RELATIVE ATTITUDE DETERMINATION USING MULTIPLE CONSTRAINTS

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In this paper a relative attitude determination solution of a formation of two vehicles with multiple constraints is shown. The solution for the relative attitude between the two vehicles is obtained only using line-of-sight measurements between them and common (unknown) objects observed by both vehicles. The constraints used in the solution are formed from triangles on the vector observations. Multiple constraints are used for each object and the solution is cast into a Wahba problem formulation. Simulation runs are shown that study the performance of the new approach.

INTRODUCTION

Using a set of line-of-sight (LOS) observations between vehicles in a three-vehicle formation has been shown to offer a deterministic relative attitude solution,[1] which is not possible if each vehicle is considered separately. The observability of this relative attitude solution depends on both vehicle geometry and sensor location. It is well known that the rotation around a unit vector is unobservable when that unit vector is the only observation used for attitude determination. Reference 1 shows that having only one LOS set between each of the individual vehicles provides sufficient information to determine all relative attitudes in a three-vehicle system. An unobservable case arises when all vectors are in the same plane, e.g. they form a triangle. Reference 2 extends the previous result to a two-vehicle formation with a common observed object, which can be another vehicle or a landmark, by applying a parametric constraint to the attitude solution. This constraint is based on assuming that a triangle set of observations is given. In the work of Reference 1 this issue causes problems in the solution, while in Reference 2 this constraint is forced to be true and hence relieves the arising difficulties. This results in a deterministic solution for the relative attitude with no ambiguity and no observability issues.

The triangle scenario does reflect a realistic physical situation. For example, this occurs naturally when two Unmanned Aerial Vehicles (UAVs) have a common LOS between them and measure some common object other than each other, which forms a triangle of LOS observations. It is important to note that no information on the location of the object is required in the solution, only the fact that both vehicles *observe* the common object. This constitutes a significant departure from standard navigation or attitude approaches that use *known* objects or landmarks. The triangle constraint is

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used to determine a solution, however, only one common object is considered and in many cases there may be more available. For example in a formation flying scenario there may be more than three spacecraft in the formation. In this paper the solution presented in Reference 2 is extended to multiple common observer objects.

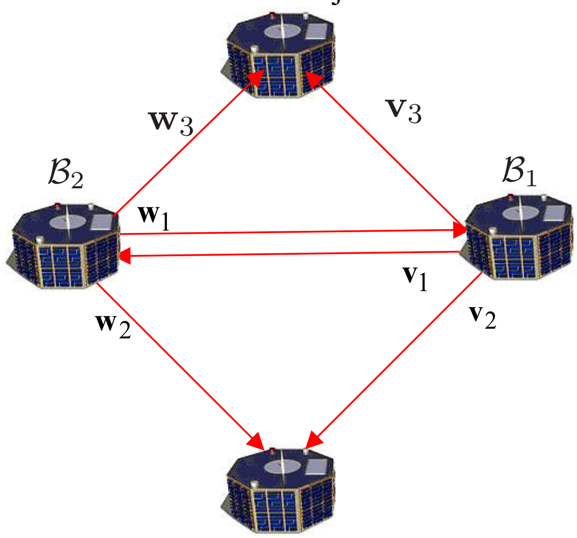


Figure 1. Vehicle Formation

PROBLEM STATEMENT

Noting Figure 1, the case of multiple vehicles in relative formation flight is considered. The relative attitude from the $\mathcal{B}_1$ frame to the $\mathcal{B}_2$ frame is sought. Both vehicles measure a LOS from itself to the other vehicles in the formation as well as a LOS to two common objects, which are shown as two spacecraft in Figure 1, labeled as common object 1 and common object 2. In the case of a formation of spacecraft the distance between the reference object and the vehicle may be comparable to the distance between the two frames. Therefore the parallax issue needs to be resolved by an origin transformation. This typically requires associated range information which introduces more error into the algorithm.[1] Here it is assumed that parallel beams between the vehicles exist, so that range information is not required. For example, for a laser communication system a feedback device can be employed to ensure parallel beams are given in realtime. The LOS vectors between vehicles are denoted as $\mathbf{v}_1$ and $\mathbf{w}_1$, as shown in Figure 1. The LOS observation vectors used in this paper are denoted by the following, as shown by Figure 2:

- The vector $\mathbf{w}_1$ is the vector from the $\mathcal{B}_2$ vehicle frame to the $\mathcal{B}_1$ vehicle frame expressed in $\mathcal{B}_2$ coordinates.
- The vector $\mathbf{v}_1$ is the vector from the $\mathcal{B}_2$ vehicle frame to the $\mathcal{B}_1$ vehicle frame expressed in $\mathcal{B}_1$ coordinates. Note that in actual practice the negative of the vector is measured by the sensor, as shown by Figure 1.

- The vector $\mathbf{w}_2$ is the vector from the $\mathcal{B}_2$ vehicle frame to the common object 1 expressed in $\mathcal{B}_2$ coordinates.
- The vector $\mathbf{v}_2$ is the vector from the $\mathcal{B}_1$ vehicle frame to the common object 1 expressed in $\mathcal{B}_1$ coordinates.
- The vector $\mathbf{w}_3$ is the vector from the $\mathcal{B}_2$ vehicle frame to the common object 2 expressed in $\mathcal{B}_2$ coordinates.
- The vector $\mathbf{v}_3$ is the vector from the $\mathcal{B}_1$ vehicle frame to the common object 2 expressed in $\mathcal{B}_1$ coordinates.

The observations $\mathbf{w}_1$ and $\mathbf{v}_1$ can be related to each other through the attitude matrix mapping:

$$\mathbf{w}_1 = A_{\mathcal{B}_1}^{\mathcal{B}_2}\mathbf{v}_1 \tag{1}$$

It is well known that using a single pair of LOS vectors between the two vehicles does not provide enough information for a complete three-axis relative attitude solution. In particular to determine the full attitude the rotation angle about the LOS direction must be determined. Consider the following property of the attitude matrix: $(A_I^{\mathcal{B}_2\,T}\mathbf{w}_2)^T A_I^{\mathcal{B}_1\,T}\mathbf{v}_2 = \mathbf{w}_2^T A_I^{\mathcal{B}_2} A_I^{\mathcal{B}_1\,T}\mathbf{v}_2 = \mathbf{w}_2^T A_{\mathcal{B}_1}^{\mathcal{B}_2}\mathbf{v}_2$, which means that the attitude matrix preserves the angle between vectors. This allows us to write $d = \mathbf{w}_2^T A_{\mathcal{B}_1}^{\mathcal{B}_2}\mathbf{v}_2$, where d is the cosine of the angle between the two LOS vectors to the common object and we denote $A_{\mathcal{B}_1}^{\mathcal{B}_2}$ as just A. In Reference 1 this angle is determined from two LOS vectors measured on the third vehicle in the three-vehicle formation. This requires an extra LOS vector between the two vehicles and the third vehicle, which is not required in Reference 2. In Reference 2 a geometric constraint is used instead of requiring additional measurement to be made by a third vehicle.

Reference 2 uses the fact that in the measurement geometry considered all measurement vectors lie on a common plane, then a planar constraint is applied to solve for unknown rotation angle. First the LOS vector between the two frames can be aligned through an initial rotation, then a rotation angle about this direction can be found such that when this rotation is applied the angle between the measurements add up to π or the vectors lie on a common plane. The third reference object in the formation doesn't need to communicate its LOS observations to the other two vehicles for the solution of their relative attitude. Therefore a very powerful conclusion can be made from this observation: choice of the common third object in the formation is arbitrary and can be any common reference point, with unknown position, when the geometrical condition is applied. In the present work with is extended for multiple common objects.

The constraint that the observation vectors constitute the legs of a triangle, or they lie on a common plane, is first applied to common object 1. The planar constraint for common object 1 can be simply written as $0 = \mathbf{w}_2^T[\mathbf{w}_1\times]A\mathbf{v}_2$. This equation is basically a condition that all the observations are perpendicular to a vector which is perpendicular to any two observations. To determine the full attitude between the $\mathcal{B}_1$ and $\mathcal{B}_2$ frames using one common object, the attitude matrix must satisfy the following equations:

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{2a}$$

$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times]A\mathbf{v}_2 \tag{2b}$$

Since only common object 1 is considered in Eq. (2) only one constraint equation is written. In the case where both common objects are considered two constraint equations can be written for both common object 1 and common object 2.

