RELATIVE ATTITUDE DETERMINATION FROM PLANAR VECTOR OBSERVATIONS

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A method for relative attitude determination from planar line-of-sight observations is presented. Three vehicles are assumed to be equipped with sensors to provide measurements of the three inter-vehicle line-of-sight vectors. The line-of-sight vectors are further assumed to be always in the same plane. This information is combined with all the available measurements to find an optimal relative attitude estimation solution. Covariance analysis is provided to help gain insight on the statistical properties of the attitude errors of the solution.

INTRODUCTION

Formation flying employs multiple vehicles to maintain specific relative attitude/position configurations. Here relative is defined as being between any two vehicles of interest. Applications of formation flying are numerous, involving all types of vehicles, including land (robotics[1]), sea (autonomous underwater vehicles[2]), space (spacecraft formations[3]) and air (uninhabited air vehicles[4]) systems. Relative attitude information is needed to maintain formation attitude through control.

The problem under consideration is to determine the relative attitude of three vehicles from inter-vehicle line-of-sight (LOS) vector measurements.[5] Each vehicle is equipped with optical-type sensors, such as a beacon or laser communication system. Through the sensors a vehicle measures the LOS vector to the two other vehicles, and this applies to each vehicle, making three pairs of LOS vector measurements. Reference 5 shows that one LOS vector between each two vehicles provides sufficient information to determine all three relative attitudes in a three-vehicle formation. Because of the symmetric role of the three relative attitudes, they can be determined individually using a deterministic method from Reference 6, which determines an attitude from a vector and the cosine of angle. Here a deterministic attitude estimation method is defined as (1) using the minimum number of measurements (three for three-axis attitude estimation) and (2) finding the attitude without need to minimize a cost function.

When all the LOS vectors are in the same plane, however, covariance analysis shows that the deterministic attitude estimation method of Reference 5 has an observability problem (to first order). Although the problem can be relieved by not placing the LOS sensors of one vehicle in the same plane as that formed by the sensors of the other two vehicles, the remedy is effective only when the vehicles are large and sufficiently close to each other. A more subtle issue with the deterministic

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method of Reference 5 is that owing to noise in the vector measurements, the solution for the relative attitude may have a two-fold ambiguity and sometimes does not exist.

Reference 7 shows that when the LOS vectors are in the same plane, by imposing this information as a planar or triangle constraint on the LOS vectors, a deterministic relative attitude solution better than that of Reference 5 exists. It uses few measurements, thus minimizing the communications burden, and is more accurate and with no observability or ambiguity issues.[7] While the deterministic method of Reference 7 was originally intended for two-vehicle formations, it can be used in three-vehicle planar formations without modification. Since planar or triangle constraint is valid as long as the size of the vehicles is much smaller than the distances between the vehicles, the deterministic method of Reference 7 is applicable in many scenarios.

While the deterministic method has many advantages, its accuracy is limited by the fact that it does not use all the information contained in the measurements. The main contribution of this paper is to present an optimal relative attitude estimation method for three-vehicle formations with planar LOS vectors, which combines all the information in an optimal manner, including the planar or triangle constraint and the LOS vector measurements. The method of this paper differs from those of References 5 and 7 in that the former is optimization-based and solve an over-determined problem for the attitude while the latter is deterministic and determines the attitude using three independent pieces of information only.

The organization of the remainder of this paper is as follows. First, the relative attitude estimation problem is defined. Second, the methods of References 5 and 7 are reviewed. Then, the optimal relative attitude estimation method is developed. Finally, a numerical example is shown, followed by the conclusions.

RELATIVE ATTITUDE ESTIMATION FOR THREE-VEHICLE FORMATIONS

The geometry of the relative attitude estimation problem is illustrated in Figure 1. There are three

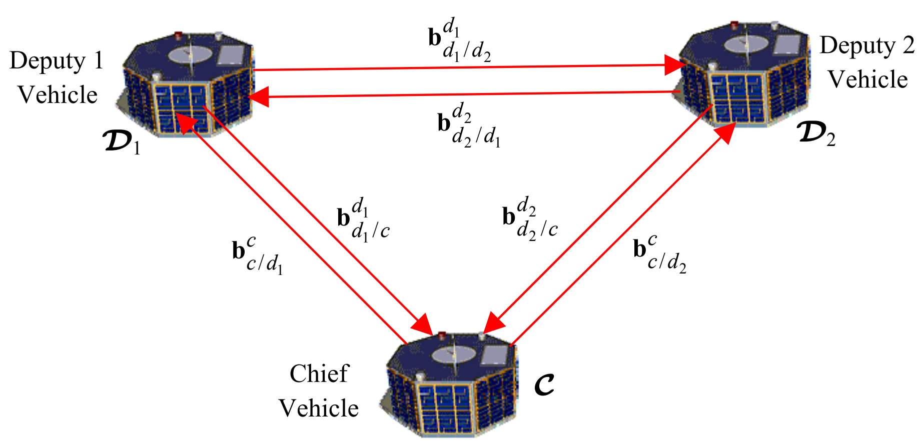

Figure 1. Vehicle Formation

vehicles flying in formation. Each vehicle measures the LOS vector to the two other vehicles. The attitude solutions are obtained using a centralized approach in the sense that information from all vehicles is assumed to be transmitted to one location. Decentralized attitude estimation is discussed in Ref. 8. Also, it is assumed that no communication delays are present in the system.

