Adaptive Disturbance Accommodating Controller for Nonlinear Stochastic Systems

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***Abstract*— This paper presents the the formulation of an adaptive disturbance accommodating controller for nonlinear stochastic systems with unknown nonlinearities and external disturbances. The disturbance accommodating control scheme presented here utilizes an estimator to determine the corrections to the nominal control input required to minimizes the effects of both unknown system nonlinearities and external disturbances on the controlled system. Similar to the disturbance accommodating controller for linear systems, performance of the nonlinear disturbance accommodating controller depends on the accuracy of the assumed disturbance model. The proposed formulation indicates that an adaptive law can be developed for updating the process noise covariance matrix associated with the disturbance model online so that the controlled system is stable.**

I. INTRODUCTION

The adaptive disturbance accommodating control scheme presented here combines two different techniques in nonlinear control: disturbance rejection and complexity mitigation. The problem of rejecting unwanted disturbances occurring in dynamical systems is a fundamental problem in control theory, with numerous technological applications in control of vibration, active noise control, and control of rotating mechanisms. There are numerous papers in the literature on the subject of disturbance rejection for nonlinear systems [1]–[4], but all of them assume perfect knowledge of the nonlinear system and/or the disturbance model. There is very little research about the rejection of stochastic disturbances in nonlinear systems. Complexity mitigation or complexity reduction involves forcing the original nonlinear system to behave like a chosen reduced complexity model by treating the effects of the original complexity-related terms as disturbances acting on the reduced-complexity system [5]. This paper presents a robust control approach where the unknown complex nonlinearities and external disturbances are lumped into a *disturbance term*, and utilizing a Kalman estimator the disturbance term is estimated in real time. The estimated disturbance term is further used as a signal synthesis adaptive correction to the nominal control input to achieve maximum performance.

The main disadvantage of the Kalman filter-based disturbance accommodating control is that the accuracy of the estimated disturbance term depends on how well one can model the model error dynamics or the dynamics of the disturbance term. The process noise covariance used in the estimator is a quantitative measure on the accuracy of the assumed disturbance term dynamics. Therefore, as shown in [6], [7], the closed-loop performance of the disturbance accommodating control system depends on the estimator parameters, such as the process noise covariance. In [6], an adaptive law is synthesized for selecting a process noise covariance so that the controlled linear system is stable. In this paper, it is shown that a similar adaptive law for the process noise can be developed based on the supermartingale theory so that the controlled nonlinear system is stable. The formulation of the nonlinear disturbance accommodating controller is given next. Afterwards, an adaptive law is synthesized for the selection of an appropriate process noise covariance associated with the disturbance term dynamics so that the controlled system is stable. Finally, simulation results and conclusion are presented.

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II. CONTROLLER FORMULATION

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ denote a complete filtered probability space, where $\mathcal{F}$ is a σ-field, $\{\mathcal{F}_t\}_{t\geq t_0}$ is a collection of sub-σ-fields called a filtration and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$. Additionally, the elements of Ω are denoted by ω and the members of $\mathcal{F}$ are called events. Now consider a nonlinear system of the following form:

$$\begin{aligned} d\mathbf{X}_1(t) &= \big\{A_1\mathbf{X}_1(t) + A_2\mathbf{X}_2(t)\big\}dt, \quad \mathbf{X}_1(t_0) = \mathbf{x}_{1_0} \\ d\mathbf{X}_2(t) &= \big\{A_3\mathbf{X}_1(t) + A_4\mathbf{X}_2(t) + B\mathbf{u}(t) + \\ &\qquad \mathbf{g}\left(\mathbf{X}(t), \mathbf{u}(t), \mathbf{W}(t)\right)\big\}dt, \quad \mathbf{X}_2(t_0) = \mathbf{x}_{2_0} \end{aligned} \tag{1}$$