An equivalent approach for expressing the constraint involves defining two vectors, $\mathbf{s}_2$ and $\mathbf{r}_2$, that are both perpendicular to the plane. These two vectors for object 1 can be written as

$$\mathbf{s}_2 = \frac{[\mathbf{w}_2\times]\mathbf{w}_1}{\|\mathbf{w}_2 \times \mathbf{w}_1\|} \tag{3a}$$

$$\mathbf{r}_2 = \frac{[\mathbf{v}_2\times]\mathbf{v}_1}{\|\mathbf{v}_2 \times \mathbf{v}_1\|} \tag{3b}$$

Then by aligning these vectors the attitude matrix can be solved. The following are now defined: $\mathbf{s}_1 = \mathbf{w}_1$ and $\mathbf{r}_1 = \mathbf{v}_1$. The equation relating $\mathbf{s}_i$ and $\mathbf{r}_i$ is given by

$$\mathbf{s}_1 = A\mathbf{r}_1 \tag{4a}$$

$$\mathbf{s}_2 = A\mathbf{r}_2 \tag{4b}$$

where $\mathbf{s}_1$ and $\mathbf{r}_1$ are equal to $\mathbf{w}_1$ and $\mathbf{v}_1$, respectively. To illustrate that solving for the attitude in Eq. (4) is equivalent to solving Eq. (2), consider the two equivalent forms of Eq. (2b):

$$0 = ([\mathbf{w}_1\times]\mathbf{w}_2)^T A\mathbf{v}_2 \tag{5a}$$

$$0 = ([\mathbf{v}_1\times]\mathbf{v}_2)^T A^T\mathbf{w}_2 \tag{5b}$$

Then noting that $\mathbf{s}_2 = [\mathbf{w}_1\times]\mathbf{w}_2$, $\mathbf{r}_2 = [\mathbf{v}_1\times]\mathbf{v}_2$, $\mathbf{r}_2 \perp \mathbf{v}_2$, and $\mathbf{s}_2 \perp \mathbf{w}_2$ the following relation is given:

$$(A\mathbf{r}_2)^T A\mathbf{v}_2 = \mathbf{r}_2^T A^T A\mathbf{v}_2 = \mathbf{r}_2^T\mathbf{v}_2 = 0 \tag{6a}$$

$$(A^T\mathbf{s}_2)^T A^T\mathbf{w}_2 = \mathbf{s}_2^T AA^T\mathbf{w}_2 = \mathbf{s}_2^T\mathbf{w}_2 = 0 \tag{6b}$$

Solving for the attitude in Eq. (4) is equivalent to solving Eq. (2), but by writing the constraint in the latter form, this solution can now be extended to multiple common objects. The following relation is given for common object 2:

$$\mathbf{s}_3 = \frac{[\mathbf{w}_3\times]\mathbf{w}_1}{\|\mathbf{w}_3 \times \mathbf{w}_1\|} \tag{7a}$$

$$\mathbf{r}_3 = \frac{[\mathbf{v}_3\times]\mathbf{v}_1}{\|\mathbf{v}_3 \times \mathbf{v}_1\|} \tag{7b}$$

The relative attitude determination problem from the formation given in Figure 1 is given by

$$\mathbf{s}_1 = A\mathbf{r}_1 \tag{8a}$$

$$\mathbf{s}_2 = A\mathbf{r}_2 \tag{8b}$$

$$\mathbf{s}_3 = A\mathbf{r}_3 \tag{8c}$$

The general problem when there exists more the two common objects is given by

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{9a}$$

$$\mathbf{w}_i = A\mathbf{v}_i \quad \text{for } i = 2, \ldots, n \tag{9b}$$

Here the $\mathbf{w}_i$ and $\mathbf{v}_i$ vectors are the n LOS observations made for the n common objects. For the case of Reference 2 a deterministic solution is possible since the number of unknowns exactly matches the number of pieces of information. However, for the case of multiple constraints this is not true and therefore no exact solutions exist. Hence, the optimal solution must be found by minimizing a cost function. In this case each equation can not be satisfied exactly and therefore error is allowed in each equation which relaxes the planar constraint but produces the least amount of attitude error.

SENSOR MODEL

Line-of-sight observations between multiple vehicles can be obtained using standard light-beam and focal-plane-detector (FPD) technology. One such system is the vision-based navigation (VISNAV) system,[3] which consists of a position sensing diode as the focal plane that captures incident light from a beacon omitted from a neighboring vehicle from which a LOS vector can be determined. The light source is such that the system can achieve selective vision. This sensor has the advantage of having a small size and a very wide field-of-view (FOV). Another system for obtaining LOS information between vehicles can be based on laser communication hardware.[4] The use of laser communication devices has increased in recent years and the accuracy of the LOS information obtained from these devices is comparable to the VISNAV system. LOS observations to the common object can be obtained through standard camera-based tracking technology. All of the aforementioned LOS observations can be modeled using the sensor model shown in this section.

The measurement can be expressed as coordinates in the focal plane, denoted by α and β. The focal plane coordinates can be written in a 2×1 vector $\mathbf{m} \equiv [\alpha \ \ \beta]^T$ and the measurement model follows

$$\tilde{\mathbf{m}} = \mathbf{m} + \mathbf{w}_m \tag{10}$$

A typical noise model used to describe the uncertainty, $\mathbf{w}_m$, in the focal-plane coordinate measurements is given as

$$\mathbf{w}_m \sim \mathcal{N}\left(\mathbf{0}, R^{\text{FOCAL}}\right) \tag{11a}$$

$$R^{\text{FOCAL}} = \frac{\sigma^2}{1 + d\left(\alpha^2 + \beta^2\right)} \begin{bmatrix} \left(1 + d\alpha^2\right)^2 & (d\alpha\beta)^2 \\ (d\alpha\beta)^2 & \left(1 + d\beta^2\right)^2 \end{bmatrix} \tag{11b}$$

where σ^2 is the variance of the measurement errors associated with α and β, and d is on the order of 1. The covariance for the focal plane measurements is a function of the true values and this covariance realistically increases as the distance from the boresight increases. The measurement error associated with the focal plane measurements results in error in the measured LOS vector. A general sensor LOS observation can be expressed in unit vector form given by

$$\mathbf{b} = \frac{1}{\sqrt{f + \alpha^2 + \beta^2}} \begin{bmatrix} \alpha \\ \beta \\ f \end{bmatrix} \tag{12}$$

where f denotes the focal length. The LOS observation has two independent parameters α and β. Therefore in the presence of random noise in these parameters the LOS vector still must maintain a unit norm. Although the LOS measurement noise must lie on the unit sphere the measurement noise can be approximated as additive noise, given by

$$\tilde{\mathbf{b}} = \mathbf{b} + \boldsymbol{\upsilon} \tag{13}$$

with

$$\boldsymbol{\upsilon} \sim \mathcal{N}\left(\mathbf{0}, \Omega\right) \tag{14}$$

where $\boldsymbol{\upsilon}$ is assumed to be a Gaussian random vector with zero mean and covariance Ω. Reference 5 has shown that the probability density for unit vector measurements lies on a sphere and can accurately be approximated by a density on a plane tangent to the vector for a small FOV sensors.

This approximation is known as the QUEST measurement model,[5] which characterizes the LOS noise process resulting from the focal plane model as

$$\Omega \equiv E\left\{\boldsymbol{\upsilon}\boldsymbol{\upsilon}^T\right\} = \sigma^2\left(I_{3\times 3} - \mathbf{b}\,\mathbf{b}^T\right) \tag{15}$$

It is clear that this is only valid for a small FOV in which a tangent plane closely approximates the surface of a unit sphere. For wide FOV sensors, a more accurate measurement covariance is shown in Reference 6. This formulation employs a first-order Taylor series approximation about the focal-plane axes. The partial derivative operator is used to linearly expand the focal-plane covariance in Eq. (11), given by (for $f = 1$)

$$J = \frac{\partial \mathbf{b}}{\partial \mathbf{m}} = \frac{1}{\sqrt{1+\alpha^2+\beta^2}}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} - \frac{1}{1+\alpha^2+\beta^2}\mathbf{b}\,\mathbf{m}^T \tag{16}$$

Then the wide-FOV covariance model is given by

$$\Omega = J\,R^{\text{FOCAL}} J^T \tag{17}$$

If a small FOV model is valid, then Eq. (17) can still be used, but is nearly identical to Eq. (15). For both equations, Ω is a 3×3 covariance matrix for a unit vector measurement with two independent parameters and therefore must be singular. A nonsingular covariance matrix for the LOS measurements can be obtained by a rank-one update to Ω:

$$\Omega_{\text{new}} = \Omega + \frac{1}{2}\text{trace}\left(\Omega\right)\mathbf{b}\mathbf{b}^T \tag{18}$$

which can be used without loss in generality to develop attitude-error covariance expressions.[1] Equation (17) represents the covariance for the LOS measurements in their respective body frame shown in Figure 2. Replacing $\mathbf{b}$ with respective true vectors and $\tilde{\mathbf{b}}$ with respective measured vectors, the measurement models are summarized by

$$\tilde{\mathbf{w}}_1 = \mathbf{w}_1 + \boldsymbol{\upsilon}_{w1}, \quad \boldsymbol{\upsilon}_{w1} \sim \mathcal{N}\left(\mathbf{0}, R_{w_1}\right) \tag{19a}$$
$$\tilde{\mathbf{w}}_i = \mathbf{w}_i + \boldsymbol{\upsilon}_{wi}, \quad \boldsymbol{\upsilon}_{wi} \sim \mathcal{N}\left(\mathbf{0}, R_{\mathbf{w}_i}\right) \quad \text{for } i = 2,\ldots,n \tag{19b}$$
$$\tilde{\mathbf{v}}_1 = \mathbf{v}_1 + \boldsymbol{\upsilon}_{v1}, \quad \boldsymbol{\upsilon}_{v1} \sim \mathcal{N}\left(\mathbf{0}, R_{v_1}\right) \tag{19c}$$
$$\tilde{\mathbf{v}}_i = \mathbf{v}_i + \boldsymbol{\upsilon}_{vi}, \quad \boldsymbol{\upsilon}_{vi} \sim \mathcal{N}\left(\mathbf{0}, R_{\mathbf{v}_i}\right) \quad \text{for } i = 2,\ldots,n \tag{19d}$$

Since in practice each vehicle will have their own set of LOS measurement devices, then the measurements in Eq. (19) can be assumed to be uncorrelated. This assumption will be used in the attitude covariance derivation.

