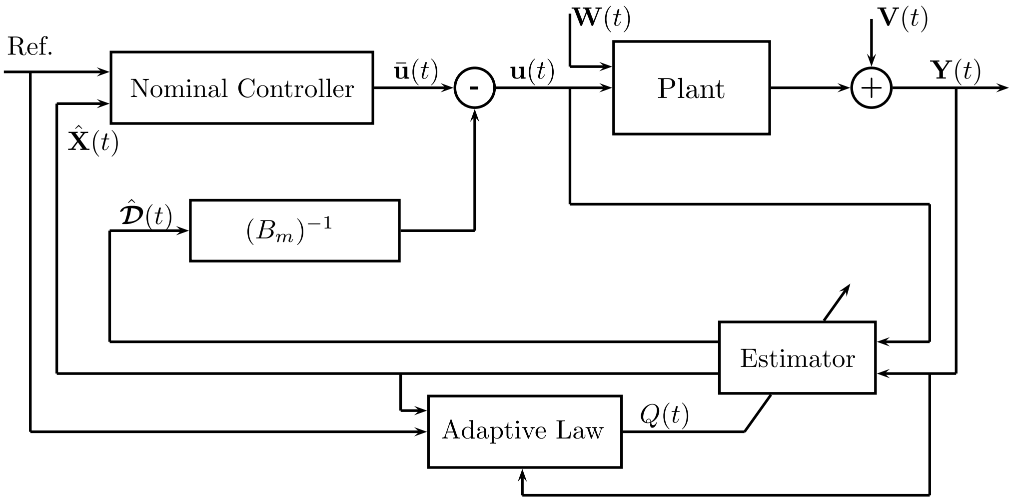


Figure 2. Adaptive DAC Block Diagram

when the nose of the helicopter goes up and yaw is defined positive for a counterclockwise rotation. Also in figure 3, there is a thrust force F_p acting on the pitch axis that is normal to the plane of the front propeller and a thrust force F_y acting on the yaw axis that is normal to the rear propeller. Therefore a pitch torque is being applied at a distance r_p from the pitch axis and a yaw torque is applied at a distance r_y from the yaw axis. The gravitational force, F_g, generates a torque at the helicopter center of mass that pulls down on the helicopter nose. As shown in figure 3, the center of mass is a distance of l_{cm} from the pitch axis along the helicopter body length.

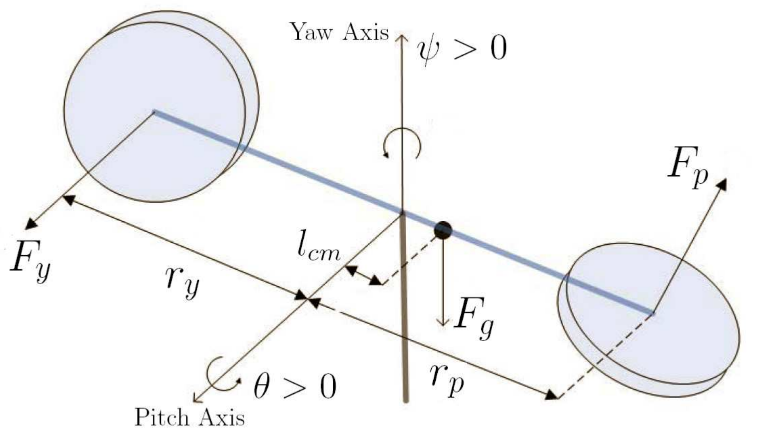


Figure 3. Two Degree of Freedom Helicopter

After linearizing about $\theta(t_0) = \psi(t_0) = \dot{\theta}(t_0) = \dot{\psi}(t_0) = 0$, the helicopter equations of motion can be written as

$$(J_{eq,p} + m_{heli} l_{cm}^2)\ddot{\theta}(t) = K_{pp} V_{m,p}(t) + K_{py} V_{m,y}(t) - B_p \dot{\theta}(t) + W_1(t) \tag{57a}$$

$$(J_{eq,y} + m_{heli} l_{cm}^2)\ddot{\psi}(t) = K_{yy} V_{m,y}(t) + K_{yp} V_{m,p}(t) - B_y \dot{\psi}(t) + W_2(t) \tag{57b}$$

A detailed description of system parameters and assumed values are given in Table 2. The system states are the pitch and yaw angles and their corresponding rates, i.e., $\theta(t)$, $\psi(t)$, $\dot{\theta}(t)$, and $\dot{\psi}(t)$. The control input to the system are the input voltages of the pitch and yaw motors, $V_{m,p}$ and

$V_{m,y}$, respectively. The external disturbances are denoted as $W_1(t)$ and $W_2(t)$. Let $\mathbf{X}_1(t) =$

Table 2. Two Degree-of-Freedom Helicopter Model Parameters

System Parameter	Description	Assumed Values	True Values	Unit
B_p	Equivalent viscous damping about pitch axis	0.8000	1	N/V
B_y	Equivalent viscous damping about yaw axis	0.3180	-0.3021	N/V
$J_{eq,p}$	Total moment of inertia about yaw pivot	0.0384	0.0288	$Kg{\cdot}m^2$
$J_{eq,y}$	Total moment of inertia about pitch pivot	0.0432	0.0496	$Kg{\cdot}m^2$
K_{pp}	Trust torque constant acting on pitch axis from pitch motor/propeller	0.2041	0.2552	$N{\cdot}m/V$
K_{py}	Trust torque constant acting on pitch axis from yaw motor/propeller	0.0068	0.0051	$N{\cdot}m/V$
K_{yp}	Trust torque constant acting on yaw axis from pitch motor/propeller	0.0219	0.0252	$N{\cdot}m/V$
K_{yy}	Trust torque constant acting on yaw axis from yaw motor/propeller	0.0720	0.0684	$N{\cdot}m/V$
m_{heli}	Total mass of the helicopter	1.3872	1.3872	Kg
l_{cm}	Location of center-of-mass	0.1857	0.1764	m