Here, the stochastic state vector, $\begin{bmatrix}\mathbf{X}_1^T(t) & \mathbf{X}_2^T(t)\end{bmatrix}^T = \mathbf{X}(t) \triangleq \mathbf{X}(t,\omega) : [t_0, t_f]\times\Omega \mapsto \Re^n$, is an n-dimensional random vector for fixed t; for convenience, the dependency of a stochastic process on ω is not explicitly shown. The state matrices and control distribution matrix, $A_1 \in \Re^{(n-r)\times(n-r)}$, $A_2 \in \Re^{(n-r)\times r}$, $A_3 \in \Re^{r\times(n-r)}$, $A_4 \in \Re^{r\times r}$, and $B \in \Re^{r\times r}$, are assumed to be fully known. Also, the input matrix, B, is assumed to be nonsingular. The control input and the stochastic external disturbance are denoted as $\mathbf{u}(t) \in \Re^r$ and $\mathbf{W}(t) \triangleq \mathbf{W}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^r$, respectively. The nonlinear term, $\mathbf{g}(\cdot) : \Re^n \times \Re^r \times \Re^r \mapsto \Re^r$, is assumed to be unknown. The unknown external disturbance model is assumed to have the following form

$$d\mathbf{W}(t) = \boldsymbol{\varphi}\left(\mathbf{W}(t)\right)dt + d\boldsymbol{\mathcal{B}}(t) \quad \mathbf{W}(t_0) = \mathbf{0}, \tag{2}$$

where $\boldsymbol{\varphi}(\cdot) : \Re^r \mapsto \Re^r$ indicates an unknown operator and $\boldsymbol{\mathcal{B}}(t) \triangleq \boldsymbol{\mathcal{B}}(t,\omega) : [t_0, t_f] \times \Omega \mapsto \Re^r$, is assumed to be a stationary Wiener process with zero-mean and the correlation of increments

$$E\left[\{\boldsymbol{\mathcal{B}}(\tau)-\boldsymbol{\mathcal{B}}(\zeta)\}\{\boldsymbol{\mathcal{B}}(\tau)-\boldsymbol{\mathcal{B}}(\zeta)\}^T\right]=\mathcal{Q}|\tau-\zeta|,$$

where $\mathcal{Q}\in\Re^{r\times r}>0$ and is unknown.

Assumption 1: The stochastic external disturbance, $\mathbf{W}(t)$, is assumed to be mean square bounded, i.e.,

$$\sup_{t\geq t_0} E\left[\mathbf{W}^T(t)\mathbf{W}(t)\right]\leq w_{\max}.$$

Assumption 2: The nonlinear term, $\mathbf{g}(\cdot)$, is assumed to satisfy the following growth condition $\forall\,\mathbf{x}\in\Re^n,\mathbf{u}\in\Re^r$ and $\mathbf{w}\in\Re^r$:

$$|\mathbf{g}(\mathbf{x},\mathbf{u},\mathbf{w})|^2\leq\kappa\left(1+|\mathbf{x}|^2+|\mathbf{u}|^2+|\mathbf{w}|^2\right),$$

where $\kappa>0$ is a known scalar constant.
The measurement equation is given as

$$\mathbf{Y}(t)=\mathbf{X}(t)+\mathbf{V}(t), \tag{3}$$

where the measurement noise, $\mathbf{V}(t)\triangleq\mathbf{V}(t,\omega):[t_0,t_f]\times\Omega\mapsto\Re^n$, is assumed to be zero-mean Gaussian white noise, i.e., $\mathbf{V}(t)\sim\mathcal{N}\big(\mathbf{0},R\delta(\tau)\big)$. Using Itô formula $d\mathbf{g}(\cdot)$ can be written as

$$d\mathbf{g}(\cdot)=\boldsymbol{\psi}(\mathbf{X},\mathbf{u},\mathbf{W})\,dt+\Sigma(\mathbf{X},\mathbf{u},\mathbf{W})\,d\boldsymbol{\mathcal{B}}(t). \tag{4}$$

Here $\boldsymbol{\psi}(\cdot):\Re^n\times\Re^r\times\Re^r\mapsto\Re^r$ and $\Sigma(\cdot):\Re^n\times\Re^r\times\Re^r\mapsto\Re^{r\times r}$ denote unknown nonlinear operators. Combining (1) and (4) yields

$$\begin{aligned}d\mathbf{X}_1(t)&=\{A_1\mathbf{X}_1(t)+A_2\mathbf{X}_2(t)\}\,dt\\ d\mathbf{X}_2(t)&=\{A_3\mathbf{X}_1(t)+A_4\mathbf{X}_2(t)+B\mathbf{u}(t)+\mathbf{g}(\cdot)\}\,dt\\ d\mathbf{g}(\cdot)&=\boldsymbol{\psi}(\mathbf{X},\mathbf{u},\mathbf{W})\,dt+\Sigma(\mathbf{X},\mathbf{u},\mathbf{W})\,d\boldsymbol{\mathcal{B}}(t)\end{aligned} \tag{5}$$