DETERMINISTIC CASE: ONE CONSTRAINT

This section summarizes the constrained attitude solution. More details can be found in Reference 2. Considering the measurements shown in Figure 2, to determine the full attitude between the $\mathcal{B}_2$ and $\mathcal{B}_1$ frames the attitude matrix must satisfy the following measurement equations:

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{20a}$$
$$d = \mathbf{w}_2^T A\mathbf{v}_2 \tag{20b}$$

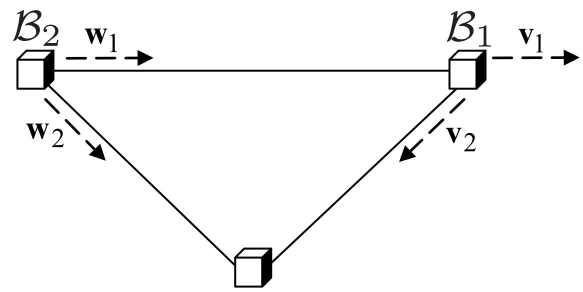


Figure 2. Observation Geometry

It is assumed that $|d| \leq 1$; otherwise a solution will not exist. Here it is assumed that the LOS vectors $\mathbf{v}_1$ and $\mathbf{w}_1$ are parallel. Also note that from Figure 2 no observation information is required from the third object to either $\mathcal{B}_1$ or $\mathcal{B}_2$. Hence, no information such as position is required for this object to determine the relative attitude. A solution for the attitude satisfying Eq. (20) is discussed in Reference 7 and will be utilized to form a solution for the constrained problem discussed here. The solution for the rotation matrix that satisfies Eq. (20) can be found by first finding a rotation matrix that satisfies that first equation and then finding the angle that one must rotate about the reference direction to align the two remaining vectors such that their dot product is equivalent to that measured in the remaining frame in the formation. The first rotation can be found by rotating about any direction by any angle, where $B = R\left(\mathbf{n}_1, \theta\right)$ is a general rotation about some axis rotation that satisfies Eq. (20a). The choice of the initial rotation axis is arbitrary, here the vector between the two reference direction vectors is used and the rotation is as follows:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T \mathbf{w}_1)} - I_{3\times 3} \tag{21}$$

where $\mathbf{n}_1 = (\mathbf{w}_1 + \mathbf{v}_1)$ and $\theta = \pi$. This rotation matrix will align the LOS vectors between frames, but the frames could still have some rotation about this vector, so therefore the angle about this axis must be determined to solve the second equation. To do so the vector $\mathbf{w}^*$ is first defined, which is the vector produced after applying the rotation B on the vector $\mathbf{v}_2$. This will allow us to determine the second rotation needed to map $\mathbf{v}_2$ properly to the $\mathcal{B}_2$ frame with $\mathbf{w}^* = B\,\mathbf{v}_2$. Since the rotation axis is the $\mathbf{w}_1$ vector, this vector will be invariant under this transformation and the solution to the full attitude can be written as $A = R\left(\mathbf{n}_2, \theta\right) B$.

Consider solving for the rotation angle using the planar constraint, the constraint can be written as the following:

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times] R\left(\mathbf{n}_2, \theta\right) \mathbf{w}^* \tag{22}$$

Substituting the second rotation matrix into Eq. (22), and with $\mathbf{n}_2 = \mathbf{w}_1$, leads to

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times][\mathbf{w}_1 \mathbf{w}_1^T - \cos(\theta)[\mathbf{w}_1 \times]^2 \mathbf{w}^* - \sin(\theta)[\mathbf{w}_1 \times]\mathbf{w}^*] \tag{23}$$

Expanding out this expression gives

$$\left(\mathbf{w}_2^T [\mathbf{w}_1 \times]\mathbf{w}^*\right)\cos(\theta) = \left(\mathbf{w}_2^T [\mathbf{w}_1 \times]^2 \mathbf{w}^*\right)\sin(\theta) \tag{24}$$

Notice that if Eq. (24) is divided by -1 then the resulting equation would be unchanged but the solution for the angle θ would differ by π. Therefore, using the planar constraint the solution for the angle θ can be written as $\theta = \beta + \phi$, where

$$\beta = \text{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) \tag{25}$$

and $\phi = 0$ or π. An ambiguity exists when using this approach but it is important to note that one of the possible solutions for this approach is equivalent to the triangle constraint case.

Finally the solution for the attitude is given by $A = R(\mathbf{w}_1, \theta)\, B$. The solution is now summarized:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T \mathbf{w}_1)} - I_{3\times3} \tag{26a}$$

$$R(\mathbf{w}_1, \theta) = I_{3\times3}\cos(\theta) + (1 - \cos(\theta))\mathbf{w}_1\mathbf{w}_1^T - \sin(\theta)[\mathbf{w}_1\times] \tag{26b}$$

$$\theta = \text{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \pi \tag{26c}$$

$$A = R(\mathbf{w}_1, \theta)\, B \tag{26d}$$

This result shows that for any formation of two vehicles a deterministic solution will exist using one direction and one angle. Due to the fact that this case is truly deterministic there is no need to minimize a cost function and the solution will always be the maximum likelihood one. It is very important to note that without the resolution of the attitude ambiguity any covariance development might not have any meaning since although the covariance might take a small value if the wrong possible attitude is used then the error might be fairly large and not bounded by the attitude covariance.

The solution in Eq. (26) can be rewritten without the use of any transcendental functions. The following relationships can be derived:

$$\cos(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{27a}$$

$$\sin(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{27b}$$

This leads to $\cos(\theta) = -b/c$ and $\sin(\theta) = -a/c$ with

$$a = \mathbf{w}_2^T[\mathbf{w}_1\times]\left([\mathbf{w}_1\times] + [\mathbf{v}_1\times]\right)[\mathbf{v}_1\times]\mathbf{v}_2 \tag{28a}$$

$$b = \mathbf{w}_2^T[\mathbf{w}_1\times]\left([\mathbf{w}_1\times][\mathbf{v}_1\times] - I_{3\times3}\right)[\mathbf{v}_1\times]\mathbf{v}_2 \tag{28b}$$

$$c = (1 + \mathbf{v}_1^T\mathbf{w}_1)\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\| \tag{28c}$$

Note that $c = \sqrt{a^2 + b^2}$. Then the matrix R is given by

$$R = -\frac{b}{c}\, I_{3\times3} + \left(1 + \frac{b}{c}\right)\mathbf{w}_1\mathbf{w}_1^T + \frac{a}{c}\,[\mathbf{w}_1\times] \tag{29}$$

Noting that $\mathbf{w}_1\mathbf{w}_1^T B = \mathbf{w}_1\mathbf{v}_1^T$ then the solution in Eq. (26d) can be rewritten as

$$\begin{aligned} A = \frac{b}{c}&\left(I_{3\times3} - \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} + \mathbf{w}_1\mathbf{v}_1^T\right) \\ &+ \frac{a}{c}\,[\mathbf{w}_1\times]\left(\frac{\mathbf{v}_1\mathbf{w}_1^T + \mathbf{v}_1\mathbf{v}_1^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times3}\right) + \mathbf{w}_1\mathbf{v}_1^T \end{aligned} \tag{30}$$

Note in practice the measured quantities from the previous section are used in place of the observed quantities shown in Eq. (26), and Eqs. (28) and (30).

The covariance matrix for an attitude estimate is defined as the covariance of a small angle rotation taking the true attitude to the estimated attitude. Typically small Euler angles are used to parameterize the attitude error-matrix. Reference 2 derives the attitude error-covariance for the constrained solution by using the attitude matrix with respect to the small angle errors. The attitude error-covariance is given by

$$P = \left(\begin{bmatrix} -[A_{\text{true}}\mathbf{v}_2\times] \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix} \begin{bmatrix} R_{\Delta_1} & R_{\Delta_1\Delta_2} \\ R_{\Delta_1\Delta_2}^T & R_{\Delta_2} \end{bmatrix}^{-1} \begin{bmatrix} -[A_{\text{true}}\mathbf{v}_2\times] \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix}^T \right)^{-1} \tag{31}$$

where

$$R_{\Delta_1} = R_{w_1} + A_{\text{true}} R_{v_1} A_{\text{true}}^T \tag{32a}$$

$$\begin{aligned} R_{\Delta_2} = {} & \mathbf{w}_2^T[A_{\text{true}}\mathbf{v}_2\times]R_{w_1}[A_{\text{true}}\mathbf{v}_2\times]\mathbf{w}_2 + (A_{\text{true}}\mathbf{v}_2)^T[\mathbf{w}_1\times]R_{w_2}[\mathbf{w}_1\times](A_{\text{true}}\mathbf{v}_2) \\ & + \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}R_{v_2}A_{\text{true}}^T[\mathbf{w}_1\times]\mathbf{w}_2 \end{aligned} \tag{32b}$$

$$R_{\Delta_1\Delta_2} = -R_{w_1}[A_{\text{true}}\mathbf{v}_2\times]\mathbf{w}_2 \tag{32c}$$

This expression is a function of the true attitude, A_{true}, but the true attitude can effectively be replaced with the estimated attitude to within first order.

