Modified Rodrigues Parameter Averaging in the Presence of Large Orientation Ambiguity

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This paper presents a closed-form method to average Modified Rodrigues Parameters. This method is compared to quaternion averaging through Monte Carlo simulations. Although both approaches exhibit good performance for small attitude errors the Modified Rodrigues Parameter averaging method shows considerable better performance characteristics for larger angle errors.

Key Words: Modified Rodrigues Parameter, Attitude Estimation

1. Introduction

Computing the weighted mean of an attitude parameterization has many applications, including determining the attitude estimate in a particle filters[1] or in multiple-model adaptive estimatiors,[2] and decentralized filtering approaches.[3] Many parameterizations for the attitude exist, such as the attitude matrix, Euler angles, quaternions, Gibbs vector, and others.[4] Here Modified Rodrigues Parameters (MRPs) are used for an implementation that avoids making a small angle approximation for the attitude ambiguity. The direct averaging of MRPs is inaccurate because the distance metric between two MRPs is nonlinear to second order. However, the distance metric between two quaternions can be shown to be linear with the addition of a unit norm constraint on the quaternion's magnitude. The unit norm constraint present in quaternions forces quaternion parameter dependence, thereby violating fundamental assumptions used to represent associated ambiguity (e.g. covariance).

The MRP representation avoids the use of a unit norm constraint. As a consequence, MRPs exhibit nonlinearities in their distance metric. Typically it has been assumed that these nonlinearities are minimal for small attitude errors, making estimation strategies that linearize the error about a reference orientation well suited.[5] However, in some attitude estimation scenarios this small angle assumption may be invalid. When errors are large, MRPs do not allow for direct averaging because of the nonlinearities attributed to their magnitude. This motivates an approach based on dividing the computation of the weighted mean MRP into two parts; one in the rotational angle space of the norm of the MRP and the second in the eigen-axis space of the direction of the MRP. In doing so, the challenges of having to minimize the nonlinear distance metric are avoided, while still exploiting the unconstrained nature of MRPs.

Simulation results are provided for the MRP weighted mean computation. These results are compared and contrasted against weighted mean computations for quaternions with that obtained from the proposed approach. The weighted mean MRP shows comparable results in accuracy to that from weighted quaternion while requiring less computation.

2. Problem Statement

The goal is to obtain an average attitude from a sample set of attitudes rather than an average attitude parameter vector. Following this observation, the average attitude should minimize a weighted sum of the squared Frobenius norms of attitude matrix differences:

$$\mathbf{\Theta} \triangleq \text{argmax}_{\mathbf{\Theta}\in\Omega} \sum_{i=1}^{n} w_i \|A(\mathbf{\Theta}) - A_i(\mathbf{\Theta})\|_{\text{F}}^2 \quad (1)$$

where Ω denotes the parameter space for $\mathbf{\Theta}$. For the quaternion representation the space is $\mathbb{S}^3$ the unit 3-sphere, whereas for the MRP representation this is the unconstrained $\mathcal{R}^3$.[1]

3. Quaternion

The quaternion is defined as $\boldsymbol{q} \equiv [\boldsymbol{\varrho}^T \ q_4]^T$ with $=$ $[q_1\ q_2\ q_3]^{\text{T}} = \hat{\mathbf{e}}\sin(\nu/2)$ and $q_4 = \cos(\nu/2)$, where $\hat{\mathbf{e}}$ and ν are the Euler axis of rotation and rotation angle, respectively. This vector must satisfy the constraint $\boldsymbol{q}^T\boldsymbol{q} = 1$. The attitude matrix can be written as a function of the quaternion:

$$A = \Xi(\boldsymbol{q})^{\text{T}}\Psi(\boldsymbol{q}) \quad (2)$$

where

$$\Xi(\boldsymbol{q}) = \begin{bmatrix} q_4 I_{3\times3} + [\boldsymbol{\varrho}\,\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \quad (3)$$

$$\Psi(\boldsymbol{q}) = \begin{bmatrix} q_4 I_{3\times3} - [\boldsymbol{\varrho}\,\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \quad (4)$$

and

$$[\boldsymbol{a}\times] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix} \quad \textbf{(5)}$$

for any general 3×1 vector $\boldsymbol{a}$.