Because different reference frames are used to represent the various LOS vectors, a structured notation is required here as well. A subscript will describe the vehicle for which the LOS is taken both from and to, while a superscript will denote to which reference frame the LOS is both represented by and measured in. For example, $\mathbf{b}^x_{x/y} = -\mathbf{b}^x_{y/x}$ is a LOS vector beginning at x (using an emitter) and ending at y (using a detector) while it is both expressed in and observed from frame $\mathcal{X}$. Note that from Figure 1, the detectors measure all vectors that point into the vehicles and the emitters are used to generate vectors that point away from the vehicles. For the relative attitude matrix the notation A^y_x denotes the attitude matrix that maps components expressed in $\mathcal{X}$-frame coordinates to components expressed in $\mathcal{Y}$-frame coordinates, so that $\mathbf{b}^x_{x/y} = A^x_y\,\mathbf{b}^y_{x/y}$. The inverse operator is simply $A^{y^T}_x \equiv A^x_y$. The LOS vector equations for each vehicle pair are given by

$$\mathbf{b}^c_{c/d_1} = A^c_{d_1}\mathbf{b}^{d_1}_{c/d_1} \tag{1a}$$

$$\mathbf{b}^c_{c/d_2} = A^c_{d_2}\mathbf{b}^{d_2}_{c/d_2} \tag{1b}$$

$$\mathbf{b}^{d_2}_{d_2/d_1} = A^{d_2}_c A^c_{d_1}\mathbf{b}^{d_1}_{d_2/d_1} = A^{d_2}_{d_1}\mathbf{b}^{d_1}_{d_2/d_1} \tag{1c}$$

The noise-free vectors are denoted by $\mathbf{b}$ with appropriate superscript and subscript. The model in Eq. (1) assumes parallel beams, which can be maintained through hardware calibrations.[5] The six body-vector measurements (denoted by $\tilde{\mathbf{b}}$ with appropriate superscript and subscript) from the onboard sensors, along with their respective error characteristics are given by[8]

$$\tilde{\mathbf{b}}^{d_2}_{d_2/d_1} = \mathbf{b}^{d_2}_{d_2/d_1} + \boldsymbol{\upsilon}^{d_2}_{d_2/d_1}, \quad \boldsymbol{\upsilon}^{d_2}_{d_2/d_1} \sim \mathcal{N}\left(\mathbf{0}, \Omega^{d_2}_{d_2/d_1}\right) \tag{2a}$$

$$\tilde{\mathbf{b}}^{d_1}_{d_2/d_1} = \mathbf{b}^{d_1}_{d_2/d_1} + \boldsymbol{\upsilon}^{d_1}_{d_2/d_1}, \quad \boldsymbol{\upsilon}^{d_1}_{d_2/d_1} \sim \mathcal{N}\left(\mathbf{0}, \Omega^{d_1}_{d_2/d_1}\right) \tag{2b}$$

$$\tilde{\mathbf{b}}^{c}_{c/d_1} = \mathbf{b}^{c}_{c/d_1} + \boldsymbol{\upsilon}^{c}_{c/d_1}, \quad \boldsymbol{\upsilon}^{c}_{c/d_1} \sim \mathcal{N}\left(\mathbf{0}, \Omega^{c}_{c/d_1}\right) \tag{2c}$$

$$\tilde{\mathbf{b}}^{d_1}_{c/d_1} = \mathbf{b}^{d_1}_{c/d_1} + \boldsymbol{\upsilon}^{d_1}_{c/d_1}, \quad \boldsymbol{\upsilon}^{d_1}_{c/d_1} \sim \mathcal{N}\left(\mathbf{0}, \Omega^{d_1}_{c/d_1}\right) \tag{2d}$$

$$\tilde{\mathbf{b}}^{c}_{c/d_2} = \mathbf{b}^{c}_{c/d_2} + \boldsymbol{\upsilon}^{c}_{c/d_2}, \quad \boldsymbol{\upsilon}^{c}_{c/d_2} \sim \mathcal{N}\left(\mathbf{0}, \Omega^{c}_{c/d_2}\right) \tag{2e}$$

$$\tilde{\mathbf{b}}^{d_2}_{c/d_2} = \mathbf{b}^{d_2}_{c/d_2} + \boldsymbol{\upsilon}^{d_2}_{c/d_2}, \quad \boldsymbol{\upsilon}^{d_2}_{c/d_2} \sim \mathcal{N}\left(\mathbf{0}, \Omega^{d_2}_{c/d_2}\right) \tag{2f}$$

where $\mathcal{N}(\mathbf{0}, \Omega)$ denotes a Gaussian probability density with mean zero and covariance Ω. The covariance of a LOS vector is singular (to first order), but a nonsingular covariance can be obtained by a rank-one update.[9] All noise terms (denoted by $\boldsymbol{\upsilon}$ with appropriate superscript and subscript) in Eq. (2) are assumed to be independent.

The objective of the problem is to determine the relative attitudes from the LOS vector measurements. There are only two relative attitudes to be determined for the three-vehicle system. Using the characteristic of the attitude matrix, the third attitude is easily obtainable if two relative attitudes are given. For example, knowing $A^{d_1}_c$ and $A^{d_2}_c$ gives $A^{d_2}_{d_1} = A^{d_2}_c A^{d_1^T}_c$.

Now we will show what equations are used to determine $A^{d_2}_{d_1}$. The other relative attitude can be determined following the same procedure. Taking the dot product of Eq. (1a) and Eq. (1b) gives

$$\mathbf{b}^{c^T}_{c/d_2}\mathbf{b}^c_{c/d_1} = \mathbf{b}^{d_2^T}_{c/d_2} A^{d_2}_{d_1}\mathbf{b}^{d_1}_{c/d_1} \tag{3}$$

Equations (1c) and (3) represent a direction and an angle, respectively, which, combined with the noise model in Eq. (6), can be used to determine $A^{d_2}_{d_1}$. A variant of the method of Reference 6 has been used in Reference 5 to find a deterministic attitude solution satisfying Eqs. (1c) and (3). The method works well except in a special case in which all the LOS vectors are in the same plane. The relative attitude solution becomes highly inaccurate in this case.[5]

While the LOS vectors being in the same plane causes the method of Reference 5 severe accuracy degradation, this fact can be made use of because it contains attitude information.[7] The mathematical description of the special case used in this paper is given by

$$\left(\mathbf{b}^{x}_{c/d_2}\right)^T \mathbf{b}^{x}_{c/d_1} \times \mathbf{b}^{x}_{d_2/d_1} = 0 \tag{4}$$

where the superscript x denotes a frame and the $\times$ symbol denotes the vector or cross product. To explicitly include $A^{d_2}_{d_1}$, the equation can be written as

$$\left(\mathbf{b}^{d_2}_{c/d_2}\right)^T \mathbf{b}^{d_2}_{c/d_1} \times A^{d_2}_{d_1}\mathbf{b}^{d_1}_{d_2/d_1} = 0 \tag{5}$$

DETERMINISTIC RELATIVE ATTITUDE ESTIMATION

In this section, the two deterministic attitude estimation methods in References 5 and 7, respectively, are reviewed and compared.