OVER-DETERMINISTIC CASE: MULTIPLE CONSTRAINTS

When multiple common objects are acquired a deterministic solution is no longer achievable and therefore the solution given previously is no longer valid. The measurement geometry for the over-deterministic case is given in Figure 1. The observation is written using the following notation:

$$\tilde{\mathbf{s}}_1 = \tilde{\mathbf{w}}_1, \quad \tilde{\mathbf{r}}_1 = \tilde{\mathbf{v}}_1 \tag{33a}$$

$$\tilde{\mathbf{s}}_i = \frac{[\tilde{\mathbf{w}}_i\times]\tilde{\mathbf{w}}_1}{\|\tilde{\mathbf{w}}_i \times \tilde{\mathbf{w}}_1\|}, \quad \tilde{\mathbf{r}}_i = \frac{[\tilde{\mathbf{v}}_i\times]\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_i \times \tilde{\mathbf{v}}_1\|} \tag{33b}$$

where $\tilde{\mathbf{s}}_1$ and $\tilde{\mathbf{r}}_1$ denote the vectors that are derived from the LOS observations and therefore are corrupted with measurement noise, denoted by the $\tilde{\cdot}$ notation. The vectors $\tilde{\mathbf{s}}_i$ and $\tilde{\mathbf{r}}_i$, for $i > 1$, are the vectors associated with the multiple planar constraints. Notice that all vectors in Eq. (33) use the observations $\tilde{\mathbf{w}}_1$ or $\tilde{\mathbf{v}}_1$, resulting in cross-correlation between the vectors in Eq. (33). Then the attitude estimation problem can be written as follows:

$$\tilde{\mathbf{s}}_1 = A\tilde{\mathbf{r}}_1 \tag{34a}$$

$$\tilde{\mathbf{s}}_i = A\tilde{\mathbf{r}}_i \tag{34b}$$

The measurement models for $\tilde{\mathbf{s}}_i$ and $\tilde{\mathbf{r}}_i$ are given by

$$\tilde{\mathbf{s}}_1 = \mathbf{s}_1 + \mathbf{\Delta s}_1, \quad \tilde{\mathbf{r}}_1 = \mathbf{r}_1 + \mathbf{\Delta r}_1 \tag{35a}$$

$$\tilde{\mathbf{s}}_i = \mathbf{s}_i + \mathbf{\Delta s}_i, \quad \tilde{\mathbf{r}}_i = \mathbf{r}_i + \mathbf{\Delta r}_i \tag{35b}$$

Here both the measurement LOS vectors and the reference vectors contain uncertainty and therefore two noise terms are needed, denoted by $\boldsymbol{\Delta}_{\mathbf{s}_i}$ and $\boldsymbol{\Delta}_{\mathbf{r}_i}$. Equations (34) and (33) are used to write the effective noise of the LOS equations as

$$\boldsymbol{\Delta}_1 = \boldsymbol{\Delta}\mathbf{s}_1 - A\boldsymbol{\Delta}\mathbf{r}_1 \tag{36a}$$

$$\boldsymbol{\Delta}_i = \boldsymbol{\Delta}\mathbf{s}_i - A\boldsymbol{\Delta}\mathbf{r}_i \tag{36b}$$

where $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_i$ denote the noise in the first and i^{th} LOS equation, respectively, and the noise terms are functions of the measurements $\{\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i\}$. To describe the optimal attitude estimation problem the noise statistics of $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_i$ need to be described with respect to the known noise statistics of the measurements $\{\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i\}$. For now it is assumed that $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_i$ are both zero-mean Gaussian random variables.

Under this assumption the covariance expression can be written as

$$R_{\boldsymbol{\Delta}_1} = R_{\boldsymbol{\Delta}\mathbf{s}_1} + AR_{\boldsymbol{\Delta}\mathbf{r}_1}A^T \tag{37a}$$

$$R_{\boldsymbol{\Delta}_i} = R_{\boldsymbol{\Delta}\mathbf{s}_i} + AR_{\boldsymbol{\Delta}\mathbf{r}_i}A^T \tag{37b}$$

Then the LOS equations can be cast in matrix vector form, given by

$$\mathbf{y}(\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_n) = \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_n) + \boldsymbol{\Delta} \tag{38}$$

where $\mathbf{h}(A, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_n) = \begin{bmatrix}(\hat{A}\tilde{\mathbf{r}}_1)^T & \dots & (\hat{A}\tilde{\mathbf{r}}_n)^T\end{bmatrix}^T$, $\mathbf{y}(\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_n) = \begin{bmatrix}\tilde{\mathbf{s}}_1^T & \dots & \tilde{\mathbf{s}}_n^T\end{bmatrix}^T$, and $\boldsymbol{\Delta} = \begin{bmatrix}\boldsymbol{\Delta}_1^T & \dots & \boldsymbol{\Delta}_i^T\end{bmatrix}^T$. The covariance for $\boldsymbol{\Delta}$ can be written in the following form:

$$\mathcal{R} = \begin{bmatrix} R_{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_1} & \dots & R_{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_i} & \dots & R_{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_n} \\ \vdots & \ddots & \cdot & \cdot & \vdots \\ R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_1} & \cdot & R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_i} & \cdot & R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_n} \\ \vdots & \cdot & \cdot & \ddots & \vdots \\ R_{\boldsymbol{\Delta}_n\boldsymbol{\Delta}_1} & \dots & R_{\boldsymbol{\Delta}_n\boldsymbol{\Delta}_i} & \dots & R_{\boldsymbol{\Delta}_n\boldsymbol{\Delta}_n} \end{bmatrix} \tag{39}$$

where the terms of this covariance matrix can be found by taking $E\{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_1^T\}$. This covariance matrix will be derived in the next section. Then given this representation the optimal attitude estimation problem can be stated as

$$\begin{aligned} \min J &= \Big(\mathbf{y}(\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_n) - \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_n)\Big)^T \mathcal{R}^{-1} \Big(\mathbf{y}(\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_n) - \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_n)\Big) \\ \text{s.t. } & I_{3\times3} - \hat{A}\,\hat{A}^T = 0_{3\times3} \end{aligned} \tag{40}$$

The issue with solving Eq. (40) for A is that the vector components of $\mathbf{y}(\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_n) - \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_n)$ are correlated because each $\mathbf{w}_i$ and $\mathbf{v}_i$ is calculated using $\mathbf{w}_1$ and $\mathbf{v}_1$. Therefore $\mathcal{R}$ is not a diagonal matrix and the objective function can not be written as a sum of individual vector residuals, which will prevent writing the objective function in the form given by Wahba's problem.[8] Specifically, the cost function is quartic when the quaternion parameterization is used for the attitude matrix. Unfortunately, there exists no solution in the open literature for this type of problem. This motivates using an approximate weighting matrix of the form

$$W = \begin{bmatrix} \sigma_1^2 I_{3\times3} & \sigma_1^2 I_{3\times3} & \sigma_1^2 I_{3\times3} \\ \sigma_1^2 I_{3\times3} & \sigma_1^2 I_{3\times3} & \sigma_1^2 I_{3\times3} \\ \sigma_1^2 I_{3\times3} & \sigma_1^2 I_{3\times3} & \sigma_1^2 I_{3\times3} \end{bmatrix}^{-1} \tag{41}$$

where $W \approx \mathcal{R}^{-1}$ is the approximation of the weighting matrix in Eq. (40). Using Eq. (41) as the weighting matrix in the cost function also maintains the linear structure allowing an analytic solution for the attitude using the QUEST approach,[5] while still maintaining some of the correlation information contained in the exact problem. The question still remains how to choose σ_i^2 and $\sigma_{i,j}^2$ in the approximated weighting matrix. This work will implement the approximation of using $\sigma_{i,j}^2 = \text{trace}(\Omega_{i,j})$ as the weighting matrix. The exact problem will be solved using an iterative nonlinear least squares approach and compared to the approximate solution using the weighting function given in Eq. (41) and obtained in closed form via the QUEST solution.