4. Modified Rodrigues Parameters

The modified Rodrigues parameters (MRPs) have been used for spacecraft estimation and control because the unique properties. Schaub and Junkins[6] have shown that MRPs, along with the classic Rodrigues parameters, belong to a more general group called stereographic parameters. These parameters are the result of a stereographic projection of the quaternion onto a three-dimensional hyperplane. In particular, the Rodrigues and MRPs are part of a subset of stereographic parameters which are referred to as symmetric. The MRP representations have the benefit of providing a minimal attitude representation by using 3 parameters. The general form of the symmetric stereographic parameters are defined in terms of the quaternion components as

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\varrho}}{q_4 - a} \quad \textbf{(6)}$$

where $a = \cos(\phi/2)$ is the projection point which is determined by the singular rotation, ϕ_s. Choosing the singularity to lay at $\pm\pi$, we have that $a = 0$ and we recover the classic Rodrigues parameters. The MRPs, $\boldsymbol{p}$, are singular at $\pm 2\pi$ which corresponds to $a = -1$. In particular, MRPs are given by

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \frac{\boldsymbol{\varrho}}{1 + q_4} \quad \textbf{(7)}$$

The inverse transformation from MRPs to quaternion components are given by

$$\boldsymbol{\varrho} = \frac{2\boldsymbol{p}}{1 + \boldsymbol{p}^T\boldsymbol{p}} \quad \textbf{(8)}$$

$$q_4 = \frac{1 - \boldsymbol{p}^T\boldsymbol{p}}{1 + \boldsymbol{p}^T\boldsymbol{p}} \quad \textbf{(9)}$$

The MRPs can also be written in terms of the Euler axis and angle:

$$\boldsymbol{p} = \hat{\mathbf{e}}\tan(\nu/4) \quad \textbf{(10)}$$

While MRPs are an unconstrained parameter set, they do not obey a strictly additive error representation. An error representation for MRPs is derived by considering a rotation of $\boldsymbol{\delta p}$ about $\boldsymbol{p}_a$

$$\boldsymbol{p}_b = \boldsymbol{\delta p} \odot \boldsymbol{p}_a \quad \textbf{(11)}$$

where the MRP composition operator is given by

$$\boldsymbol{p}_b = \frac{(1 - |\boldsymbol{p}_a|^2)\boldsymbol{\delta p} + (1 - |\boldsymbol{\delta p}|^2)\boldsymbol{p}_a - \mathbf{2}[\boldsymbol{\delta p}\times]\boldsymbol{p}_a}{1 + |\boldsymbol{p}_a|^2|\boldsymbol{\delta p}|^2 - 2\boldsymbol{\delta p}^T\boldsymbol{p}_a} \quad (12)$$

where $|\cdot|$ expresses the L_2 norm of the argument, i.e. $|\boldsymbol{u}|^2 = \boldsymbol{u}^T\boldsymbol{u}$. Assuming that $\boldsymbol{\delta p}$ is small allows for the following approximation

$$\begin{aligned} \boldsymbol{p}_b \approx (1 + 2\boldsymbol{\delta p}^T\boldsymbol{p}_a)[(1 - \boldsymbol{p}_a^T\boldsymbol{p}_a)\boldsymbol{\delta p} + \ & \boldsymbol{p}_a \\ & - \mathbf{2}[\boldsymbol{\delta p}\times]\boldsymbol{p}_a] \\ & = \boldsymbol{p}_a \\ & + [(1 - \boldsymbol{p}_a^T\boldsymbol{p}_a)\boldsymbol{I}_{3\times 3} + 2[\boldsymbol{p}_a\times] \\ & - \mathbf{2}\boldsymbol{p}_a\boldsymbol{p}_a^T]\boldsymbol{\delta p} \end{aligned} \quad \textbf{(13)}$$

The error MRP can then be expressed as

$$\boldsymbol{\delta p} = \boldsymbol{B}^{-1}(\boldsymbol{p}_a)(\boldsymbol{p}_b - \boldsymbol{p}_a) \quad \textbf{(14)}$$

where

$$\boldsymbol{B}(\boldsymbol{p}_a) = [(1 - \boldsymbol{p}_a^T\boldsymbol{p}_a)\boldsymbol{I}_{3\times 3} + 2[\boldsymbol{p}_a\times] - \mathbf{2}\boldsymbol{p}_a\boldsymbol{p}_a^T] \quad \textbf{(15)}$$

From Eq. (12) it can be seen that using the MRP composition will lead to a nonlinear expression for the terms in Eq. (1) and therefore requires a nonlinear optimization approach to solve for the optimal averaged MRP vector. This motivates the development of a simplified approximation to obtain the average MRP vector.