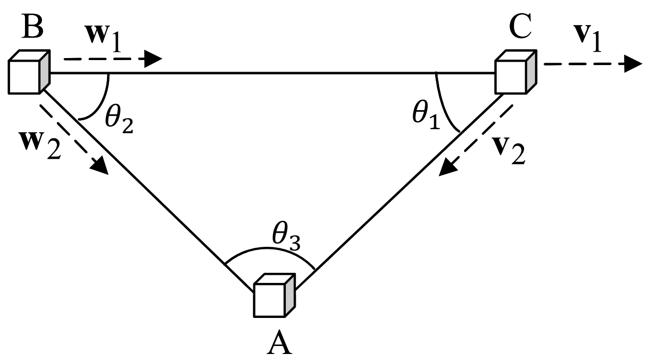

Figure 2. Vectors Used for Attitude Solution

Figure 2 shows the vector configuration (when the attitude matrix A is the identity matrix), where $\mathbf{v}_1$ and $\mathbf{v}_2$ are vector representations in Frame $\mathcal{D}_1$ (C in the figure) and $\mathbf{w}_2$ and $\mathbf{w}_1$ are vector representations in Frame $\mathcal{D}_2$ (B in the figure). The attitude A to be determined is that of Frame $\mathcal{D}_2$ (B in the figure) relative to Frame $\mathcal{D}_1$ (C in the figure). The LOS vectors $\mathbf{w}_2$ and $\mathbf{v}_2$ are nonparallel ($\theta_3 \neq 0$) unless A is infinitely far away from B and C. The three angles are denoted by θ_1, θ_2, and θ_3, respectively. When the LOS vectors are in the same plane and forms a triangle, $\theta_1+\theta_2+\theta_3 = \pi$.

The four measurement models are given by[5]

$$\tilde{\mathbf{w}}_1 = \mathbf{w}_1 + \boldsymbol{\upsilon}_{w1}, \quad \boldsymbol{\upsilon}_{w1} \sim \mathcal{N}\left(\mathbf{0}, R_{w_1}\right) \tag{6a}$$
$$\tilde{\mathbf{w}}_2 = \mathbf{w}_2 + \boldsymbol{\upsilon}_{w2}, \quad \boldsymbol{\upsilon}_{w2} \sim \mathcal{N}\left(\mathbf{0}, R_{w_2}\right) \tag{6b}$$
$$\tilde{\mathbf{v}}_1 = \mathbf{v}_1 + \boldsymbol{\upsilon}_{v1}, \quad \boldsymbol{\upsilon}_{v1} \sim \mathcal{N}\left(\mathbf{0}, R_{v_1}\right) \tag{6c}$$
$$\tilde{\mathbf{v}}_2 = \mathbf{v}_2 + \boldsymbol{\upsilon}_{v2}, \quad \boldsymbol{\upsilon}_{v2} \sim \mathcal{N}\left(\mathbf{0}, R_{v_2}\right) \tag{6d}$$

Since in practice each vehicle will have their own set of LOS measurement devices, the measurements in Eq. (6) can be assumed to be independent. The measurement $\tilde{d}$ on $d = \cos\theta_3$ is provided by A. This measurement used by the method of Reference 5, but not by that of Reference 7. Its model is given by

$$\tilde{d} = d + v_d, \quad v_d \sim \mathcal{N}\left(0, R_d\right) \tag{7}$$

Method of Reference 5

This method is based on the following equations:

$$\mathbf{w}_1 = A\,\mathbf{v}_1 \tag{8a}$$

$$d = \mathbf{w}_2^T A\,\mathbf{v}_2 \tag{8b}$$

The attitude estimate is required to satisfy:

$$\tilde{\mathbf{w}}_1 = \hat{A}\,\tilde{\mathbf{v}}_1 \tag{9a}$$

$$\tilde{d} = \tilde{\mathbf{w}}_2^T \hat{A}\,\tilde{\mathbf{v}}_2 \tag{9b}$$

The solution is summarized by:

$$B = \frac{(\tilde{\mathbf{w}}_1 + \tilde{\mathbf{v}}_1)(\tilde{\mathbf{w}}_1 + \tilde{\mathbf{v}}_1)^T}{(1 + \tilde{\mathbf{v}}_1^T \tilde{\mathbf{w}}_1)} - I_{3\times 3} \tag{10a}$$

$$\mathbf{w}^* = B\tilde{\mathbf{v}}_2 \tag{10b}$$

$$\theta = \tan^{-1}(\tilde{\mathbf{w}}_2^T[\tilde{\mathbf{w}}_1\times]\mathbf{w}^*, \tilde{\mathbf{w}}_2^T[\tilde{\mathbf{w}}_1\times]^2\mathbf{w}^*) + \cos^{-1}\left[\frac{\left(\tilde{\mathbf{w}}_2^T\tilde{\mathbf{w}}_1\right)\left(\tilde{\mathbf{w}}_1^T\mathbf{w}^*\right) - \tilde{d}}{||\tilde{\mathbf{w}}_2 \times \tilde{\mathbf{w}}_1||\,||\tilde{\mathbf{w}}_1 \times \mathbf{w}^*||}\right] \tag{10c}$$

$$R\left(\tilde{\mathbf{w}}_1, \theta\right) = \cos(\theta) I_{3\times 3} + (1 - \cos(\theta))\tilde{\mathbf{w}}_1\mathbf{w}_1^T - \sin(\theta)[\tilde{\mathbf{w}}_1\times] \tag{10d}$$

$$\hat{A} = R\left(\tilde{\mathbf{w}}_1, \theta\right) B \tag{10e}$$

where $\tan^{-1}$ is the four-quadrant inverse tangent function, $I_{3\times 3}$ is the 3×3 identity matrix, and the matrix $[\tilde{\mathbf{w}}_1\times]$ is the cross product matrix. The definition of this matrix for a general 3×1 vector $\boldsymbol{\alpha}$ is given by

$$[\boldsymbol{\alpha}\times] \equiv \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} \tag{11}$$

Define

$$F = \frac{\left(\tilde{\mathbf{w}}_2^T\tilde{\mathbf{w}}_1\right)\left(\tilde{\mathbf{w}}_1^T\mathbf{w}^*\right) - \tilde{d}}{||\tilde{\mathbf{w}}_2 \times \tilde{\mathbf{w}}_1||\,||\tilde{\mathbf{w}}_1 \times \mathbf{w}^*||} \tag{12}$$

There is no solution, one solution, or two solutions for θ (mod 2π) when $|F| > 1$, $|F| = 1$, or $|F| < 1$, respectively. Additional information is needed to resolve the ambiguity when the solution is not unique.