Solution to the Suboptimal Problem

The general problem with the general weighting function can be written as

$$J(A) = \sum_{i=1}^{n} \frac{1}{2} (\mathbf{s}_i - A\mathbf{r}_i)^T W_{ii} (\mathbf{s}_i - A\mathbf{r}_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq 1}^{n} \frac{1}{2} (\mathbf{s}_i - A\mathbf{r}_i)^T W_{ij} (\mathbf{s}_j - A\mathbf{r}_j) \tag{42}$$

The issue with solving this problem is the structure of W_{ij} and W_{ii} which are fully populated matrices. In general $W = \mathcal{R}^{-1}$ is chosen to define the optimal problem. But this problem is difficult to solve so the approximations $W_{ii} = a_{ii}^{-1} I_{3\times 3}$ and $W_{ij} = a_{ij}^{-1} I_{3\times 3}$ are chosen instead. Then the cost function can be written as

$$J(A) = \sum_{i=1}^{n} \frac{1}{2a_{ii}} (\mathbf{s}_i - A\mathbf{r}_i)^T (\mathbf{s}_i - A\mathbf{r}_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq 1}^{n} \frac{1}{2a_{ij}} (\mathbf{s}_i - A\mathbf{r}_i)^T (\mathbf{s}_j - A\mathbf{r}_j) \tag{43}$$

The cost function can be rewritten as

$$J(A) = \sum_{i=1}^{n} \frac{1}{a_{ii}} \left(1 - \mathbf{s}_i^T A\mathbf{r}_i\right) + \sum_{i=1}^{n} \sum_{j=1, j\neq 1}^{n} \frac{1}{a_{ij}} \left(1 - \mathbf{s}_i^T A\mathbf{r}_j\right) + C \tag{44}$$

where C is given by $C = \sum_{i=1}^{n} \frac{1}{2a_{ij}} \left(\mathbf{s}_i^T \mathbf{s}_j - (A\mathbf{r}_i)^T A\mathbf{r}_j - 2\right)$ and is not a function of the attitude. Thus this term can be neglected. Using the quaternion parameterization leads to

$$J(A) = \sum_{i=1}^{n} \frac{1}{a_{ii}} \left(1 - \mathbf{s}_i^T \Xi^T(\mathbf{q})\Psi(\mathbf{q})\mathbf{r}_i\right) + \sum_{i=1}^{n} \sum_{j=1, j\neq 1}^{n} \frac{1}{a_{ij}} \left(1 - \mathbf{s}_i^T \Xi^T(\mathbf{q})\Psi(\mathbf{q})\mathbf{r}_j\right) \tag{45}$$

Next the identities $\Xi(\mathbf{q})\mathbf{s}_i = \Omega(\mathbf{s}_i)\mathbf{q}$ and $\Psi(\mathbf{q})\mathbf{r}_i = \Gamma(\mathbf{r}_i)\mathbf{q}$ are used, where $\Omega(\mathbf{b}_i)$ and $\Gamma(\mathbf{b}_i)$ are defined by

$$\Omega(\mathbf{b}_i) \equiv \begin{bmatrix} -[\mathbf{b}_i \times] & \mathbf{b}_i \\ -\mathbf{b}_i^T & 0 \end{bmatrix} \quad \Gamma(\mathbf{b}_i) \equiv \begin{bmatrix} [\mathbf{b}_i \times] & \mathbf{b}_i \\ -\mathbf{b}_i^T & 0 \end{bmatrix} \tag{46}$$

This allows Eq. (45) to be written as

$$J(\mathbf{q}) = \mathbf{q}^T K \mathbf{q} \tag{47}$$

where

$$K \equiv -\sum_{i=1}^{n} \frac{1}{a_{ii}} \Omega(\mathbf{s}_i)\Gamma(\mathbf{r}_i) - \sum_{i=1}^{n} \sum_{j=1, j\neq 1}^{n} \frac{1}{a_{ij}} \Omega(\mathbf{s}_i)\Gamma(\mathbf{r}_j) \tag{48}$$

The goal is to minimize $J(A)$ or $J(\mathbf{q})$ but it must also be ensured that the quaternion constraint is maintained. Lagrange multipliers are employed to impose the quaternion constraint on the solution. The loss function is augmented as follows

$$J(\mathbf{q}) = \mathbf{q}^T K \mathbf{q} + \lambda(1 - \mathbf{q}^T \mathbf{q}) \tag{49}$$

where λ is the Lagrange multiplier. Then the optimality condition can be imposed on the loss function, mainly $\frac{\partial J(\mathbf{q})}{\partial \mathbf{q}} = 0$, which leads to

$$K\mathbf{q} = \lambda \mathbf{q} \tag{50}$$

This is an eigenvalue problem with four possible solutions. Substituting Eq. (50) into Eq. (49) leads to $J(\mathbf{q}) = \lambda$, hence the eigenvector associated with the smallest eigenvalue is chosen. It can be shown that as long as there exist at least two non-collinear vectors there will be a distinct and real minimum eigenvalue. The solution to Eq. (50) can be found using a number of existing efficient algorithms that do not involve a full eigenvalue/eigenvector decomposition, e.g. QUEST.[5]

All that remains is how to select the values of a_{ii} and a_{ij}. One approach is to determine the covariance for the approximate solution, P_{aprox}, and select the values for a_{ii} and a_{ij} such that the trace of P_{aprox} is minimized. This is extremely complex approach, therefore the expression for the approximate covariance, P_{aprox}, and the optimal covariance, P_{opt}, are used. The error attitude covariance for the general/optimal cost function can be derived using the results from maximum likelihood estimate.[9] The Fisher information matrix for a parameter vector $\mathbf{x}$ is given by

$$F_{\mathbf{xx}} = E\left\{ \frac{\partial^2}{\partial \mathbf{x} \partial \mathbf{x}} J(\mathbf{x}) \right\} \tag{51}$$

Then the error covariance for $\mathbf{x}$ can written as

$$P_{\mathbf{xx}} = F_{\mathbf{xx}}^{-1} \tag{52}$$

Equation (40) can used to derive the attitude error covariance using Eq. (51). Because the attitude error is not expected to be large, a small error angle assumption is made in Eq. (40). The attitude estimate can be expressed in terms of the true attitude and the angle errors, $\boldsymbol{\delta\alpha}$: $\hat{A} = e^{-[\boldsymbol{\delta\alpha}\times]} A_{\text{true}} \approx (I_{3\times3} - [\boldsymbol{\delta\alpha}\times])\, A_{\text{true}}$, where $\boldsymbol{\delta\alpha}$ represents the small roll, pitch and yaw error rotations. The attitude covariance is defined as

$$P_{\boldsymbol{\delta\alpha\delta\alpha}} = E\left\{ \boldsymbol{\delta\alpha\delta\alpha}^T \right\} \tag{53}$$

The attitude angle error covariance for the two vector case can be accomplished using the Cramèr-Rao inequality:

$$F = -E\left\{ \frac{\partial^2}{\partial \mathbf{x} \partial \mathbf{x}^T} \ln L\left(\tilde{\mathbf{y}}; \mathbf{x}\right) \right\} \tag{54}$$

where $L\left(\tilde{\mathbf{y}}; \mathbf{x}\right)$ is the likelihood function for a measurement $\tilde{\mathbf{y}}$. Taking the appropriate partials with respect to $\boldsymbol{\delta\alpha}$ leads to the following covariance:

$$P_{opt} \equiv E\left\{ \boldsymbol{\delta\alpha\delta\alpha}^T \right\} = \left(H \mathcal{R}^{-1} H^T \right)^{-1} \tag{55}$$

where $H = -\left[[\mathbf{s}_1\times] \ldots [\mathbf{s}_n\times]\right]^T$. Note that this expression is evaluated at the true values due to the expectation and the assumption that the observation are unbiased. The same procedure can be followed for the approximate cost function to derive the attitude error covariance for the approximate problem. Using the cost function in Eq. (43) and small angle approximation for the error the attitude error covariance can be written as

$$P_{\text{aprox}} \equiv E\left\{\boldsymbol{\delta\alpha\delta\alpha}^T\right\} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{a_{ij}}[\mathbf{s}_i\times][\mathbf{s}_j\times]^T\right)^{-1} \tag{56}$$

Note from Eq. (55) and Eq. (56) that the only difference between the two expressions is the matrix $R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_j}$. In fact if $R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_j} = \sigma_{ij}^2 I_{3\times3}$ then Eq. (55) reduces to Eq. (56). Hence the a_{ij} are chosen to minimize some matrix norm of the following form:

$$J(a_{ij}) = \|R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_j} - a_{ij}I_{3\times3}\| \tag{57}$$

For example if the norm in Eq. (57) is chosen to be the Frobenius norm then the values for a_{ij} that minimize Eq. (57) are

$$a_{ij} = \frac{1}{3}\text{trace}\left(R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_j}\right) \tag{58}$$

Then once the proper weights are determined the approximate attitude solution can be found by solving Eq. (50) and the attitude error covariance for this solution can be computed using Eq. (56). Furthermore the performance of the approximate solution can be compared with the optimal solution by comparing Eq. (55) and Eq. (56). Finally to compute the solution the $R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_j}$ covariance terms must be known, which will be derived in the next section.