5. The Average Quaternion

The goal is to determine an average quaternion given a set of n quaternions $\boldsymbol{q}_i$ with associated scalar weights w_i. When considering the average quaternion, the simple normalized weighted sum does not provide an average that retains the properties of a quaternion and therefore introduces undesirable error. As Ref. 7 notes, the simple procedure of determining the average quaternion via

$$\boldsymbol{q}_{\mathrm{bad}} = \frac{1}{w_{tot}}\sum_{i=1}^{n}\boldsymbol{q}_i \quad \textbf{(16)}$$

where $w_{tot} = \sum_{i=1}^{n} w_i$ presents two notable issues. First, the average quaternion is not necessarily unit norm. Second, since $\boldsymbol{q}_i$ and $-\boldsymbol{q}_i$ represent the same attitude, an averaging algorithm should not be susceptible to sign changes in $\boldsymbol{q}_i$. Therefore, an averaging algorithm for quaternions that preserves the unit norm property of the quaternion is required. Reference 7 solves this problem for the quaternion representation and determines a proper average quaternion. To determine the average quaternion, first compute a matrix $\boldsymbol{M} \in \mathcal{R}^{4\times 4}$ as

$$\boldsymbol{M} = \sum_{i=1}^{n} w_i \boldsymbol{q}_i \boldsymbol{q}_i^T \tag{17}$$

Then, it can be shown that the average quaternion is given by the maximization problem

$$\boldsymbol{q}_{\text{avg}} = \text{argmax}_{\boldsymbol{q}_i \in \mathbb{S}^3}\, \boldsymbol{q}_i \boldsymbol{M} \boldsymbol{q}_i^T \tag{18}$$

The maximization problem of Eq. (19) can then be cast in terms of

$$J = \boldsymbol{q}_i \boldsymbol{M} \boldsymbol{q}_i^T \tag{19}$$

However, the constraint on the norm of the quaternion must be accounted for, which can be accomplished by use of a Lagrange multiplier, λ. The augmented performance index is given by

$$J = \boldsymbol{q}_i \boldsymbol{M} \boldsymbol{q}_i^T + \lambda(\boldsymbol{q}_i \boldsymbol{q}_i^T - 1) \tag{20}$$

It is then straightforward to show that the optimality conditions lead to an eigenvalue problem of the form

$$\boldsymbol{M}\boldsymbol{q}_i = \lambda \boldsymbol{q}_i \tag{21}$$

and that the performance index of Eq. (19) is given by

$$J = \lambda \tag{22}$$

Therefore to maximize the performance index, the average quaternion is select to be the eigenvector of $\boldsymbol{M}$ corresponding to the largest eigenvalue of $\boldsymbol{M}$. It is noted then that this procedure not only leads to an average quaternion which is unit norm, but also leads to a process that is not affected by a sign change in any of the $\boldsymbol{q}_i$ terms since the performance index is in quadratic form.

6. Proposed MRP Approach

One cannot simply add the three elements of the MRP vector components together and take the average because it will destroy the physical representation what the actual averaged attitude is. This is due to the fact that MRPs are scaled by a nonlinear function of the rotation angle times the eigenvector. An example of how this will fall is in the averaging of two parameters, given by

$$\sigma_1 = \tan\left(\frac{\pi}{4}\right), \sigma_2 = \tan\left(\frac{-\pi}{4}\right) \text{ and } \frac{\sigma_2 + \sigma_1}{2} = 0 \tag{23}$$

Both are the same angle ($\pi = -\pi$) in terms of what attitude they represent, but when averaging the zero MRP is obtained which is obviously incorrect. This way of averaging MRPs will fail even when the angles are not positive and negative but also when they are large and small. In simulation section this shows an enormous error using large angles and uncertainties with none of the angles negative.