Noting that

$$\tilde{\mathbf{w}}_1 = B\tilde{\mathbf{v}}_1, \quad \mathbf{w}^* = B\tilde{\mathbf{v}}_2 \tag{13}$$

and examining Figure 2, we have

$$||\tilde{\mathbf{w}}_1 \times \mathbf{w}^*|| = ||\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{v}}_2|| = \sin\tilde{\theta}_1 \tag{14}$$

$$\tilde{\mathbf{w}}_1^T \mathbf{w}^* = \tilde{\mathbf{v}}_1^T \tilde{\mathbf{v}}_2 = -\cos\tilde{\theta}_1 \tag{15}$$

$$||\tilde{\mathbf{w}}_2 \times \tilde{\mathbf{w}}_1|| = \sin\tilde{\theta}_2 \tag{16}$$

$$\tilde{\mathbf{w}}_2^T \tilde{\mathbf{w}}_1 = \cos\tilde{\theta}_2 \tag{17}$$

where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ denotes the angles calculated from the noisy LOS vector measurements. Then, F can be rewritten as

$$F = \frac{-\cos\tilde{\theta}_1 \cos\tilde{\theta}_2 - \tilde{d}}{\sin\tilde{\theta}_1 \sin\tilde{\theta}_2} \tag{18}$$

Note that F is independent of the attitude and is fully determined by the LOS vector measurements. For the triangle configuration in Figure 2, $F = -1$ when $\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3 = \pi$. The solution is unique and the estimated LOS vectors $\tilde{\mathbf{w}}_1$, $\tilde{\mathbf{w}}_2$, and $\hat{A}\tilde{\mathbf{v}}_2$ are in the same plane. When $F < -1$, $\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3 < \pi$, which is possible in the presence of noise. Obviously, there is no solution in this case. When $\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3 > \pi$, there are two solutions. Both solutions ensure that $\tilde{d} = \tilde{\mathbf{w}}_2^T \hat{A}\, \tilde{\mathbf{v}}_2$ holds, but the estimated LOS vectors $\tilde{\mathbf{w}}_1$, $\tilde{\mathbf{w}}_2$, and $\hat{A}\tilde{\mathbf{v}}_2$ will not be in the same plane.

Method of Reference 7

This method is based on the following equations:

$$\mathbf{w}_1 = A\,\mathbf{v}_1 \tag{19a}$$

$$d = \mathbf{w}_2^T \mathbf{w}_1 \mathbf{v}_1^T \mathbf{v}_2 + \|\mathbf{w}_1 \times \mathbf{w}_2\| \|\mathbf{v}_1 \times \mathbf{v}_2\| = \mathbf{w}_2^T A\, \mathbf{v}_2 \tag{19b}$$

Note that the second equation is the triangle constraint,[7] which is equivalent to the planar constraint, given by[7]

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times] A \mathbf{v}_2 \tag{20}$$

The attitude estimate is required to satisfy:

$$\tilde{\mathbf{w}}_1 = \hat{A}\, \tilde{\mathbf{v}}_1 \tag{21a}$$

$$\tilde{\mathbf{w}}_2^T \tilde{\mathbf{w}}_1 \tilde{\mathbf{v}}_1^T \tilde{\mathbf{v}}_2 + \|\tilde{\mathbf{w}}_1 \times \tilde{\mathbf{w}}_2\| \|\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{v}}_2\| = \tilde{\mathbf{w}}_2^T \hat{A}\, \tilde{\mathbf{v}}_2 \tag{21b}$$

The solution is summarized by:

$$B = \frac{(\tilde{\mathbf{w}}_1 + \tilde{\mathbf{v}}_1)(\tilde{\mathbf{w}}_1 + \tilde{\mathbf{v}}_1)^T}{(1 + \tilde{\mathbf{v}}_1^T \tilde{\mathbf{w}}_1)} - I_{3\times3} \tag{22a}$$

$$\mathbf{w}^* = B \tilde{\mathbf{v}}_2 \tag{22b}$$

$$\theta = \tan^{-1}(\tilde{\mathbf{w}}_2^T [\tilde{\mathbf{w}}_1 \times] \mathbf{w}^*, \tilde{\mathbf{w}}_2^T [\tilde{\mathbf{w}}_1 \times]^2 \mathbf{w}^*) + \pi \tag{22c}$$

$$R\,(\tilde{\mathbf{w}}_1, \theta) = \cos(\theta) I_{3\times3} + (1 - \cos(\theta)) \tilde{\mathbf{w}}_1 \mathbf{w}_1^T - \sin(\theta)[\tilde{\mathbf{w}}_1 \times] \tag{22d}$$

$$\hat{A} = R\,(\tilde{\mathbf{w}}_1, \theta)\, B \tag{22e}$$

Comparisons of the Deterministic Methods

1. Both attitude estimates satisfy $\tilde{\mathbf{w}}_1 = \hat{A}\tilde{\mathbf{v}}_1$ exactly.

2. Both methods calculate the attitude using $\hat{A} = R\left(\tilde{\mathbf{w}}_1, \theta\right) B$. However, they differ in how θ is calculated. The method of Reference 5 calculates θ as a function of the angle measurement $\tilde{d}$, but the method of Reference 7 does not use the angle measurement $\tilde{d}$. In the presence of measurement noise, the method of Reference 5 may have one solution, two solutions, or no solution for θ, but the method of Reference 7 always has a unique solution.

3. The method of Reference 5 is inaccurate when the LOS vectors are in the same plane. In contrast, the accuracy of the method of Reference 5 is the highest when the LOS vectors are in the same plane and degrades when they are not.