Covariance Expressions

For simplicity consider the following notation:

$$\tilde{\mathbf{s}}_1 = \boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i) \tag{59}$$

This simplifies the derivation of the covariance matrix. The terms needed for the covariance derivation are given by

$$\begin{aligned}\boldsymbol{\alpha}(\mathbf{w}_1,\mathbf{w}_i) = \|\mathbf{w}_i\times\mathbf{w}_1\|^{-1},\quad \boldsymbol{\alpha}_{\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_i) &= \frac{1}{2}\|\mathbf{w}_i\times\mathbf{w}_1\|^{-3}([\mathbf{w}_i\times]\mathbf{w}_1)^T[\mathbf{w}_i\times],\\ \boldsymbol{\alpha}_{\mathbf{w}_i}(\mathbf{w}_1,\mathbf{w}_i) &= -\frac{1}{2}\|\mathbf{w}_i\times\mathbf{w}_1\|^{-3}([\mathbf{w}_i\times]\mathbf{w}_1)^T[\mathbf{w}_1\times]\end{aligned} \tag{60a}$$

$$\boldsymbol{\beta}(\mathbf{w}_1,\mathbf{w}_i) = [\mathbf{w}_i\times]\mathbf{w}_1,\quad \boldsymbol{\beta}_{\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_i) = [\mathbf{w}_i\times],\quad \boldsymbol{\beta}_{\mathbf{w}_i}(\mathbf{w}_1,\mathbf{w}_i) = -[\mathbf{w}_1\times] \tag{60b}$$

A first-order Taylor series expansion about the error statistics of the terms $\boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ and $\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ yields:

$$\boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i) = \boldsymbol{\alpha}(\mathbf{w}_1,\mathbf{w}_i) + \boldsymbol{\alpha}_{\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_i)\boldsymbol{\upsilon}_{w1} + \boldsymbol{\alpha}_{\mathbf{w}_i}(\mathbf{w}_1,\mathbf{w}_i)\boldsymbol{\upsilon}_{wi} \tag{61a}$$

$$\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i) = \boldsymbol{\beta}(\mathbf{w}_1,\mathbf{w}_i) + \boldsymbol{\beta}_{\mathbf{w}_1}(\mathbf{w}_1,\mathbf{w}_i)\boldsymbol{\upsilon}_{w1} + \boldsymbol{\beta}_{\mathbf{w}_i}(\mathbf{w}_1,\mathbf{w}_i)\boldsymbol{\upsilon}_{wi} \tag{61b}$$

Note that the expressions for $\boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ and $\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ can b written in terms of the true quantities $\mathbf{w}_1$, $\mathbf{w}_i$, and error terms $\boldsymbol{\upsilon}_{wi}$ and $\boldsymbol{\upsilon}_{w1}$. The notation is simplified by removing the first function dependence in $\mathbf{w}_1$ since all terms share this and display the functional dependence of the second term as a superscript, i.e. $\boldsymbol{\beta}^i = \boldsymbol{\beta}(\mathbf{w}_1,\mathbf{w}_i)$.

To determine the error term in $\tilde{\mathbf{s}}_1 = \mathbf{s}_1 + \mathbf{\Delta s}_i$ a Taylor series expansion in Eq. (61) is used and the result is substituted into Eq. (59), giving

$$\mathbf{s}_i + \mathbf{\Delta s}_i = \left(\boldsymbol{\alpha}^i + \boldsymbol{\alpha}^i_{\mathbf{w}_1}\boldsymbol{\upsilon}_{w1} + \boldsymbol{\alpha}^i_{\mathbf{w}_i}\boldsymbol{\upsilon}_{wi}\right) \cdot \left(\boldsymbol{\beta}^i + \boldsymbol{\beta}^i_{\mathbf{w}_1}\boldsymbol{\upsilon}_{w1} + \boldsymbol{\beta}^i_{\mathbf{w}_i}\boldsymbol{\upsilon}_{wi}\right) \tag{62}$$

Rearranging terms and using the fact that $\mathbf{s}_i = \boldsymbol{\alpha}^i\boldsymbol{\beta}^i$, and neglecting higher order terms $\boldsymbol{\upsilon}_{w1}\boldsymbol{\upsilon}_{wi}$, $\boldsymbol{\upsilon}_{w1}\boldsymbol{\upsilon}_{w1}$, and $\boldsymbol{\upsilon}_{wi}\boldsymbol{\upsilon}_{wi}$, then the following expression is given:

$$\mathbf{\Delta s}_i = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_1})\boldsymbol{\upsilon}_{w1} + (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})\boldsymbol{\upsilon}_{wi} \tag{63}$$

Then the covariance for $\mathbf{\Delta s}_i$ can be calculated as

$$\begin{aligned} R_{\mathbf{\Delta s}_i\mathbf{\Delta s}_i} &= E\{\mathbf{\Delta s}_i\mathbf{\Delta s}_i^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})R_{\mathbf{w}_1}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})^T \\ &+ (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})R_{\mathbf{w}_i}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})^T \end{aligned} \tag{64}$$

The cross-correlation between the $\mathbf{\Delta s}_i$ and $\mathbf{\Delta s}_j$ terms can also be written as

$$R_{\mathbf{\Delta s}_i\mathbf{\Delta s}_j} = E\{\mathbf{\Delta s}_i\mathbf{\Delta s}_j^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_1})R_{\mathbf{w}_1}(\boldsymbol{\alpha}^j\boldsymbol{\beta}^j_{\mathbf{w}_1} + \boldsymbol{\beta}^j\boldsymbol{\alpha}^j_{\mathbf{w}_1})^T \tag{65}$$

Finally the cross-correlation between the $\mathbf{\Delta s}_1$ and $\mathbf{\Delta s}_i$ terms can also be written as

$$R_{\mathbf{\Delta s}_i\mathbf{\Delta s}_1} = E\{\mathbf{\Delta s}_i\mathbf{\Delta s}_1^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_1})R_{\mathbf{w}_1} \tag{66}$$

A similar procedure can be used to compute $\mathbf{\Delta r}_i$ terms, which is summarized here. Similarly, for simplicity the following notation is considered:

$$\tilde{\mathbf{r}}_1 = \alpha(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i)\boldsymbol{\alpha}(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i) \tag{67}$$

The terms need for the covariance derivation for the terms $\mathbf{\Delta r}_i$ are given by

$$\begin{aligned} \boldsymbol{\alpha}(\mathbf{v}_1, \mathbf{v}_i) = \|\mathbf{v}_i \times \mathbf{v}_1\|^{-1}, \quad \boldsymbol{\alpha}_{\mathbf{v}_1}(\mathbf{v}_1, \mathbf{v}_i) &= \frac{1}{2}\|\tilde{\mathbf{v}}_i \times \mathbf{v}_1\|^{-3}([\mathbf{v}_i\times]\mathbf{v}_1)^T[\mathbf{v}_i\times], \\ \boldsymbol{\alpha}_{\mathbf{v}_i}(\mathbf{v}_1, \mathbf{v}_i) &= -\frac{1}{2}\|\mathbf{v}_i \times \mathbf{v}_1\|^{-3}([\mathbf{v}_i\times]\mathbf{v}_1)^T[\mathbf{v}_1\times] \end{aligned} \tag{68a}$$

$$\boldsymbol{\beta}(\mathbf{v}_1, \mathbf{v}_i) = [\mathbf{v}_i\times]\mathbf{v}_1, \quad \boldsymbol{\beta}_{\mathbf{v}_1}(\mathbf{v}_1, \mathbf{v}_i) = [\mathbf{v}_i\times], \quad \boldsymbol{\beta}_{\mathbf{v}_i}(\mathbf{v}_1, \mathbf{v}_i) = -[\mathbf{v}_1\times] \tag{68b}$$

Then the error term $\mathbf{\Delta r}_i$ can be written in terms of the noise terms $\boldsymbol{\upsilon}_{v1}$ and $\boldsymbol{\upsilon}_{vi}$, resulting in the following expression:

$$\mathbf{\Delta r}_i = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})\boldsymbol{\upsilon}_{w1} + (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_i})\boldsymbol{\upsilon}_{vi} \tag{69}$$

Then the covariance term for $\mathbf{\Delta r}_i$: $R_{\mathbf{\Delta r}_i\mathbf{\Delta r}_i}$, $R_{\mathbf{\Delta r}_i\mathbf{\Delta r}_j}$, and $R_{\mathbf{\Delta r}_i\mathbf{\Delta r}_1}$ are given as follows

$$\begin{aligned} R_{\mathbf{\Delta r}_i\mathbf{\Delta r}_i} &= E\{\mathbf{\Delta r}_i\mathbf{\Delta r}_i^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})R_{\mathbf{v}_1}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})^T \\ &+ (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_i})R_{\mathbf{v}_i}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_i})^T \end{aligned} \tag{70a}$$

$$R_{\mathbf{\Delta r}_i\mathbf{\Delta r}_j} = E\{\mathbf{\Delta r}_i\mathbf{\Delta r}_j^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})R_{\mathbf{v}_1}(\boldsymbol{\alpha}^j\boldsymbol{\beta}^j_{\mathbf{v}_1} + \boldsymbol{\beta}^j\boldsymbol{\alpha}^j_{\mathbf{v}_1})^T \tag{70b}$$

$$R_{\mathbf{\Delta r}_i\mathbf{\Delta r}_1} = E\{\mathbf{\Delta r}_i\mathbf{\Delta r}_1^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})R_{\mathbf{v}_1} \tag{70c}$$

Then the covariance in Eq. (39) can be formed by using Eq. (60) to form the terms in Eqs. (64), (65) and (66), and using Eq. (67) to form the terms in Eqs. (70a), (70b) and (70c), and combining it all together to solve for the terms in Eq. (37). These terms can be collected into one matrix given by Eq. (39). Once the full covariance matrix is found then each of the weights given in Eq. (58) can be calculated and the approximate solution given in Eqs. (50) and (48) can be formed. The covariance for the approximate solution can be formed using the weight to quantify the error in the solution. In the next section a numerical simulation example is considered to study the performance of the proposed solution.