Let $\mathbf{Z}(t)=\begin{bmatrix}\mathbf{X}_1^T(t) & \mathbf{X}_2^T(t) & \mathbf{g}^T(\cdot)\end{bmatrix}^T$. Now the system in (1) is rewritten as the following extended dynamically equivalent form:

$$d\mathbf{Z}(t)=\{F\mathbf{Z}(t)+D\mathbf{u}(t)+L\boldsymbol{\psi}(\cdot)\}\,dt+L\Sigma(\cdot)d\boldsymbol{\mathcal{B}}(t), \tag{6}$$

where

$$F=\begin{bmatrix}A_1 & A_2 & 0_{(n-r)\times r}\\ A_3 & A_4 & I_{r\times r}\\ 0_{r\times(n-r)} & 0_{r\times r} & 0_{r\times r}\end{bmatrix},\quad D=\begin{bmatrix}0_{(n-r)\times r}\\ B\\ 0_{r\times r}\end{bmatrix},$$

and $L=\begin{bmatrix}0_{(n-r)\times r}\\ 0_{r\times r}\\ I_{r\times r}\end{bmatrix}$. Now the measurement equation can be rewritten as

$$\mathbf{Y}(t)=H\mathbf{Z}(t)+\mathbf{V}(t), \tag{7}$$

where $H=\begin{bmatrix}I_{n\times n} & 0_{n\times r}\end{bmatrix}$.

Assumption 3: The nonlinear terms, $\boldsymbol{\psi}(\cdot)$ and $\Sigma(\cdot)$, are assumed to be locally Lipschitz continuous, i.e.,

$$|\boldsymbol{\psi}(\mathbf{z}_1)-\boldsymbol{\psi}(\mathbf{z}_2)|\leq c_\alpha|\mathbf{z}_1-\mathbf{z}_2|,\quad\text{and}$$

$$\|\Sigma(\mathbf{z}_1)-\Sigma(\mathbf{z}_2)\|\leq c_\alpha|\mathbf{z}_1-\mathbf{z}_2|,\quad\forall\,\mathbf{z}_1,\mathbf{z}_2\in\Re^{n+2r}$$

such that $|\mathbf{z}_1|,|\mathbf{z}_2|\leq\alpha$, where $|\cdot|$ denotes the usual Euclidean norm, $\|\cdot\|$ denotes the matrix two-norm, and $c_\alpha>0$ is a scalar constant.

Assumption 4: The nonlinear terms, $\boldsymbol{\psi}(\cdot)$ and $\Sigma(\cdot)$, are assumed to satisfy some growth condition, i.e.,

$$|\boldsymbol{\psi}(\mathbf{z})|^2\leq k\left(1+|\mathbf{z}|^2\right)\quad\text{and}$$

$$\|\Sigma(\mathbf{z})\|^2\leq k\left(1+|\mathbf{z}|^2\right),\quad\forall\,\mathbf{z}\in\Re^{n+2r}$$

where $k>0$ is a scalar constant.
Assuming $\boldsymbol{\mathcal{B}}(t_0)=\mathbf{0}$, $\mathbf{Z}(t_0)$ is $\mathcal{F}_0$-measurable, and the nonlinear terms are not explicit functions of time, the local Lipshitz condition (assumption 3) and the growth condition (assumption 4) provide the sufficient conditions for the existence and uniqueness of a continuous solution of (6) [8].

The total control law, $\mathbf{u}(t)$, consists of a nominal control and necessary corrections to the nominal control to compensate for system nonlinearities and external disturbances. Thus the disturbance accommodating control law, $\mathbf{u}(t)$, is selected as

$$\mathbf{u}(t)=\bar{\mathbf{u}}(t)-B^{-1}\mathbf{g}(\cdot) \tag{8}$$

where $\bar{\mathbf{u}}(t)$ is a nominal controller. Though the term $\mathbf{g}(\cdot)$ is unknown, an estimator such as a Kalman filter can be implemented in the feedback loop to estimate this nonlinear term. For this purpose, an assumed model of the extended system in (6) is constructed as

$$d\mathbf{Z}_m(t)=\{F\mathbf{Z}_m(t)+D\mathbf{u}(t)\}\,dt+Ld\boldsymbol{\mathcal{W}}(t), \tag{9}$$