4. The method of Reference 5 ensures that $\tilde{d} = \tilde{\mathbf{w}}_2^T \hat{A}\, \tilde{\mathbf{v}}_2$ holds exactly, but the vectors $\tilde{\mathbf{w}}_1$, $\tilde{\mathbf{w}}_2$, and $\hat{A}\tilde{\mathbf{v}}_2$ are in generally not in the same plane. Equivalently, the vectors $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$, and $\hat{A}^T\tilde{\mathbf{w}}_2$ are in general not in the same plane. The method of Reference 7 ensures that $\tilde{d} = \tilde{\mathbf{w}}_2^T \hat{A}\, \tilde{\mathbf{v}}_2$, or equivalently the vectors $\tilde{\mathbf{w}}_1$, $\tilde{\mathbf{w}}_2$, and $\hat{A}\tilde{\mathbf{v}}_2$ (or $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$, and $\hat{A}^T\tilde{\mathbf{w}}_2$) are in the same plane, but the cosine of the angle between $\tilde{\mathbf{w}}_2$ and $\hat{A}\tilde{\mathbf{v}}_2$ in general does not equal $\tilde{d}$.

OPTIMAL RELATIVE ATTITUDE ESTIMATION FROM PLANAR VECTORS

Wen the LOS vectors are in the same plane, the method of Reference 7 is superior to the method of Reference 5, but it is not optimal because it does not use the information about the attitude contained in the scalar measurement $\tilde{d}$, regardless of the accuracy of scalar measurement.

Optimal Attitude Estimation

The optimal attitude can be found by solving a new attitude estimation problem, which will be defined in this section. When the LOS vectors are in the same plane, the true attitude satisfies the following equations:

$$\mathbf{w}_1 = A\,\mathbf{v}_1 \tag{23a}$$

$$d = \mathbf{w}_2^T A\,\mathbf{v}_2 \tag{23b}$$

$$\mathbf{w}_2^T \mathbf{w}_1 \times A\mathbf{v}_2 = 0 \tag{23c}$$

One may be tempted to require that the optimal attitude estimate $\hat{A}$ satisfy

$$\tilde{\mathbf{w}}_1 = \hat{A}\,\tilde{\mathbf{v}}_1 \tag{24a}$$

$$\tilde{d} = \tilde{\mathbf{w}}_2^T \hat{A}\,\tilde{\mathbf{v}}_2 \tag{24b}$$

$$\tilde{\mathbf{w}}_2^T \tilde{\mathbf{w}}_1 \times \hat{A}\tilde{\mathbf{v}}_2 = 0 \tag{24c}$$

However, the above three equations represent four independent pieces of information on the attitude matrix, which only has three degrees of freedom. Therefore, the attitude determination problem is over-determined and there is in general no attitude matrix that can simultaneously satisfy all the three equations exactly. For example, given $\tilde{\mathbf{w}}_1 = \hat{A}\,\tilde{\mathbf{v}}_1,$ the other two equations cannot be satisfied simultaneously. The attitude solution of Reference 5 satisfies Eq. (24b) and that of Reference 7 satisfies Eq. (24c). The optimal attitude should be obtained by solving a minimization problem. The following quadratic cost function is chosen to be minimized:

$$J(A) = \frac{1}{2}\boldsymbol{\Delta}^T W \boldsymbol{\Delta} \tag{25}$$

subject to

$$A^T A = AA^T = I_{3\times 3}, \quad \det A = 1 \tag{26}$$

where

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}(A) = \begin{bmatrix} \tilde{\mathbf{w}}_1 - A\tilde{\mathbf{v}}_1 \\ \tilde{d} - \tilde{\mathbf{w}}_2^T A\tilde{\mathbf{v}}_2 \\ (\tilde{\mathbf{w}}_1 \times \tilde{\mathbf{w}}_2)^T A\tilde{\mathbf{v}}_2 \end{bmatrix} \tag{27}$$

and the weighting matrix $W = \mathcal{R}^{-1}$, with $\mathcal{R}$ the covariance matrix of $\boldsymbol{\Delta}$. In addition, we define

$$\boldsymbol{\Delta}_1 = \tilde{\mathbf{w}}_1 - A\tilde{\mathbf{v}}_1 \tag{28a}$$

$$\Delta_2 = \tilde{d} - \tilde{\mathbf{w}}_2^T A\tilde{\mathbf{v}}_2 \tag{28b}$$

$$\Delta_3 = (\tilde{\mathbf{w}}_1 \times \tilde{\mathbf{w}}_2)^T A\tilde{\mathbf{v}}_2 \tag{28c}$$

Note that the optimal solution is optimal in the sense that it minimizes the cost function. It may not be the maximum likelihood estimate, however, because the quadratic cost is just an approximation to the true negative log-likelihood function, which is non-Gaussian.

There are two equivalent forms of the planar or triangle constraint, respectively given by

$$\mathbf{w}_2^T \mathbf{w}_1 \mathbf{v}_1^T \mathbf{v}_2 + \|\mathbf{w}_1 \times \mathbf{w}_2\| \|\mathbf{v}_1 \times \mathbf{v}_2\| = \mathbf{w}_2^T A\, \mathbf{v}_2 \tag{29a}$$

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times] A\mathbf{v}_2 \tag{29b}$$

with the former the "cosine" form and the latter the "sine" form.[7] Thus, an alternative to the cost function is to replace Δ_3 with

$$\Delta_3' = \tilde{\mathbf{w}}_2^T \tilde{\mathbf{w}}_1 \tilde{\mathbf{v}}_1^T \tilde{\mathbf{v}}_2 + \|\tilde{\mathbf{w}}_1 \times \tilde{\mathbf{w}}_2\| \|\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{v}}_2\| - \tilde{\mathbf{w}}_2^T A\, \tilde{\mathbf{v}}_2 \tag{30}$$

The weighting matrix will then be the inverse of the covariance matrix of $\boldsymbol{\Delta}' = [\boldsymbol{\Delta}_1^T, \Delta_2, \Delta_3']^T$. This alternative form is not used, however because (1) the covariance matrix of $\boldsymbol{\Delta}'$ is more complex to derive and (2) the attitude error covariance matrix is singular owing to the rank deficiency of the sensitivity matrix,[7] which indicates that the "sine" form of the constraint may be inappropriate for the covariance analysis based on the quadratic cost.