SIMULATIONS

Two simulation scenarios are studied: a static formation configuration and a dynamic formation configuration of two vehicles and multiple common targets are considered. Each vehicle has light source devices and FPDs, which produce a set of parallel LOS measurements. In the static formation each vehicle is observing two common objects. As mentioned previously the location of these objects is not required for the attitude solution, only the LOS vectors from each vehicle to the objects are needed. In the dynamic configuration is is assumed that the two vehicle are formation flying spacecraft in a close relative orbit. The two common targets in the dynamic case are two other spacecraft that are also in the formation. They do not measure or communicate LOS observations to the two vehicles for which the relative attitudes are determined.

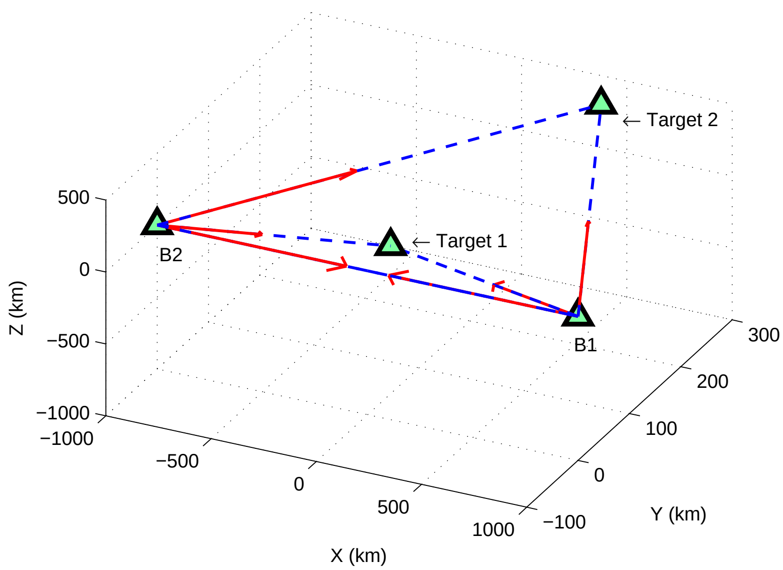


Figure 3. Dynamic Trajectories

Static Formation Simulation

The formation configuration uses the following true location of vehicle and targets:

$$\mathbf{x}_1 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \text{ m}, \quad \mathbf{x}_2 = \begin{bmatrix} -1000 \\ 0 \\ 0 \end{bmatrix} \text{ m}, \quad \mathbf{x}_3 = \begin{bmatrix} 500 \\ 250 \\ 500 \end{bmatrix} \text{ m}, \quad \mathbf{x}_4 = \begin{bmatrix} -500 \\ 250 \\ -800 \end{bmatrix} \text{ m} \tag{71}$$

Here vehicle one is at $\mathbf{x}_1$ and vehicle two is at $\mathbf{x}_2$. Targets one and two are at $\mathbf{x}_3$ and $\mathbf{x}_4$, respectively. The LOS truth vectors are determined from locations listed in Eq. (71), others can be found by using the appropriate attitude transformation without noise added. For this configuration the true relative attitude is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{72}$$

For the simulation the LOS vectors are converted into focal-plane coordinates and random noise is added to the true values having covariances described previously, with $\sigma = 17 \times 10^{-6}$ rad. Since each FPD has its own boresight axis, and the measurement covariance in Eq. (11) is described with respect to the boresight, individual sensor frames must be defined to generate the FPD measurements. The measurement error-covariance for each FPD is determined with respect to the corresponding sensor frames and must be rotated to the vehicle's body frame as well. The letter S is used to denote sensor frame. The orthogonal transformations for their respective sensor frames, denoted by the subscript, used to orientate the FPD to the specific vehicle, denoted by the superscript, are given by

$$A_{s_{\mathcal{B}_1}}^{\mathbf{v}_1} = \begin{bmatrix} -0.8373 & -0.2962 & 0.4596 \\ -0.2962 & -0.4609 & 0.8366 \\ 0.4596 & -0.8366 & 0.2981 \end{bmatrix}, \quad A_{s_{\mathcal{B}_1}}^{\mathbf{v}_2} = \begin{bmatrix} -0.8069 & 0.4487 & 0.3843 \\ 0.4487 & -0.0423 & 0.8927 \\ 0.3843 & 0.8927 & -0.2355 \end{bmatrix} \tag{73a}$$

$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_1} = \begin{bmatrix} -0.8889 & 0.0644 & 0.4535 \\ 0.0644 & -0.9626 & 0.2630 \\ 0.4535 & 0.2630 & 0.8515 \end{bmatrix}, \quad A_{s_{\mathcal{B}_2}}^{\mathbf{w}_2} = \begin{bmatrix} 0.4579 & -0.0169 & 0.8888 \\ -0.0169 & -0.9998 & -0.0103 \\ 0.8888 & 0.0103 & -0.4581 \end{bmatrix} \tag{73b}$$

$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_3} = \begin{bmatrix} -0.8121 & -0.1287 & -0.5692 \\ -0.1287 & -0.9119 & 0.3898 \\ -0.5692 & 0.3898 & 0.7240 \end{bmatrix}, \quad A_{s_{\mathcal{B}_1}}^{\mathbf{v}_3} = \begin{bmatrix} -0.1171 & -0.1582 & 0.9804 \\ -0.1582 & -0.9716 & -0.1757 \\ 0.9804 & -0.1757 & 0.0887 \end{bmatrix} \tag{73c}$$

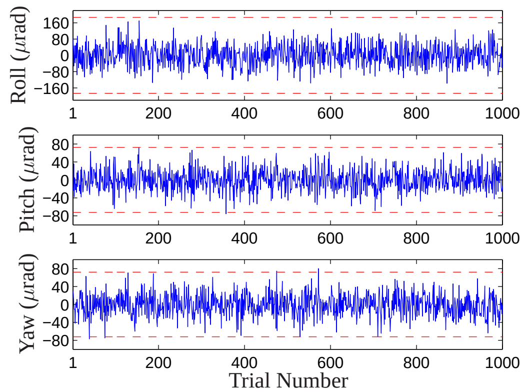
The configuration is considered for 1,000 Monte Carlo trials. Measurements are generated in the sensor frame and rotated to the body frame to be combined with the other measurements to determine the full relative attitudes. The wide-FOV measurement model for the FPD LOS covariance is used. Relative attitude angle errors are displayed in Figure 4. Three case are shown: two cases involve only one target and the LOS vectors between the vehicle, and one case uses both targets and the LOS vectors between the vehicles. The first case uses only target one, $\mathbf{x}_3$, which is shown in Figure 4(a). Case two uses target 2, $\mathbf{x}_4$, which is shown in Figure 4(b), and Case 3 uses both targets which is shown in Figure 4(c).

Better performance characteristics are given when both targets are used in the constrained solution. Figure 4 shows that the derived attitude-error covariance does indeed bound these errors in a 3σ sense. It is seen that Case 1 and Case 2 have larger roll errors than Case 3. In this scenario since the LOS between the vehicle is in the x-direction, rotation about this direction is not resolved by the LOS observation. The common target observations provide the missing information. Therefore Case 3, which uses both targets, has smaller errors in the roll direction than Case 1 or Case 2 which only use one target. The covariance values for the three cases are given below

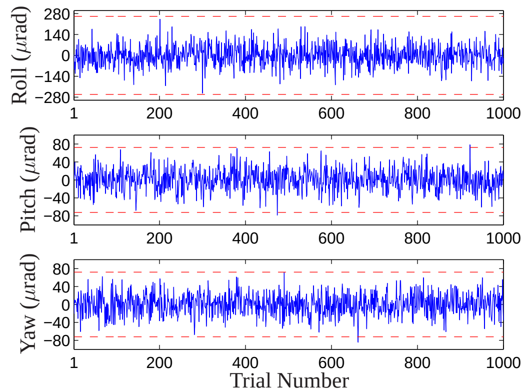
$$P_{\text{case1}} = 1 \times 10^{-8} \begin{bmatrix} 0.2599 & -0.0124 & 0.0335 \\ -0.0124 & 0.0583 & -0.0008 \\ 0.0335 & -0.0008 & 0.0579 \end{bmatrix} \text{rad}^2 \tag{74a}$$

$$P_{\text{case2}} = 1 \times 10^{-8} \begin{bmatrix} 0.5011 & 0.0243 & 0.0470 \\ 0.0243 & 0.0583 & -0.0008 \\ 0.0470 & -0.0008 & 0.0579 \end{bmatrix} \text{rad}^2 \tag{74b}$$

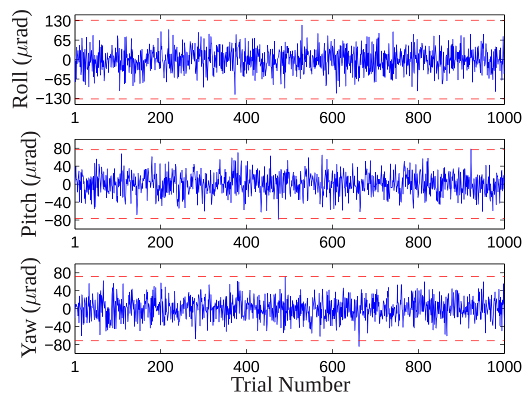
$$P_{\text{case3}} = 1 \times 10^{-8} \begin{bmatrix} 0.1925 & 0.0243 & 0.0470 \\ 0.0243 & 0.0583 & -0.0008 \\ 0.0470 & -0.0008 & 0.0579 \end{bmatrix} \text{rad}^2 \tag{74c}$$