$\begin{bmatrix}\theta(t) \; \psi(t)\end{bmatrix}^T$, $\mathbf{X}_2(t) = \begin{bmatrix}\dot\theta(t) \; \dot\psi(t)\end{bmatrix}^T$, $\mathbf{u}(t) = \begin{bmatrix}V_{m,p}(t) \; V_{m,y}(t)\end{bmatrix}^T$, and $\mathbf{W}(t) = \begin{bmatrix}W_1(t) \; W_2(t)\end{bmatrix}^T$. For simulation purposes, the external disturbance $\mathbf{W}(t)$ is selected to be

$$\begin{aligned}\dot W_1(t) &= 2.43\dot\theta(t) - 1.3\dot\psi(t) - W_1(t) + 2W_2(t) + \mathcal{V}_1(t)\\ \dot W_2(t) &= -0.34\dot\theta(t) + 1.92\dot\psi(t) + W_1(t) - 3W_2(t) + \mathcal{V}_2(t)\end{aligned} \tag{58}$$

and

$$\begin{bmatrix}\mathcal{V}_1(t)\\ \mathcal{V}_2(t)\end{bmatrix} = \boldsymbol{\mathcal{V}}(t) \sim \mathcal{N}\big(\mathbf{0}, 1\times 10^{-2} I_{2\times 2}\delta(\tau)\big)$$

Now the state-space representation of the plant can be written as

$$\begin{aligned}\dot{\mathbf{X}}_1(t) &= \mathbf{X}_2(t)\\ \dot{\mathbf{X}}_2(t) &= A_4\mathbf{X}_2(t) + B\mathbf{u}(t) + \mathbf{W}(t)\\ \dot{\mathbf{W}}(t) &= A_{\mathbf{w}_2}\mathbf{X}_2(t) + A_{\mathbf{w}_3}\mathbf{W}(t) + \boldsymbol{\mathcal{V}}(t)\end{aligned} \tag{59}$$

where

$$A_{\mathbf{w}_2} = \begin{bmatrix}2.43 & -1.3\\ -0.34 & 1.92\end{bmatrix}, \quad A_{\mathbf{w}_3} = \begin{bmatrix}-1 & 2\\ 1 & -3\end{bmatrix}, \quad A_4 = \begin{bmatrix}a_1 & 0\\ 0 & a_2\end{bmatrix}, \quad B = \begin{bmatrix}b_1 \; b_2\\ b_3 \; b_4\end{bmatrix}$$

and the system parameters are given as

$$a_1 = \frac{-B_p}{(J_{eq,p} + m_{heli} l_{cm}^2)} \qquad a_2 = \frac{-B_y}{(J_{eq,y} + m_{heli} l_{cm}^2)}$$
$$b_1 = \frac{K_{pp}}{(J_{eq,p} + m_{heli} l_{cm}^2)} \qquad b_2 = \frac{K_{py}}{(J_{eq,p} + m_{heli} l_{cm}^2)}$$
$$b_3 = \frac{K_{yp}}{(J_{eq,y} + m_{heli} l_{cm}^2)} \qquad b_4 = \frac{K_{yy}}{(J_{eq,y} + m_{heli} l_{cm}^2)}$$

The state-space representation of the assumed system model is

$$\dot{\mathbf{X}}_{1_m}(t) = \mathbf{X}_{2_m}(t)$$
$$\dot{\mathbf{X}}_{2_m}(t) = A_{4_m} \mathbf{X}_{2_m}(t) + B_m \mathbf{u}(t)$$

where

$$A_{4_m} = \begin{bmatrix} a_{1_m} & 0 \\ 0 & a_{2_m} \end{bmatrix}, \quad B_m = \begin{bmatrix} b_{1_m} & b_{2_m} \\ b_{3_m} & b_{4_m} \end{bmatrix}$$

The measured output equations are given as

$$\mathbf{Y}(t) = C\mathbf{X}(t) + \mathbf{V}(t)$$

where $\mathbf{X}(t) = \left[\mathbf{X}_1^T(t)\ \mathbf{X}_2^T(t)\right]^T$ and $C = \left[I_{2\times2}\ 0_{2\times2}\right]$. Note that the disturbance term, $\boldsymbol{\mathcal{D}}(t) = [\mathcal{D}_{\dot{\theta}}(t) \quad \mathcal{D}_{\dot{\psi}}(t)]^T$, can be written as

$$\mathcal{D}_{\dot{\theta}}(t) = \triangle a_1 \dot{\theta}(t) + \triangle b_1 u_1(t) + \triangle b_2 u_2(t) + W_1(t)$$
$$\mathcal{D}_{\dot{\psi}}(t) = \triangle a_2 \dot{\psi}(t) + \triangle b_3 u_1(t) + \triangle b_4 u_2(t) + W_2(t)$$

The assumed disturbance term dynamics is modeled as

$$\dot{\mathcal{D}}_{\dot{\theta}_m}(t) = -\mathcal{D}_{\dot{\theta}_m}(t) + \mathcal{W}_1(t)$$
$$\dot{\mathcal{D}}_{\dot{\psi}_m}(t) = -3\mathcal{D}_{\dot{\psi}_m}(t) + \mathcal{W}_2(t)$$

Let the extended assumed state vector be $\mathbf{Z}_m(t) = \left[\mathbf{X}_m^T(t)\ \mathcal{D}_{\dot{\theta}_m}(t)\ \mathcal{D}_{\dot{\psi}_m}(t)\right]^T$. Now the assumed extended state-space equation can be written as

$$\dot{\mathbf{Z}}_m(t) = F_m \mathbf{Z}_m(t) + D_m \mathbf{u}(t) + G\boldsymbol{\mathcal{W}}(t)$$

where

$$F_m = \begin{bmatrix} 0_{2\times2} & I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & A_{4_m} & I_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & A_{\mathcal{D}_m} \end{bmatrix}, \quad D_m = \begin{bmatrix} 0_{2\times2} \\ B_m \\ 0_{2\times2} \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0_{4\times2} \\ I_{2\times2} \end{bmatrix}$$