Covariance Matrix of the Effective Measurement Noise

Substituting Eqs. (6),(7), (23) into Eqs. (24) and ignoring second and higher order terms gives

$$\boldsymbol{\Delta}_1 = -A\boldsymbol{\upsilon}_{v1} + \boldsymbol{\upsilon}_{w1} \tag{31a}$$

$$\Delta_2 = -\mathbf{w}_2^T A\boldsymbol{\upsilon}_{v2} - \mathbf{v}_2^T A^T \boldsymbol{\upsilon}_{w2} + \upsilon_d \tag{31b}$$

$$\Delta_3 = (\mathbf{w}_1 \times \mathbf{w}_2)^T A\boldsymbol{\upsilon}_{v2} + (\mathbf{w}_2 \times A\mathbf{v}_2)^T \boldsymbol{\upsilon}_{w1} + [(A\mathbf{v}_2) \times \mathbf{w}_1]^T \boldsymbol{\upsilon}_{w2} \tag{31c}$$

To first order approximation, $E\{\boldsymbol{\Delta}\} = \mathbf{0}$ and the covariance of $\boldsymbol{\Delta}$ is defined as

$$\mathcal{R} = \begin{bmatrix} R_{\Delta_1} & R_{\Delta_1\Delta_2} & R_{\Delta_1\Delta_3} \\ R_{\Delta_1\Delta_2}^T & R_{\Delta_2} & R_{\Delta_2\Delta_3} \\ R_{\Delta_1\Delta_3}^T & R_{\Delta_2\Delta_3}^T & R_{\Delta_3} \end{bmatrix} \tag{32}$$

where

$$R_{\Delta_1} = E\{\boldsymbol{\Delta}_1 \boldsymbol{\Delta}_1^T\} = R_{w_1} + A R_{v_1} A^T \tag{33}$$

$$R_{\Delta_2} = E\{\Delta_2^2\} = \mathbf{w}_2^T A R_{v_2} A^T \mathbf{w}_2 + \mathbf{v}_2^T A R_{w_2} A^T \mathbf{v}_2 + R_d \tag{34}$$

$$\begin{aligned} R_{\Delta_3} = {} & (\mathbf{w}_1 \times \mathbf{w}_2)^T A R_{v_2} A^T (\mathbf{w}_1 \times \mathbf{w}_2) + (\mathbf{w}_2 \times A\mathbf{v}_2)^T R_{w_1} (\mathbf{w}_2 \times A\mathbf{v}_2) \\ & + \left[(A\mathbf{v}_2) \times \mathbf{w}_1\right]^T R_{w_2} \left[(A\mathbf{v}_2) \times \mathbf{w}_1\right] \end{aligned} \tag{35}$$

$$R_{\Delta_1 \Delta_2} = E\{\boldsymbol{\Delta}_1 \Delta_2\} = \mathbf{0} \tag{36}$$

$$R_{\Delta_1 \Delta_3} = E\{\boldsymbol{\Delta}_1 \Delta_3\} = -R_{w_1} \left[(A\mathbf{v}_2) \times \mathbf{w}_2\right] \tag{37}$$

$$R_{\Delta_2 \Delta_3} = E\{\Delta_2 \Delta_3\} = -\mathbf{w}_2^T A R_{v_2} A^T (\mathbf{w}_1 \times \mathbf{w}_2) - (A\mathbf{v}_2)^T R_{w_2} [(A\mathbf{v}_2) \times \mathbf{w}_1] \tag{38}$$

All expressions but $R_{\Delta_2 \Delta_3}$ have been derived in References 5 and 7.

Nonlinear Least-Squares Solution

A nonlinear least-squares solution is used to find the optimal attitude estimate. The attitude matrix is parameterized by the error-angle vector $\boldsymbol{\delta\alpha}$ taking the estimated attitude to the true attitude given by

$$A = e^{-[\boldsymbol{\delta\alpha}\times]} \hat{A} \approx \left(I_{3\times 3} - [\boldsymbol{\delta\alpha}\times]\right) \hat{A} \tag{39}$$

The nonlinear least-squares method is iterative. The initial guess of $\hat{A}$ is obtained using the deterministic method of Reference 7. Given the current attitude estimate $\hat{A}$, the estimated attitude error-angle vector $\widehat{\boldsymbol{\delta\alpha}}$ is calculated as

$$\widehat{\boldsymbol{\delta\alpha}} = \left[H^T(\hat{A}) \mathcal{R}^{-1} H(\hat{A})\right]^{-1} H^T(\hat{A}) \mathcal{R}^{-1} \boldsymbol{\Delta}(\hat{A}) \tag{40}$$

where the sensitivity matrix $H(\hat{A})$ is given by

$$H(\hat{A}) = -\left.\frac{\partial \Delta(\hat{A})}{\partial \boldsymbol{\delta\alpha}^T}\right|_{\boldsymbol{\delta\alpha}=\mathbf{0}} = \begin{bmatrix} \left[\hat{A}\tilde{\mathbf{v}}_1 \times\right] \\ \tilde{\mathbf{w}}_2^T \left[\hat{A}\tilde{\mathbf{v}}_2 \times\right] \\ -\left(\tilde{\mathbf{w}}_1 \times \tilde{\mathbf{w}}_2\right)^T \left[\hat{A}\tilde{\mathbf{v}}_2 \times\right] \end{bmatrix} \tag{41}$$

Then, a new attitude estimate is calculated as $\exp(-[\widehat{\boldsymbol{\delta\alpha}}\times])\hat{A}$, which will be used as the "current" estimate in the next iteration. This procedure repeats until $\widehat{\boldsymbol{\delta\alpha}}$ is sufficiently small.

The attitude error covariance, defined as

$$P_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} = E\left\{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}^T\right\} \tag{42}$$

is approximately given by

$$P_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} \approx \left[H^T(\hat{A}) \mathcal{R}^{-1} H(\hat{A})\right]^{-1} \tag{43}$$

Relation with the Deterministic Methods

When only $\mathbf{\Delta}_1$ and Δ_3 are used to determine the attitude, the cost given by Eq. (25) reduces to

$$J = \frac{1}{2} \begin{bmatrix} \mathbf{\Delta}_1 \\ \Delta_3 \end{bmatrix}^T \begin{bmatrix} R_{\Delta_1} & R_{\Delta_1 \Delta_3} \\ R^T_{\Delta_1 \Delta_3} & R_{\Delta_3} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{\Delta}_1 \\ \Delta_3 \end{bmatrix} \tag{44}$$

The deterministic method of Reference 7 solves this minimization problem and the corresponding minimum cost is zero. Any positive definite $\begin{bmatrix} R_{\Delta_1} & R_{\Delta_1 \Delta_3} \\ R^T_{\Delta_1 \Delta_3} & R_{\Delta_3} \end{bmatrix}^{-1}$ leads to the same attitude solution.