Now if all negative rotation angles are forced to be positive by adding 2π to them when they are negative, the angles and the eigenvectors can be averaged separately to obtain the averaged MRP as

$$\bar{\sigma} = \tan\left(\frac{\sum_{i=1}^{k} \frac{\nu_i}{k}}{4}\right)\left(\sum_{i=1}^{k} \frac{n_i}{k}\right) \frac{1}{\left\|\left(\sum_{i=1}^{k} \frac{n_i}{k}\right)\right\|} \tag{24}$$

The method in Eq. (25) requires renormalization of the eigenvector. This is not an optimal approach. However, because of the highly nonlinear form of the error quaternion in Eq. (12), formulating this problem minimizing a cost based off the error quaternion will lead to a highly complex expression for the optimal solution. In addition, constraints would needed to be set on the magnitude of the MRP to avoid asingularity further complicating this solution.

6. Numerical Results

Monte Carlo simulations are run for both cases of quaternion averaging and MRP averaging to evaluate and compare both approaches. For each Monte Carlo simulation the true attitude is varied by varying the Euler rotation angle about a fix Euler rotation axis. The true attitude can be represented by the three angles using to form the true quaternion, denoted $\boldsymbol{\theta}_{\text{true}} = [\Phi \quad \beta \quad \phi]^T$. The true attitude quaternion can be written as

$$\boldsymbol{q}_{\text{true}} = \begin{bmatrix} \sin\left(\frac{\Phi}{2}\right)\hat{\mathbf{e}} \\ \cos\left(\frac{\Phi}{2}\right) \end{bmatrix} \tag{25}$$

where

$$\hat{\mathbf{e}} = \begin{bmatrix} \cos(\phi)\cos(\beta) \\ \cos(\phi)\sin(\beta) \\ \sin(\phi) \end{bmatrix} \tag{26}$$

The angle $\boldsymbol{\Phi}$ is varied from -180 deg to 180 deg, while both $\boldsymbol{\beta}$ and $\boldsymbol{\phi}$ are held constant at $\boldsymbol{\pi/18}$. Then 500 Monte Carlo samples are generated by sampling a vector $\boldsymbol{\upsilon} = [\boldsymbol{\upsilon_1} \quad \boldsymbol{\upsilon_2} \quad \boldsymbol{\upsilon_3}]^{\boldsymbol{T}}$ from a normal distribution given by $\boldsymbol{\mathcal{N}}(\boldsymbol{0}, \boldsymbol{P})$, zero mean with a covariance matrix given by $\boldsymbol{P} = \mathbf{diag}([\boldsymbol{\sigma}^2 \quad \boldsymbol{\sigma}^2 \quad \boldsymbol{\sigma}^2])$. The vector the $\boldsymbol{\upsilon}$ is then used to generate attitude samples distributed around the true attitude given by

$$\widetilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_{\text{true}} + \boldsymbol{\upsilon} \tag{27}$$

Then Eqs. (26) and (27) are used to generate sample quaternions from the $\widetilde{\boldsymbol{\theta}}$ samples. Similar sample MRPs are generated using Eq. (7) from the quaternion samples. Then for each value for Φ a mean is calculated from the 500

Monte Carlo samples using the quaternion averaging approach and the proposed MRP averaging approach. Then a direction cosine matrix (DCM) is determined for both the quaternion average and the MRP average. The error DCM is calculated for each using

$$\begin{aligned}\delta A_q &= A(\boldsymbol{q}_{\text{true}})A(\boldsymbol{q})^T \\ \delta A_{\text{MRP}} &= A(\boldsymbol{q}_{\text{true}})A(\boldsymbol{p})^T\end{aligned} \tag{28}$$

Then from each error DCM the rotation vector is determined, denoted by $\boldsymbol{\delta\alpha}$, and the magnitude is plotted for varying $\boldsymbol{\sigma}$ and Φ for quaternion averaging and MRP averaging in Figure 1 and Figure 2, respectively.

The results for MRP averaging in Figure 1 show comparable performance to those for quaternion averaging in Figure 2 and for small angles. The MRP averaging performs better for larger standard deviations in the distributions of the eigen-axes shown by the difference between the error of Figures 1 and 2 in Figure 3.