The estimator dynamics can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = F_m\hat{\mathbf{Z}}(t) + D_m\mathbf{u}(t) + K(t)[C\mathbf{X}(t) - H\hat{\mathbf{Z}}(t)] + K(t)\mathbf{V}(t) \tag{60}$$

where $H = [C \quad 0_{2\times2}]$. The nominal controller is a linear quadratic regulator which minimizes the cost function

$$J = \frac{1}{2}E\left[\int_0^\infty \big((\mathbf{X}_m(t) - \mathbf{x}_d)^T\mathcal{Q}_{\mathbf{X}}(\mathbf{X}_m(t) - \mathbf{x}_d) + \mathbf{u}^T(t)\mathcal{R}_{\mathbf{u}}\mathbf{u}(t)\big)dt\right] \tag{61}$$

where $\mathbf{x}_d^T = [\theta_d \quad \psi_d \quad 0 \quad 0]$, θ_d and ψ_d are some desired final values of θ and ψ, respectively, and $\mathcal{Q}_{\mathbf{X}}$ and $\mathcal{R}_{\mathbf{u}}$ are two symmetric positive definite matrices. The nominal control that minimizes the above cost function is

$$\bar{\mathbf{u}}(t) = -K_m(\mathbf{X}_m(t) - \mathbf{x}_d)$$

where K_m is the feedback gain. Now the disturbance accommodating control law can be written in terms of the estimated states and the estimated disturbance term as

$$\mathbf{u}(t) = \begin{bmatrix} -K_m & -(B_m)^{-1}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}(t) - \mathbf{x}_d \\ \hat{\mathcal{D}}_{\dot\theta}(t) \\ \hat{\mathcal{D}}_{\dot\psi}(t)\end{bmatrix} = S\hat{\mathbf{Z}}(t) + K_m\mathbf{x}_d$$

After substituting the above control law, the true extended system dynamics can be written as

$$\begin{bmatrix}\dot{\mathbf{X}}_1(t)\\ \dot{\mathbf{X}}_2(t)\\ \dot{\boldsymbol{\mathcal{D}}}(t)\end{bmatrix} = \begin{bmatrix}0_{2\times2} & I_{2\times2} & 0_{2\times2}\\ 0_{2\times2} & A_{4_m} & I_{2\times2}\\ A_{\mathcal{D}_1}(t) & A_{\mathcal{D}_2}(t) & A_{\mathcal{D}_3}(t)\end{bmatrix}\begin{bmatrix}\mathbf{X}_1(t)\\ \mathbf{X}_2(t)\\ \boldsymbol{\mathcal{D}}(t)\end{bmatrix} + \begin{bmatrix}0_{2\times6}\\ B_mS\\ B_{\mathcal{D}_1}(t)\end{bmatrix}\hat{\mathbf{Z}}(t) + \begin{bmatrix}0_{2\times4}\\ B_mK_m\\ B_{\mathcal{D}_2}\end{bmatrix}\mathbf{x}_d + \begin{bmatrix}\mathbf{0}_{2\times1}\\ \mathbf{0}_{2\times1}\\ \boldsymbol{\mathcal{W}}_a(t)\end{bmatrix} \tag{62}$$

where $A_{\mathcal{D}_1}(t) = \Delta BSK(t)$, $A_{\mathcal{D}_2}(t) = \Delta A_2A_{4_m} + A_{\mathbf{w}_2} - A_{\mathbf{w}_3}\Delta A_2$, $A_{\mathcal{D}_3}(t) = \Delta A_2 + A_{\mathbf{w}_3}$,

$$B_{\mathcal{D}_1}(t) = \{\Delta BS\,[F_m + D_mS - K(t)H] + \Delta A_2B_mS - A_{\mathbf{w}_3}\Delta BS\},$$

and $B_{\mathcal{D}_2}(t) = \Delta A_2B_mK_m - A_{\mathbf{w}_3}\Delta BK_m$. Here $\Delta B = B - B_m$ and $\Delta A_2 = A_4 - A_{4_m}$.

Table 3. Nominal Controller/Estimator Matrices

LQR Weighting Matrices	Covariance Matrices
$\mathcal{R}_{\mathbf{u}} = 10 \times I_{2\times2}$	$Q = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4\end{bmatrix}$, $R = 10^{-3} \times I_{2\times2}$,
$\mathcal{Q}_{\mathbf{x}} = \begin{bmatrix} 500 \times I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 100 \times I_{2\times2}\end{bmatrix}$	$P(t_0) = \begin{bmatrix} 10^{-1} \times I_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 10^2 \times I_{2\times2}\end{bmatrix}$

Table 3 shows the nominal controller and estimator matrices. Since the measurement noise covariance, R, can be obtained from sensor calibration, the process noise matrix, Q, is treated as a tuning parameter. Based on the weighting matrices given in Table 3, the feedback gain is calculated to be

$$K_m = \begin{bmatrix} 7.0229 & 0.8239 & 1.6691 & 0.3310 \\ -0.8239 & 7.0229 & -0.0830 & 2.4486\end{bmatrix}$$

For simulation purposes the initial states are selected to be $[\theta_0 \quad \psi_0 \quad \dot{\theta}_0 \quad \dot{\psi}_0]^T = [-45^o \quad 0 \quad 0 \quad 0]^T$ and the desired states θ_d and ψ_d are selected to be 45^o and 30^o, respectively.

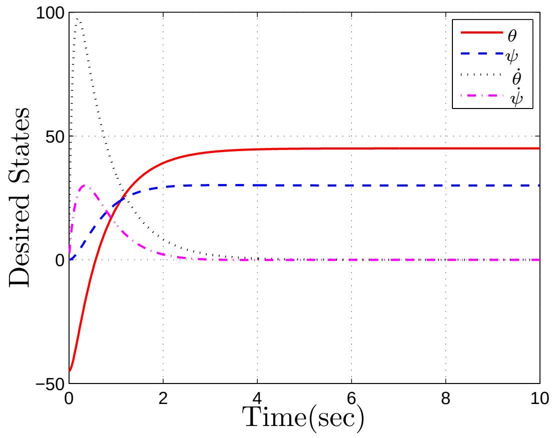
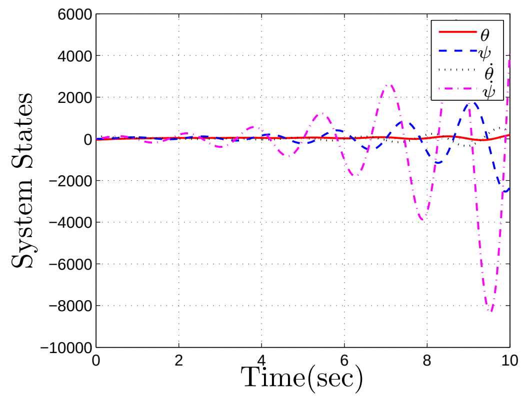


Figure 4. Desired System Response

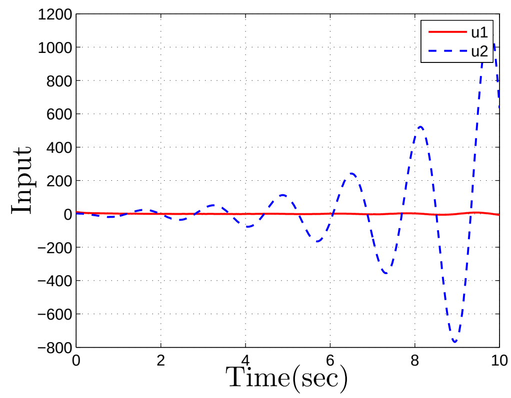
The desired response given in figure 4 is the system response to the nominal control when there is no model error and external disturbance. For illustrative purposes, simulations are conducted using the traditional disturbance accommodating control as well as the proposed adaptive disturbance accommodating control. Results obtained using the traditional DAC is given first.