When only $\mathbf{\Delta}_1$ and Δ_2 are used to determine the attitude, the cost given by Eq. (25) reduces to

$$J = \frac{1}{2} \begin{bmatrix} \mathbf{\Delta}_1 \\ \Delta_2 \end{bmatrix}^T \begin{bmatrix} R^{-1}_{\Delta_1} & \mathbf{0} \\ \mathbf{0}^T & R^{-1}_{\Delta_2} \end{bmatrix} \begin{bmatrix} \mathbf{\Delta}_1 \\ \Delta_2 \end{bmatrix} \tag{45}$$

The deterministic method of Reference 5 solves this minimization problem when it has one or two solutions and the corresponding minimum cost is zero. Any positive definite $\begin{bmatrix} R^{-1}_{\Delta_1} & \mathbf{0} \\ \mathbf{0}^T & R^{-1}_{\Delta_2} \end{bmatrix}$ leads to the same attitude solution. When the deterministic method has no solution, however, which occurs when $\tilde{\theta}_1 + \tilde{\theta}_2 + \cos^{-1}(\tilde{d}) < \pi$, there is still a solution to the minimization problem, but it will satisfy neither $\mathbf{\Delta}_1 = \tilde{\mathbf{w}}_1 - \hat{A}\tilde{\mathbf{v}}_1 = 0$ nor $\Delta_2 = \tilde{d} - \tilde{\mathbf{w}}_2^T A \tilde{\mathbf{v}}_2 = 0$ exactly. The corresponding cost is strictly positive.

NUMERICAL EXAMPLE

The true attitude matrix is assumed to be the identity matrix. The true LOS vectors are in the x-y plane, given by:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -\cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \tag{46}$$

with $\alpha = \pi/3$. The angles between the LOS vectors are $\theta_1 = \theta_2 = \theta_3 = \pi/3$. The configuration is symmetric. All LOS vector measurement noise covariance matrices are assumed to be $\sigma^2(I_{3\times3} - \mathbf{b}\mathbf{b}^T)$, with $\mathbf{b}$ the true LOS direction and $\sigma = 0.001$. The LOS vector measurements are generated by first perturbing the true LOS vectors with zero-mean Gaussian noise of covariance $\sigma^2 I_{3\times3}$ and then normalizing the perturbed vectors. The scalar measurements $\tilde{d}$ are generated using $\tilde{d} = \tilde{\mathbf{w}}_2'^T \tilde{\mathbf{v}}_2'$, where $\tilde{\mathbf{w}}_2'$ and $\tilde{\mathbf{v}}_2'$ are two unit vectors generated from $\tilde{\mathbf{w}}_2$ and $\tilde{\mathbf{v}}_2$ using the same perturbing method, but the measurement noise used to generate $\tilde{\mathbf{w}}_2'$ and $\tilde{\mathbf{v}}_2'$ are independent of that used to generate $\tilde{\mathbf{w}}_2$ and $\tilde{\mathbf{v}}_2$.

The optimal attitude estimator is tested with 1,000,000 sets of LOS and scalar measurements. It is initialized with the deterministic solution of Reference 7. The termination criterion is $\|\boldsymbol{\delta\alpha}\| < 0.001\sigma$, which is found to be satisfied within three iterations. The sample means and covariance matrices for the deterministic attitude estimator and the optimal method are calculated. They are

$$\widehat{\boldsymbol{\delta\alpha}}^{\text{deterministic}} = 1 \times 10^{-3} \begin{bmatrix} -0.5321 \\ 0.9213 \\ -0.8141 \end{bmatrix} \tag{47a}$$

$$\widehat{\boldsymbol{\delta\alpha}}^{\text{optimal}} = 1 \times 10^{-3} \begin{bmatrix} -0.5320 \\ 0.9214 \\ -0.5428 \end{bmatrix} \tag{47b}$$

$$\hat{P}^{\text{deterministic}}_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.3009 & 0.0579 & 0.0000 \\ 0.0579 & 0.1003 & 0.0000 \\ 0.0000 & 0.0000 & 0.1004 \end{bmatrix} \tag{48a}$$

$$\hat{P}^{\text{optimal}}_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.3010 & 0.0579 & 0.0000 \\ 0.0579 & 0.1003 & 0.0000 \\ 0.0000 & 0.0000 & 0.0888 \end{bmatrix} \tag{48b}$$

The mean square error matrices from the samples, defined by $\hat{M}_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} = \hat{P}_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} + \widehat{\boldsymbol{\delta\alpha}}\widehat{\boldsymbol{\delta\alpha}}^T$, are also calculated. They are

$$\hat{M}^{\text{deterministic}}_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.3293 & 0.0089 & 0.0433 \\ 0.0089 & 0.1852 & -0.0750 \\ 0.0433 & -0.0750 & 0.1667 \end{bmatrix} \tag{49a}$$

$$\hat{M}^{\text{optimal}}_{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.3293 & 0.0089 & 0.0289 \\ 0.0089 & 0.1852 & -0.0500 \\ 0.0289 & -0.0500 & 0.1183 \end{bmatrix} \tag{49b}$$

The result shows that the optimal solution has the same attitude accuracy as the deterministic solution in the x and y directions but reduces the root-mean-square error in the z direction by 15%.