where $\mathbf{Z}_m(t_0)=\begin{bmatrix}\mathbf{x}_0^T & \mathbf{0}^T\end{bmatrix}^T$ and $\boldsymbol{\mathcal{W}}(t)\triangleq\boldsymbol{\mathcal{W}}(t,\omega):[t_0,t_f]\times\Omega\mapsto\Re^r$ is a stationary Wiener process with zero-mean and the correlation of increments

$$E\left[\{\boldsymbol{\mathcal{W}}(\tau)-\boldsymbol{\mathcal{W}}(\zeta)\}\{\boldsymbol{\mathcal{W}}(\tau)-\boldsymbol{\mathcal{W}}(\zeta)\}^T\right]=Q|\tau-\zeta|,$$

where $Q\in\Re^{r\times r}>0$. Note that the measurement noise, $\mathbf{V}(t)$, can be formally written as [9]

$$\mathbf{V}(t)\sim\frac{d\boldsymbol{\mathcal{V}}(t)}{dt}$$

where $\boldsymbol{\mathcal{V}}(t)\triangleq\boldsymbol{\mathcal{V}}(t,\omega):[t_0,t_f]\times\Omega\mapsto\Re^n$ is a stationary Wiener process with zero-mean and the correlation of increments

$$E\left[\{\boldsymbol{\mathcal{V}}(\tau)-\boldsymbol{\mathcal{V}}(\zeta)\}\{\boldsymbol{\mathcal{V}}(\tau)-\boldsymbol{\mathcal{V}}(\zeta)\}^T\right]=R|\tau-\zeta|,$$

where $R\in\Re^{n\times n}>0$. Now an estimator is designed based on the assumed model in (9) to estimate the system states and the disturbance term. The estimator dynamics can now be written as

$$\begin{aligned}d\hat{\mathbf{Z}}(t)=&\left\{F\hat{\mathbf{Z}}(t)+D\mathbf{u}(t)+K(t)H[\mathbf{Z}(t)-\hat{\mathbf{Z}}(t)]\right\}dt\\ &+K(t)d\boldsymbol{\mathcal{V}}(t)\end{aligned} \tag{10}$$

where the matrix, $K(t)\in\Re^{(n+r)\times n}$, is the observer gain and it is calculated as

$$K(t)=P(t)H^TR^{-1} \tag{11}$$

where $P(t) \in \Re^{(n+r)\times(n+r)}$ can be obtained by solving the continuous-time matrix differential Riccati equation:

$$\dot{P}(t) = FP(t) + P(t)F^T - P(t)H^T R^{-1} HP(t) + LQL^T \tag{12}$$

For the purpose of analysis, the nominal control is assumed to be a state feedback control that would guarantee asymptotic stability of the assumed system, i.e.,

$$\bar{\mathbf{u}}(t) = -\begin{bmatrix} K_{m_1} & K_{m_2} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}}_1(t) \\ \hat{\mathbf{X}}_2(t) \end{bmatrix} = -K_m \hat{\mathbf{X}}(t)$$

where $\hat{\mathbf{X}}_1(t)$ and $\hat{\mathbf{X}}_2(t)$ are the estimates of $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$, respectively. The feedback gain, $K_m \triangleq \begin{bmatrix} K_{m_1} & K_{m_2} \end{bmatrix}$, is selected so that $(\mathcal{A} - \mathcal{B}K_m)$ is Hurwitz, where

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0_{(n-r)\times r} \\ B \end{bmatrix}$$

Now the disturbance accommodating control can be written as

$$\mathbf{u}(t) = -\begin{bmatrix} K_m & B^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}}(t) \\ \hat{\mathbf{g}}(\cdot) \end{bmatrix} = S\hat{\mathbf{Z}}(t) \tag{13}$$

After substituting the control law in (13), the estimator dynamics in (10) can be written as

$$\begin{aligned} d\hat{\mathbf{Z}}(t) = \Big\{ [F + DS]\, \hat{\mathbf{Z}}(t) + K(t)H[\mathbf{Z}(t) - \hat{\mathbf{Z}}(t)] \Big\}\, dt \\ + K(t) d\boldsymbol{\mathcal{V}}(t) \end{aligned} \tag{14}$$