6.1 *DAC Results*

Figure 5(a) shows the unstable system response obtained for the first simulation where the disturbance term process noise covariance is selected to be $Q = 10^3 \times I_{2\times2}$. Figure 5(b) shows the input corresponding to the first simulation scenario. Figures 6(a) and 6(b) contain the estimated



(a)

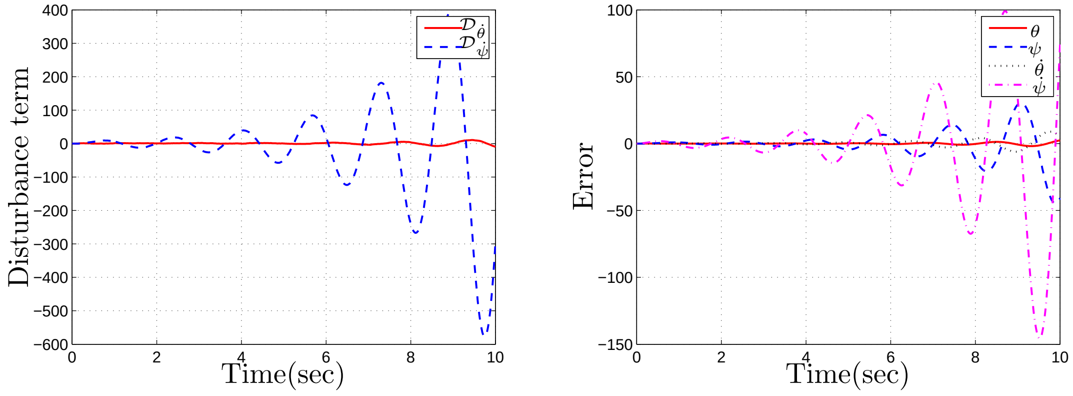


(b)

Figure 5. Actual States and Input: $Q = 10^3 \times I_{2\times2}$

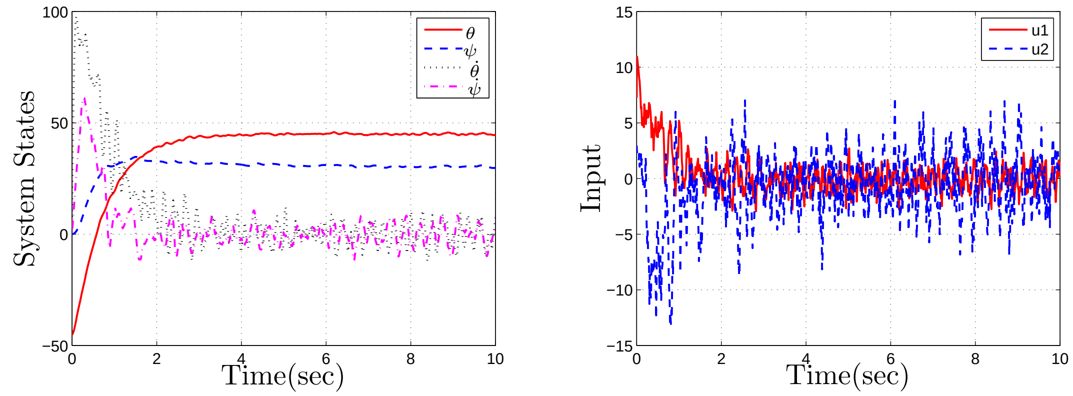
disturbance term and the error between the desired states and the true states corresponding to the first simulation. Note that the first simulation results given in figures 5 and 6 are unstable due to the low value of Q selected.

A second simulation is conducted using $Q = 10^5 \times I_{2\times2}$. The system response obtained for the second simulation is given in figure 7. Figure 8 shows the estimated disturbance term and state error obtained for the second simulation. Note that the estimated system rates, estimated disturbance term and the control input are highly noisy because of the large Q selected.



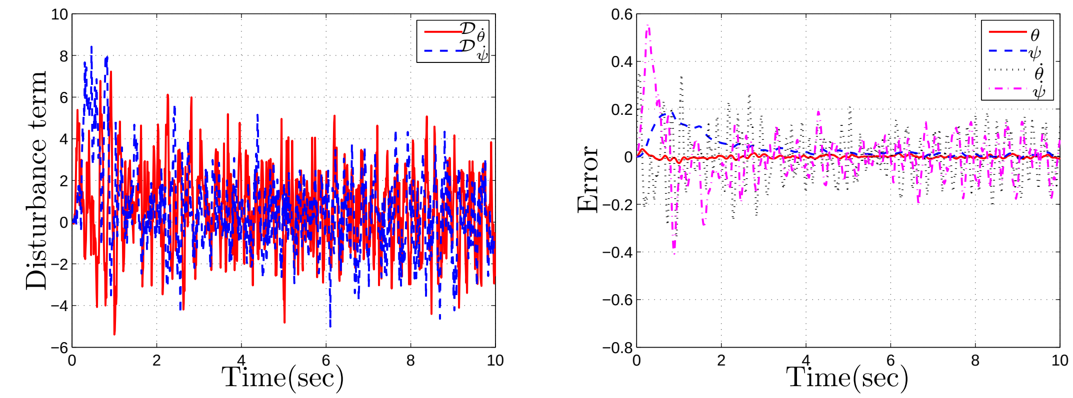
(a) (b)

Figure 6. Disturbance Term and State Error: $Q = 10^3 \times I_{2\times 2}$



(a) (b)

Figure 7. Actual States and Input: $Q = 10^5 \times I_{2\times 2}$



(a) (b)

Figure 8. Disturbance Term and State Error: $Q = 10^5 \times I_{2\times 2}$

The results shown here indicate that for a small value of process noise covariance, the controlled system is unstable. Though a large value of process noise covariance stabilizes the controlled system, it also results in highly noisy estimates. The direct dependency of the controlled system's stability on the process noise covariance is more evident in the simulation results given next.