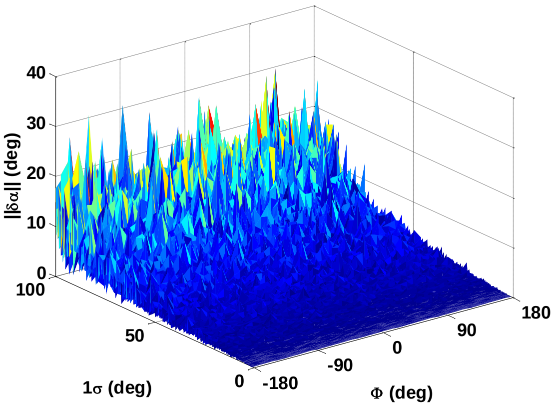

Fig. 1. Error using quaternion averaging (isometric view).

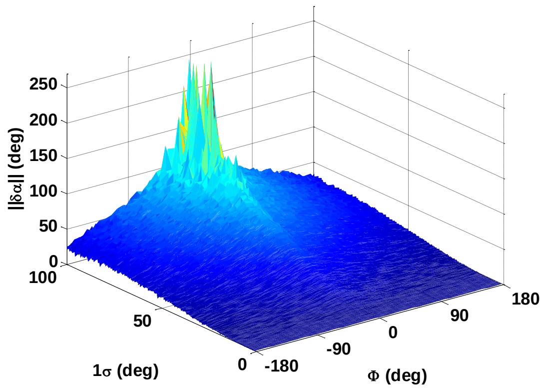

Fig. 2. Error from difference in quaternion averaging from direct MRP Averaging (isometric view).

There is lower error sensitivity in the eigen-angle as uncertainty is increased among all choices of rotation angle for the MRP averaging versus quaternion averaging, as shown in Figure 3 and Figure 4. This may be due to the fact that the optimization solved for the quaternion average is over the cost functions of L2 norm distance from DCMs in Eq. (1). The error between DCMs is not additive and thus with larger standard deviations, the cost function in Eq. (1) violates the error between attitudes. In addition, the increase in the error in quaternion averaging in Figure 2 around the zero rotation angle is due to the ambiguity of the quaternion at 0 and 2π.

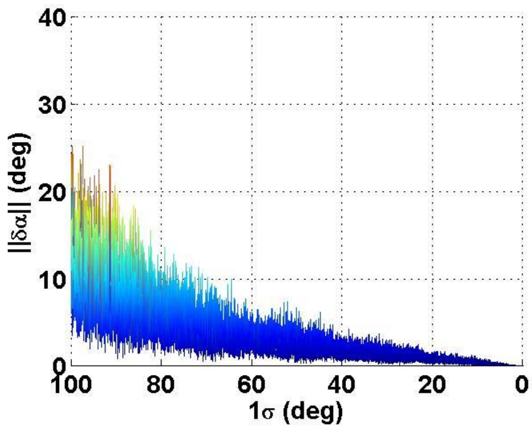

Fig. 3. Error using quaternion averaging (side view).

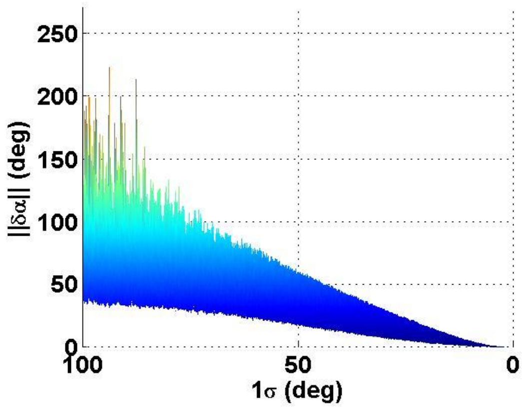

Fig. 4. Error from difference in quaternion averaging from direct MRP Averaging (side view).

6. Conclusion and Future Work

A real-time closed-form method to average MRPs was developed and compared to an optimal method of averaging quaternions. In comparison both of these methods had good results for small angles, but as the uncertainty was increased in the rotation angle and the direction of the eigen-vectors, the MRP averaging scheme showed considerable better results.

Future work will consist of applying this averaging scheme to the first and second moments of an Unscented Kalman Filter and particle filtering for representing large attitude ambiguities. With a proper way to average MRPs and lack of the unit-norm constraint, this method could provide great benefits to estimation methods with a large initial uncertainty.

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