Note that (14) can also can be written as

$$\dot{\hat{\mathbf{Z}}}(t) = [F + DS]\, \hat{\mathbf{Z}}(t) + K(t)\widetilde{\mathbf{Y}}(t) \tag{15}$$

where $\widetilde{\mathbf{Y}}(t) = \mathbf{Y}(t) - \hat{\mathbf{Y}}(t)$. Notice that the performance of the proposed control scheme depends on how well the estimator could estimate the extended states. The estimator given in (15) is optimal only if the extended plant dynamics given in (6) is a fully known linear system with additive Gaussian white noise processes as shown in (9). Though none of these assumptions are true here, just as in the case of an extended Kalman filter, the estimator parameters such as the process noise covariance, Q, and the measurement noise covariance, R, may be selected so that that the estimates are fairly accurate. For example, if the assumed process noise covariance Q is selected to be really small, i.e., $Q \approx 0$, then the estimated nonlinear disturbance term $\hat{\mathbf{g}}(\cdot) \approx \mathbf{0}$, $\forall t \geq t_0$. Also note that as R, the measurement noise covariance, increases, the observer gain decreases, and thus the observer fails to update the propagated disturbance term based on the measurements. Therefore, for small values of Q or for large values of R, the total control law given in (13) becomes just the nominal control and the system nonlinearities could degrade the closed-loop system performance. On the other hand, selecting a large Q value would compel the estimator to completely rely upon the measurement signal and therefore the noise associated with the measurement signal is directly transmitted into the estimates. This could result in a noisy control signal, which could lead to problems such as chattering and controller saturation. For a system with large uncertainties, a small Q or a large R would also result in an undesirable closed-loop performance as shown in [6].

III. Adaptive Scheme

If the estimator in (15) is able to obtain accurate estimates of the system states and the disturbance term, then the control law in (13) guarantees the desired closed-loop performance. The accuracy of the estimated system states and the disturbance term depends on the estimator parameters such as the process noise covariance, Q, and the measurement noise covariance, R. Since the measurement noise covariance can be obtained from sensor calibration, the process noise matrix is treated as a tuning parameter. Based on the stochastic Lyapunov analysis, an adaptive law is developed here for updating the selected process noise covariance online so that the controlled system is stable.

After substituting the control law given in (13), the system in (1) can be written as

$$d\mathbf{X}(t) = \Big\{ \mathcal{A}\mathbf{X}(t) - \mathcal{B}K_m\hat{\mathbf{X}}(t) + G\left[\mathbf{g}(\cdot) - \hat{\mathbf{g}}(\cdot)\right] \Big\}\, dt \tag{16}$$

where $G = \begin{bmatrix} 0_{r\times(n-r)} & I_{r\times r} \end{bmatrix}^T$. Based on (14), $d\hat{\mathbf{X}}(t)$ can be written as

$$\begin{aligned} d\hat{\mathbf{X}}(t) = \Big\{ [\mathcal{A} - \mathcal{B}K_m]\, \hat{\mathbf{X}}(t) + K_x(t)H\widetilde{\mathbf{Z}}(t) \Big\}\, dt \\ + K_x(t) d\boldsymbol{\mathcal{V}}(t) \end{aligned} \tag{17}$$

where $\widetilde{\mathbf{Z}}(t) = \mathbf{Z}(t) - \hat{\mathbf{Z}}(t)$ and $K_x(t) \in \Re^{n\times n}$ denotes the first n-rows of the $(n + r) \times n$ matrix $K(t)$ given in (11). Define $\widetilde{\mathbf{X}}(t) = \mathbf{X}(t) - \hat{\mathbf{X}}(t)$. Now subtracting (17) from (16) yields

$$d\widetilde{\mathbf{X}}(t) = \Big\{ [\mathcal{A} - K_x(t)]\, \widetilde{\mathbf{X}}(t) + G\widetilde{\mathbf{g}}(\cdot) \Big\}\, dt - K_x(t) d\boldsymbol{\mathcal{V}}(t) \tag{18}$$

where $\widetilde{\mathbf{g}}(\cdot) = \mathbf{g}(\cdot) - \hat{\mathbf{g}}(\cdot)$. Note that (17) can also be written as

$$\dot{\hat{\mathbf{X}}}(t) = [\mathcal{A} - \mathcal{B}K_m]\, \hat{\mathbf{X}}(t) + K_x(t)\widetilde{\mathbf{Y}}(t) \tag{19}$$

and (19) can be written as

$$\dot{\widetilde{\mathbf{X}}}(t) = \mathcal{A}\widetilde{\mathbf{X}}(t) - K_x(t)\widetilde{\mathbf{Y}}(t) + G\widetilde{\mathbf{g}}(\cdot) \tag{20}$$

Assumption 5: Without loss of generality, it can be assumed that the matrix $\mathcal{A}$ is Hurwitz. In the event that is not the case, one could always make appropriate changes to matrices A_3 and A_4 and corresponding changes to $\mathbf{g}(\cdot)$ so that $\mathcal{A}$ is Hurwitz.