Combining the plant dynamics in (59) and the estimator dynamics in (60), the closed-loop system dynamics can be written as

$$\begin{bmatrix} \dot{\mathbf{X}}_1(t) \\ \dot{\mathbf{X}}_2(t) \\ \dot{\mathbf{W}}(t) \\ \dot{\hat{\mathbf{Z}}}(t) \end{bmatrix} = \begin{bmatrix} 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times6} \\ 0_{2\times2} & A_4 & I_{2\times2} & BS \\ 0_{2\times2} & A_{\mathbf{w}_2} & A_{\mathbf{w}_3} & 0_{2\times6} \\ K(t) & 0_{2\times2} & 0_{2\times2} & (F_m + D_m S - K(t)H) \end{bmatrix} \begin{bmatrix} \mathbf{X}_1(t) \\ \mathbf{X}_2(t) \\ \mathbf{W}(t) \\ \hat{\mathbf{Z}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{2\times1} \\ BK_m\mathbf{x}_d \\ \boldsymbol{\mathcal{V}}(t) \\ D_m K_m \mathbf{x}_d + K(t)\mathbf{V}(t) \end{bmatrix} \tag{63}$$

Since we are considering a time-invariant system here, the Kalman gain $K(t)$ converges to its

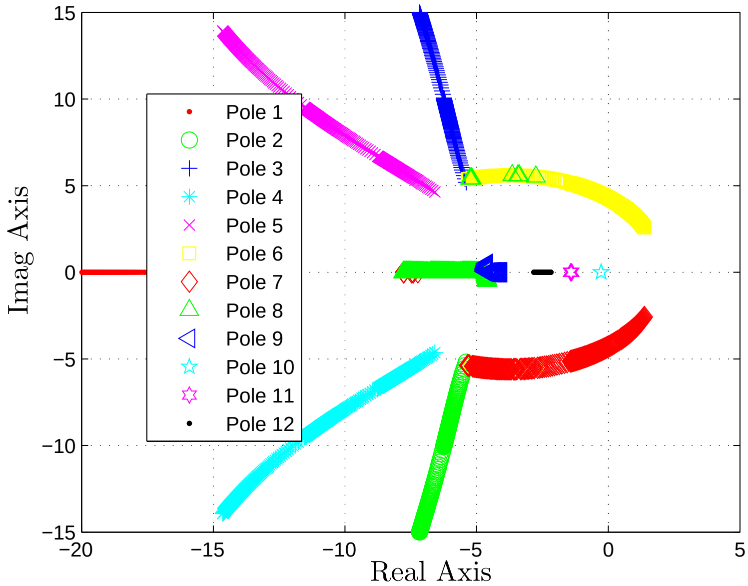


Figure 9. System Closed-loop Poles for Q Varying from 1×10^2 to 1×10^5

steady-state value fairly quickly. Using the steady-state Kalman gain, the stability of the closed-loop system can be easily be verified. Figure 9 shows the closed-loop poles of the system for different values of Q ranging from $1 \times 10^2 \times I_{2\times2}$ to $1 \times 10^5 \times I_{2\times2}$. Figure 9 indicates that the controlled system is unstable for the initial small values of Q and the closed-loop poles migrate into the stable region as Q increases.

6.2 *Adaptive DAC Results*

Results obtained by implementing the proposed adaptive disturbance accommodating scheme is presented in this subsection. Based on the assumed system parameters and controller design matrices given in Tables 2 and 3, the assumed state matrix, $\mathcal{A}_m$, the assumed input matrix, $\mathcal{B}_m$, and the DAC matrix, S, can be calculated as

$$\mathcal{A}_m = \begin{bmatrix} 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 1.00 \\ 0 & 0 & -9.28 & 0 \\ 0 & 0 & 0 & -3.50 \end{bmatrix}, \quad \mathcal{B}_m = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2.37 & 0.08 \\ 0.24 & 0.79 \end{bmatrix}, \quad \text{and}$$

$$S = \begin{bmatrix} -7.02 & -0.82 & -1.67 & -0.33 & -0.42 & 0.04 \\ 0.82 & -7.02 & 0.08 & -2.45 & 0.13 & -1.28 \end{bmatrix}$$

As shown in section 6, matrix $\mathrm{A}_{\mathcal{D}_m}$ is given as

$$\mathrm{A}_{\mathcal{D}_m} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

Now the matrix $[F_m + D_m S]$ can be calculated as

$$[F_m + D_m S] = \begin{bmatrix} 0 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.00 & 0 & 0 \\ -16.56 & -2.50 & -13.22 & -0.98 & 0 & 0 \\ -1.04 & -5.756 & -0.34 & -5.51 & -0 & 0 \\ 0 & 0 & 0 & 0 & -1.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.00 \end{bmatrix}$$

Let $\mathcal{M} = 10^{-1} \times I_6$, now the positive definite symmetric matrix $\mathcal{X}$ that satisfies the following matrix inequality

$$\mathcal{X}\,[F_m + D_m S] + [F_m + D_m S]^T\,\mathcal{X} + \mathcal{X}\mathcal{X} + \mathcal{M} \le 0$$

can be calculated as

$$\mathcal{X} = \begin{bmatrix} 0.1117 & 0.0005 & 0.0035 & -0.0011 & -0.0000 & -0.0000 \\ 0.0005 & 0.1165 & -0.0008 & 0.0102 & -0.0000 & -0.0000 \\ 0.0035 & -0.0008 & 0.0041 & -0.0005 & -0.0000 & -0.0000 \\ -0.0011 & 0.0102 & -0.0005 & 0.0110 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0513 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0167 \end{bmatrix}$$

Since the number of inputs and the number of outputs are the same here, the matrix Π is selected as the identity matrix $I_{2\times 2}$. For the implementation of the adaptive law, the following parameters are selected:

$$A_Q = -0.25 \times I_{2\times 2} \quad \text{and} \quad \gamma = \Big(\parallel K(t) \parallel + \mathrm{Tr}\left\{P(t)H^T R^{-1} H P(t) - 2F_m P(t)\right\} \mathrm{Tr}\left\{R\right\}^{-1} + 10^3 \Big)$$