(a) **Relative Attitude Estimate Errors Case 1: Using Target 1**



(b) **Relative Attitude Estimate Errors Case 2: Using Target 2**



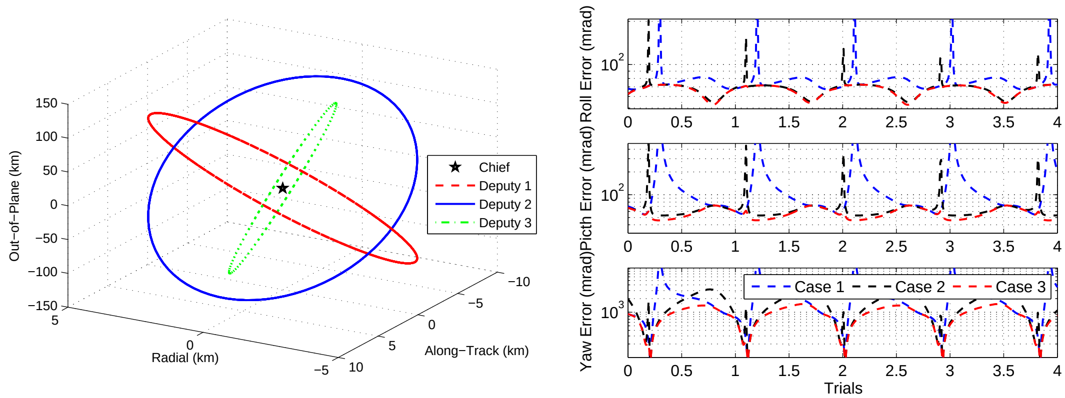
(c) **Relative Attitude Estimate Errors Case 3: Using both Targets**

Figure 4. Relative Attitude Estimate Errors for the Three Cases

Dynamic Formation Simulation

In the dynamic configuration four spacecraft are flying in a closed formation, Their relative positions are unknown as they measure LOS vectors to each other. In this configuration it is assumed that the FPD devices are gimballed allowing constant visibility and, for simplicity, it is assumed that the rotation matrix between the body frames to the sensor frames are those listed for the static simulations.

The relative motion trajectories are displayed in Figure 3. The chief spacecraft is in a low-Earth orbit and the orbital element differences are selected such that the deputies relative motion around the chief have both in-plane and out-plane relative motion. The orbital elements of the chief and the orbital element difference for the deputies are listed in Table 1. These values can be used to produce the orbits given in 3 a using linearized solution.[10] The sizes of the orbits are chosen such that deputy 1 has a smaller orbit than deputy 2 and 3 who both have similar size orbits in different orientations. Deputy 1 has 5 km of in-plane motion and 100 km of out-of-plane motion, as opposed to the size for deputy 2 and 3 which have 10 km of in-plane motion and 150 km of out-of-plane motion. LOS vectors are converted into focal-plane coordinates and random noise is added to the



(a) **Dynamic Trajectories** (b) **Comparison of 3σ Bounds for three Cases**

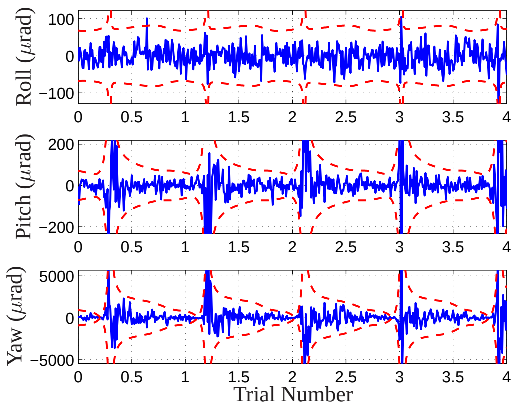
Figure 5. Dynamic Scenario

true values having covariances described previously, with $\sigma = 17 \times 10^{-6}$rad. The algorithms for the deterministic case with one constraint and for the suboptimal solution case are used to provide a point-by-point solution for the relative attitude. The relative attitude that is determined solves the transformation from the chief spacecraft frame, $\mathcal{C}$, to the deputy 1 frame, $\mathcal{D}_1$. These frames are defined as the fixed body frames of each spacecraft and it is assumed both the chief and the deputy 1 spacecraft have inertial fixed attitudes (no attitude dynamics). The algorithm from the deterministic case with one constraint is used to solve for the relative attitude in the deterministic case where only one common target is considered. Here the algorithm from the deterministic case with one constraint is used to solve for the relative attitude between the chief and deputy 1 spacecraft for two cases: one case where deputy 2 is considered as the common object and the other where deputy 3 is considered as a common object. The algorithm from the suboptimal solution case is used to provide a solution for the over-deterministic case where both deputy 2 and deputy 3 are used as common objects to solve for the relative attitude between the chief spacecraft and deputy 1 spacecraft.

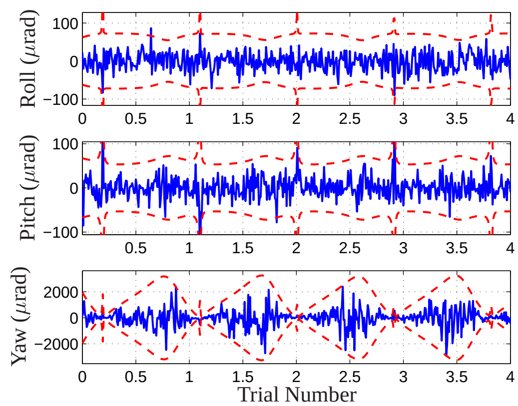
Figure 6 displays the relative attitude errors for the dynamic configuration for all three cases. The magnitude of the relative attitude errors dependence on geometry can clearly be seen. As the LOS geometry changes throughout the trajectory, the 3σ bounds of the errors also change and accurately bound the estimated attitude errors. A large increase in the relative error can be seen as the LOS configuration approaches an extreme condition where $\mathbf{w}_1$, $\mathbf{w}_2$ and $A_{\text{true}}\mathbf{v}_2$ are nearly parallel for case 1 and case 2. This results in a near unobservable situation, which is correctly depicted in the covariance. Since case 3 determines a solution using both deputy 2 and deputy 3 as common objects it avoids unobservable situations as can be seen from Figure 6. In fact case 3 doesn't have large increases in the attitude error, as in case 1 and case 2, since it considers both common objects. When one common object has large error due it approaching a unobservable situation the second common object provides this information and avoids any of the errors getting large. Figure 5(b) shows the 3σ bounds for the three cases plotted in log scale. From this figure it is seen that case 3 gives the optimal error for two targets and consistently gives the best estimate as expected, since this case uses both common objects.

Table 1. Orbital Elements of Chief and Orbital Element Differences for Deputy Spacecraft

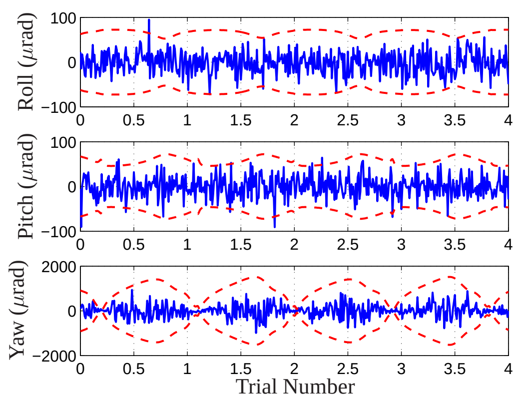
	Chief		Deputy1	Deputy2	Deputy3
a (km)	7555.0	δa (km)	0.0	0.0	0.0
e	0.03	δe	-0.477×10^{-3}	0.477×10^{-3}	-0.191×10^{-3}
i (deg)	48.0	δi (deg)	-0.60	-0.60	0.60
Ω (deg)	20.0	$\delta\Omega$ (deg)	-0.60	0.60	1
ω (Deg)	10.0	$\delta\omega$ (deg)	0	0	0
M (Deg)	0.00	δM (deg)	-3.60	3.60	3.60



(a) **Relative Attitude Estimate Errors Case 1: Using Target 1**



(b) **Relative Attitude Estimate Errors Case 2: Using Target 2**



(c) **Relative Attitude Estimate Errors Case 3: Using both Targets**

Figure 6. Relative Attitude Estimate Errors for the Three Cases

CONCLUSIONS

In this paper a new relative attitude determination approach for two vehicles with multiple common observed objects was presented. The approach requires line-of-sight information between each vehicle and line-of-sight information to multiple common objects. The attitude solution was found means of a geometric constraint applied on each common object. The optimal attitude estimation problem for multiple common object was shown but it was not in Wahba's problem form, so an approximate object function was considered. This solution provides a point-by-point solution for the relative attitude between two vehicles using multiple common objects. The advantage of this approach is that the object's position does not need to be known.

Both a static and a dynamic scenario were considered, and two common objects were used for both scenarios. Three cases were considered for both the static and the dynamic scenarios. Two of the cases used only one common object and the LOS vectors between the vehicles to solve for the relative attitude. The other case used both common objects and the LOS vectors between the vehicles to solve for the relative attitude. For the cases that used both common objects the error was shown to be less than the cases were only one target was used. The attitude error and the attitude error 3σ bounds were shown for all cases and the theoretical bounds match the Monte Carlo runs.

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