Now the theoretical covariance of the attitude solution is shown for this example. Under first-order approximation, the theoretical mean of the attitude error is zero and the mean square error matrix is identical to the covariance matrix. The nonsingular covariance matrix of $\boldsymbol{\Delta}$ is

$$\mathcal{R} = \sigma^2 \begin{bmatrix} 2I_{3\times3} & 0_{3\times1} & \begin{bmatrix} 0 \\ 0 \\ \sin(2\alpha) \end{bmatrix} \\ 0_{1\times3} & 4[1-\cos^2(2\alpha)] & 0 \\ \begin{bmatrix} 0 & 0 & \sin(2\alpha) \end{bmatrix} & 0 & 2\sin^2\alpha + \sin^2(2\alpha) \end{bmatrix} \tag{50}$$

where $2I_{3\times3}$ replaces $2(I_{3\times3} - \mathbf{w}_1\mathbf{w}_1^T)$ based on the rank-one update. The sensitivity matrix evaluated at the true attitude is

$$H = H(I_{3\times3}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \sin(2\alpha) \\ \sin^2\alpha & \frac{1}{2}\sin(2\alpha) & 0 \end{bmatrix} \tag{51}$$

Hence, for $\alpha = \pi/3$, the covariance for the optimal attitude solution using the approximation given by Eq. (43) is

$$\begin{aligned} P^{\text{optimal}}_{\boldsymbol{\delta\alpha\delta\alpha}} &= \left(H^T \mathcal{R}^{-1} H\right)^{-1} = \sigma^2 \begin{bmatrix} \frac{4}{\sin^2\alpha} - 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} \frac{10}{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \approx 1 \times 10^{-5} \begin{bmatrix} 0.3333 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1333 \end{bmatrix} \end{aligned} \tag{52}$$

Similarly, the covariance for the deterministic attitude solution is

$$\begin{aligned} P^{\text{deterministic}}_{\boldsymbol{\delta\alpha\delta\alpha}} &= \sigma^2 \begin{bmatrix} \frac{4}{\sin^2\alpha} - 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} \frac{10}{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \approx 1 \times 10^{-5} \begin{bmatrix} 0.3333 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \end{aligned} \tag{53}$$

The theoretical covariance matrices are in better agreement with the sample mean square error matrix than with the sample covariance matrix. The theoretical results also show that the optimal solution improves the accuracy in the z direction only. The slight inconsistency between the theoretical and numerical results is most likely due to the first-order approximation in the covariance analysis and the errors in the Monte Carlo simulation.

Finally, the optimal solution and the deterministic solution are compared in an asymmetric vector configuration. The true LOS vectors remain in the x-y plane, given by:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -\cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \tag{54}$$

with $\alpha = \pi/4$. The angles between the LOS vectors are $\theta_1 = \pi/4$, $\theta_2 = \pi/2$, $\theta_3 = \pi/4$. The other conditions are the same. The mean, covariance, and mean square error calculated from the samples are:

$$\widehat{\boldsymbol{\delta\alpha}}^{\text{deterministic}} = 1 \times 10^{-3} \begin{bmatrix} 0.2135 \\ -0.2147 \\ -1.1769 \end{bmatrix} \tag{55a}$$

$$\widehat{\boldsymbol{\delta\alpha}}^{\text{optimal}} = 1 \times 10^{-3} \begin{bmatrix} 0.2135 \\ -0.2148 \\ -0.7835 \end{bmatrix} \tag{55b}$$

$$\hat{P}^{\text{deterministic}}_{\boldsymbol{\delta\alpha\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.2998 & -0.0001 & -0.0001 \\ -0.0001 & 0.0999 & -0.0001 \\ -0.0001 & -0.0001 & 0.0999 \end{bmatrix} \tag{56a}$$

$$\hat{P}^{\text{optimal}}_{\boldsymbol{\delta\alpha\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.2998 & -0.0002 & -0.0000 \\ -0.0002 & 0.0999 & -0.0001 \\ -0.0000 & -0.0001 & 0.0887 \end{bmatrix} \tag{56b}$$

$$\hat{M}^{\text{deterministic}}_{\boldsymbol{\delta\alpha\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.3044 & -0.0047 & -0.0252 \\ -0.0047 & 0.1045 & 0.0251 \\ -0.0252 & 0.0251 & 0.2385 \end{bmatrix} \tag{57a}$$

$$\hat{M}^{\text{optimal}}_{\boldsymbol{\delta\alpha\delta\alpha}} = 1 \times 10^{-5} \begin{bmatrix} 0.3044 & -0.0048 & -0.0168 \\ -0.0048 & 0.1045 & 0.0168 \\ -0.0168 & 0.0168 & 0.1501 \end{bmatrix} \tag{57b}$$

The theoretical results are:

$$\mathcal{R} = \sigma^2 \begin{bmatrix} 2I_{3\times3} & 0_{3\times1} & \begin{bmatrix} 0 \\ 0 \\ \cos\alpha \end{bmatrix} \\ 0_{1\times3} & 4\cos^2\alpha & 0 \\ \begin{bmatrix} 0 & 0 & \cos\alpha \end{bmatrix} & 0 & 2 \end{bmatrix} \tag{58}$$

$$H = H(I_{3\times3}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \\ \sin\alpha & \cos\alpha & 0 \end{bmatrix} \tag{59}$$

$$\begin{aligned} P^{\text{deterministic}}_{\boldsymbol{\delta\alpha\delta\alpha}} &= \sigma^2 \begin{bmatrix} \frac{2}{\sin^2\alpha} & -\cot\alpha & 0 \\ -\cot\alpha & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 1 \times 10^{-5} \begin{bmatrix} 0.4 & -0.1 & 0 \\ -0.1 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \end{aligned} \tag{60}$$

$$\begin{aligned} P^{\text{optimal}}_{\boldsymbol{\delta\alpha\delta\alpha}} &= \sigma^2 \begin{bmatrix} \frac{2}{\sin^2\alpha} & -\cot\alpha & 0 \\ -\cot\alpha & 2 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \approx 1 \times 10^{-5} \begin{bmatrix} 0.4 & -0.1 & 0 \\ -0.1 & 0.2 & 0 \\ 0 & 0 & 0.1333 \end{bmatrix} \end{aligned} \tag{61}$$

Similar conclusions on the contribution of the scalar measurement to the accuracy of the attitude solution can be drawn. Finally, we point out that the attitude accuracy of the optimal and deterministic solutions in the z direction, the normal of the plane formed by the three LOS vectors, is independent of the vector configuration.

CONCLUSIONS

An optimization-based method for relative attitude estimation using planar LOS vector measurements between three vehicles was developed. The LOS vectors are planar when the size of the vehicles is small compared to the distances between them. The optimization-based method uses all the attitude information contained in the planar constraint and the LOS measurements and is more accurate than the deterministic methods, which only use the minimum number of measurements.

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