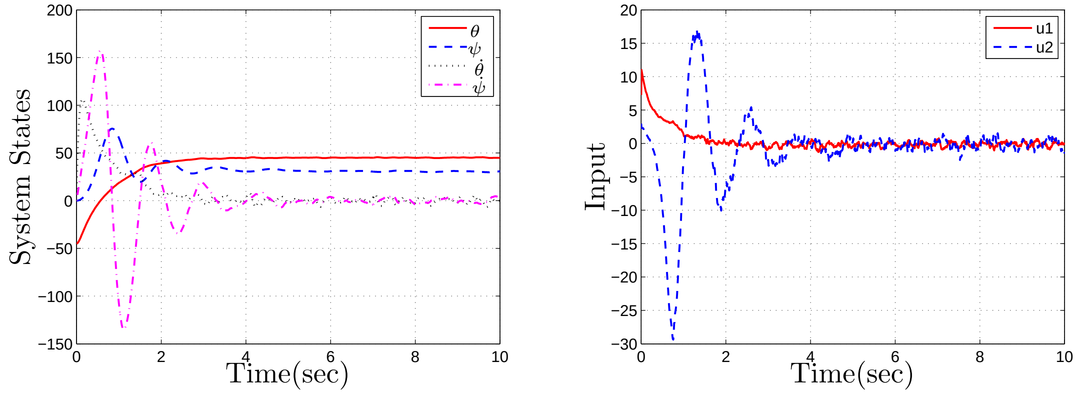
Three different simulation scenarios are considered here.

6.2.1 *Simulation I*

For the first simulation the initial process noise covariance is selected to be $Q(t_0) = 10^{-5} \times I_{2\times 2}$. Figures 10(a) and 10(b) show the system response and the disturbance accommodating control input obtained for the first simulation. Figures 11(a) and 11(b) contain the estimated disturbance term and the error between the desired states and the true states corresponding to the first simulation. Note that the first simulation results given in Figs. 10 and 11 indicate that the adaptive scheme is able to stabilize and recover the desired performance despite the initial unstable process noise covariance selected. The time varying process noise covariance obtained for the first simulation is given in figure 12.

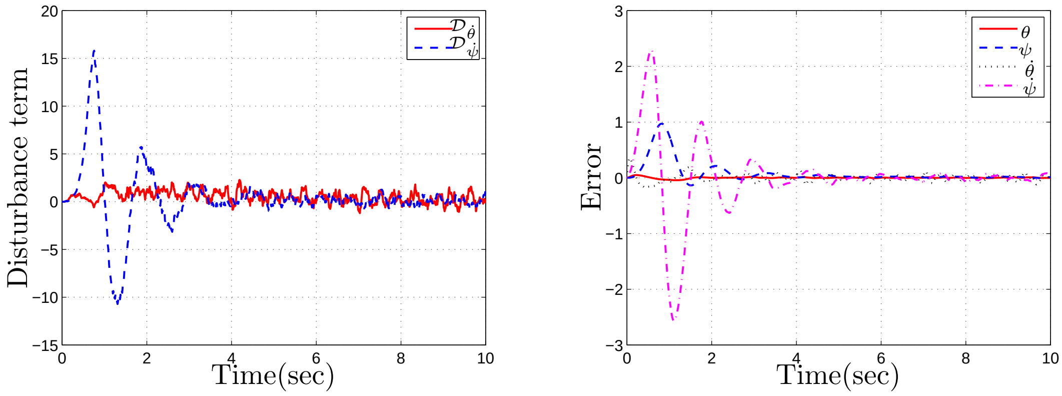
6.2.2 *Simulation II*

For the second simulation the initial process noise covariance is selected to be $Q(t_0) = I_{2\times 2}$. Figures 13(a) and 13(b) show the system response and the disturbance accommodating control input obtained for the second simulation. Figures 14(a) and 14(b) contain the estimated disturbance term and the error between the desired states and the true states corresponding to the second simulation. The simulation results given in Figs. 13 and 14 indicate that the adaptive



(a) (b)

Figure 10. Actual States and Input: $Q(t_0) = 10^{-5} \times I_{2\times2}$



(a) (b)

Figure 11. Disturbance Term and State Error: $Q(t_0) = 10^{-5} \times I_{2\times2}$

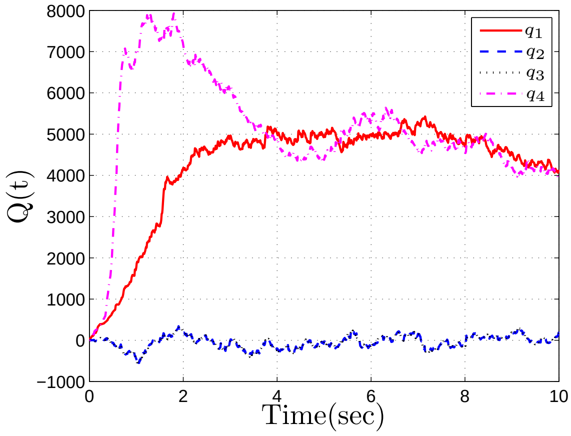
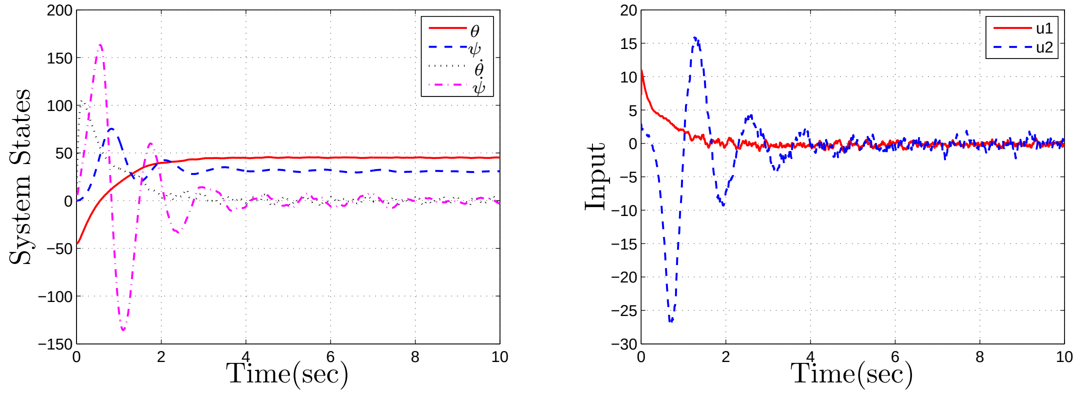


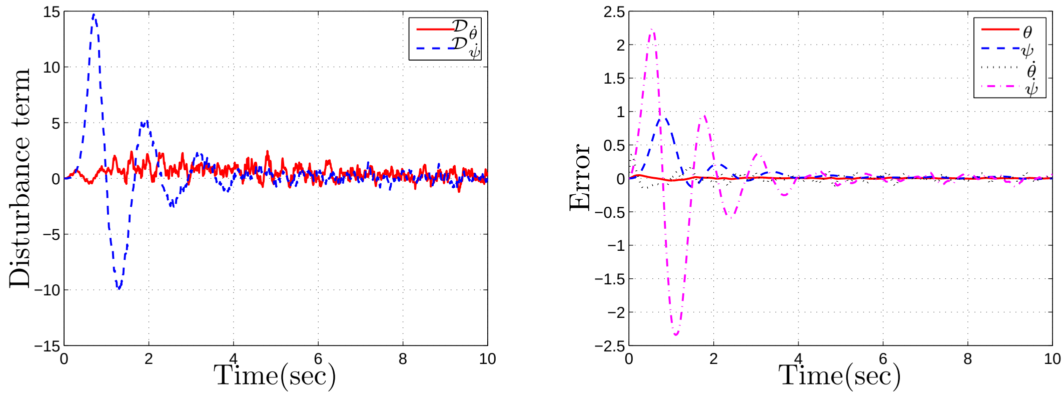
Figure 12. Adaptive Process Noise Covariance: $Q(t_0) = 10^{-5} \times I_{2\times2}$

scheme is able to stabilize and recover the desired performance despite the initial unstable process noise covariance selected. The time varying process noise covariance obtained for the second simulation is given in figure 15.



(a) (b)

Figure 13. Actual States and Input: $Q(t_0) = I_{2\times 2}$



(a) (b)

Figure 14. Disturbance Term and State Error: $Q(t_0) = I_{2\times 2}$

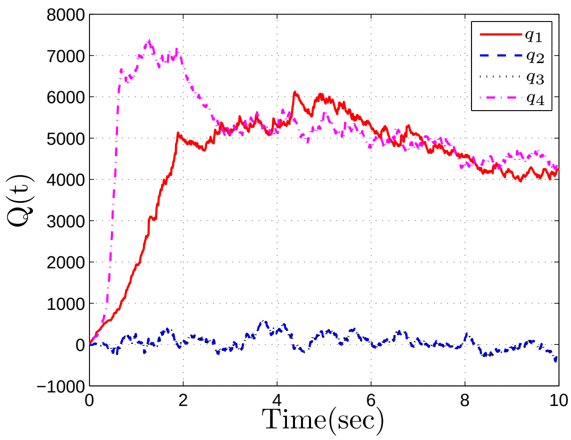
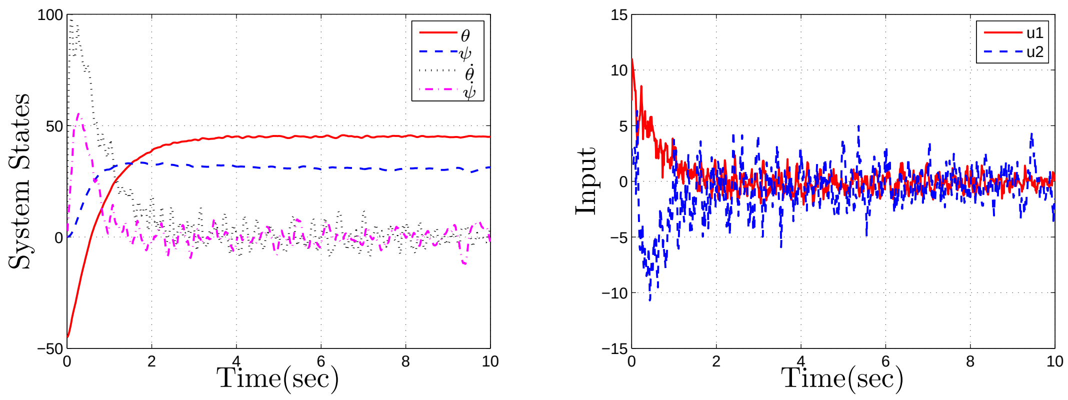


Figure 15. Adaptive Process Noise Covariance: $Q(t_0) = I_{2\times 2}$

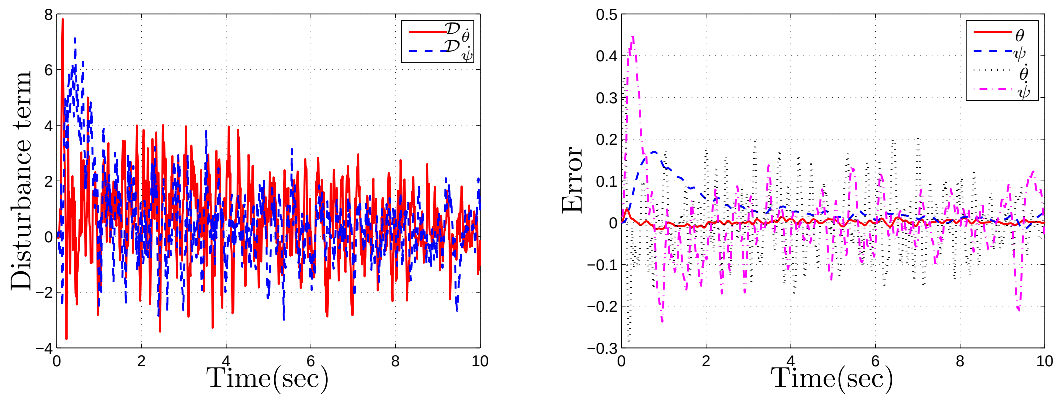
6.2.3 Simulation III

For the third simulation the initial process noise covariance is selected to be $Q(t_0) = 10^5 \times I_{2\times 2}$. Figures 16(a) and 16(b) show the system response and the disturbance accommodating control input obtained for the third simulation. Figures 17(a) and 17(b) contain the estimated disturbance term and the error between the desired states and the true states corresponding to



(a) (b)

Figure 16. Actual States and Input: $Q(t_0) = 10^5 \times I_{2\times2}$



(a) (b)

Figure 17. Disturbance Term and State Error: $Q(t_0) = 10^5 \times I_{2\times2}$

the third simulation. Figure 18 shows the time varying process noise covariance obtained for the third simulation.

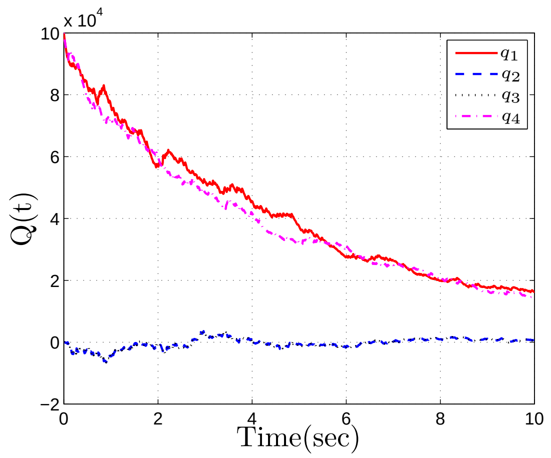
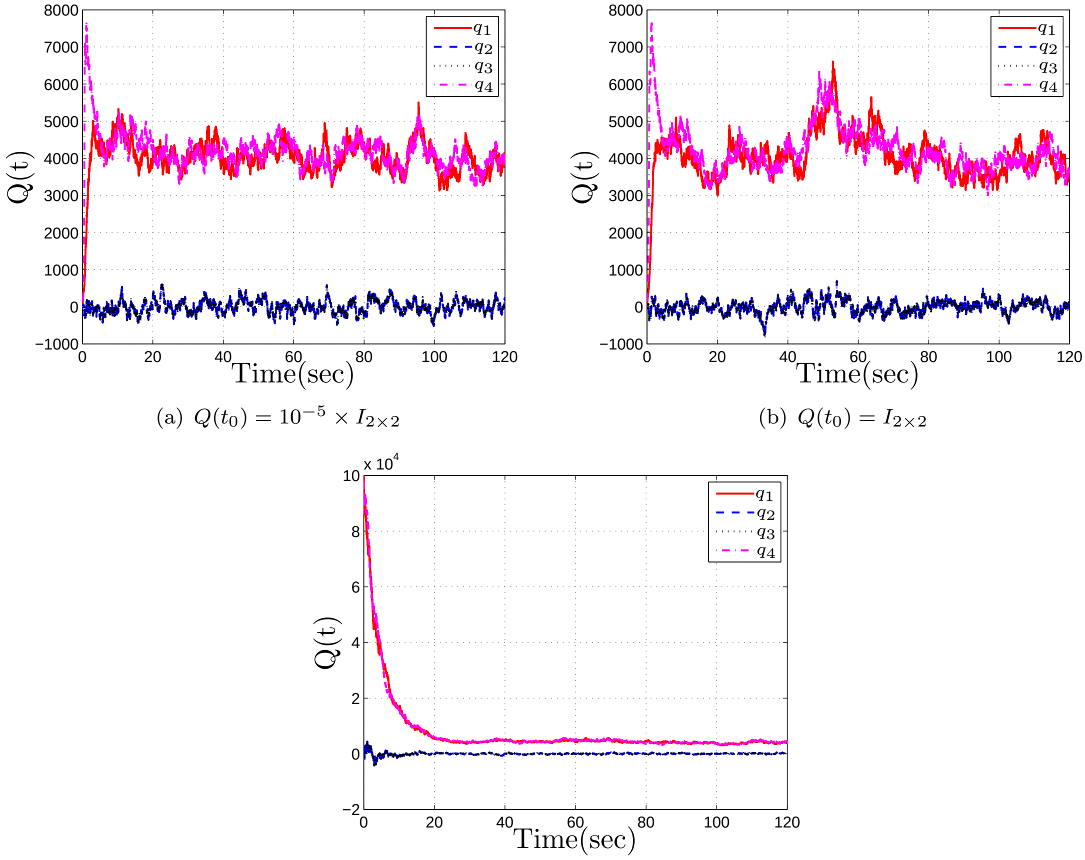


Figure 18. Adaptive Process Noise Covariance: $Q(t_0) = 10^5 \times I_{2\times2}$

Figure 19 shows the time varying process noise covariance obtained for the three simulations. Figure 19 indicates that, regardless of the initial matrix selected, the process noise covariance settles down at its steady-state value, which is around $4 \sim 5 \times 10^3$ for the present scenario.



(a) $Q(t_0) = 10^{-5} \times I_{2\times 2}$

(b) $Q(t_0) = I_{2\times 2}$

(c) $Q(t_0) = 10^{5} \times I_{2\times 2}$

Figure 19. Adaptive Process Noise Covariance Matrices

7 Conclusion

This paper presents the formulation of an observer-based stochastic disturbance accommodating control approach for linear time-invariant multi-input multi-output systems which automatically detects and minimizes the adverse effects of both model uncertainties and external disturbances. Assuming all system uncertainties and external disturbance can be lumped in a disturbance term, this control approach utilizes a Kalman estimator in the feedback loop for simultaneously estimating the system states and the disturbance term from measurements. The estimated states are then used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input to minimize the effect of system uncertainties and the external disturbances. The stochastic stability analysis conducted on the controlled system reveals a lower bound requirement on the estimator design parameters, such as the process noise covariance matrix and the measurement noise covariance matrix, in order to ensure the controlled system stability. Since the measurement noise covariance can be obtained from sensor calibration, the process noise matrix is treated as a tuning parameter. Based on the stochastic Lyapunov analysis, an adaptive law is developed for updating the selected process noise covariance online so that the controlled system is stable. The adaptive scheme introduced here guarantees asymptotic stability in the mean and the mean square stability of the controlled system. The simulation results given here explicitly reveal the direct dependency of the proposed control scheme on the process noise covariance matrix. Since the nominal control action on the true plant is unstable, selecting a very low process noise covariance resulted in an unstable system. On the other hand, selecting a large value stabilized the system but resulted in a highly noisy control input. The numerical simulations indicate that the adaptive scheme is able to stabilize and recover the desired performance despite selecting an initial unstable process noise covariance. The results also indicate that regardless of the initial matrix selected, the process noise covariance settles down to its steady-state value.

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