Theorem 1: For the nonlinear stochastic system given in (1) and the estimator given in (10), the disturbance accommodating control law given in (13) guarantees that the system states, $\mathbf{X}(t)$, is asymptotically stable in the first moment, i.e.,

$$\lim_{t\to\infty} E\left[\mathbf{X}(t)\right] = \mathbf{0}$$

and $E[\mathbf{X}(t)] \in L_2 \cap L_\infty$ if the process noise covariance is updated online using the adaptive law

$$dQ(t) = \big[A_Q Q(t) + Q(t)A_Q^T + 2\hat{\mathbf{g}}(t)\hat{\mathbf{g}}^T(t) + 2\kappa S\hat{\mathbf{Z}}(t)\hat{\mathbf{Z}}^T(t)S^T + \frac{1}{r}\Big\{\gamma\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t) + 2\kappa(1+w_{\max}) + 2\kappa\mathbf{Y}^T(t)\mathbf{Y}(t)\Big\}I_{r\times r}\big]dt \tag{21}$$

where the Hurwitz matrix $A_Q \in \Re^{r\times r}$ is selected such that $2\mathrm{Tr}\{A_Q Q(t)\} \le LQ(t)L^T$ and the adaptive gain, γ is selected such that

$$\gamma \ge \big(\mathrm{Tr}\{P(t)H^T R^{-1} HP(t) - 2FP(t)\}\,\mathrm{Tr}\{\bar{R}\}^{-1} + 1 + \parallel K_x^T(t)K_x(t)\parallel\big) \tag{22}$$

Here the matrix $\bar{R}$ denotes the lower bound

$$E\left[\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\right] \ge \bar{R} \quad \forall t \ge t_0$$

and $P(t)$ satisfies the continuous-time matrix differential Riccati equation:

$$dP(t) = \big\{FP(t) + P(t)F^T - P(t)H^T R^{-1}HP(t) + LQ(t)L^T\big\}dt, \quad P(t_0) = P_0 \tag{23}$$

Proof: Let $\mathcal{F}_t^{\widetilde{\mathbf{Y}}}$ denotes a filtration generated by $\widetilde{\mathbf{Y}}(t) = \widetilde{\mathbf{X}}(t) + \mathbf{V}(t)$ such that $\hat{\mathbf{X}}(t)$ is adapted to $\mathcal{F}_t^{\widetilde{\mathbf{Y}}}$, i.e.,

$$E\left[\hat{\mathbf{X}}(s) \mid \mathcal{F}_t^{\widetilde{\mathbf{Y}}}\right] = \hat{\mathbf{X}}(s) \qquad s \le t$$

Now consider the following nonnegative function:

$$V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q) = E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right] + \int_{t_0}^t E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right] d\tau + \mathrm{Tr}\{Q^* - Q(t)\} + \int_{t_0}^t E\left[\widetilde{\mathbf{g}}(\tau) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{g}}(\tau) \mid \mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right] d\tau + \mathrm{Tr}\{P^* - P(t)\}$$

where $Q^* \ge Q(t)\ \forall t \ge t_0$ is a stabilizing process noise covariance and P^* is selected such that $P^* \ge P(t)$, $\forall t \ge t_0$. The matrix $\mathcal{X}$ is a positive definite matrix of appropriate dimensions and it is selected so that it satisfies the following matrix inequality:

$$\mathcal{X}[\mathcal{A} - \mathcal{B}K_m] + [\mathcal{A} - \mathcal{B}K_m]^T \mathcal{X} + \mathcal{X}\mathcal{X} + \mathcal{M} \le 0$$

where $\mathcal{M} > 0$. Note $V(s,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ can be written as

$$V(\cdot) = E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] + \mathrm{Tr}\{Q^* - Q(t)\} + \int_s^t E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(\tau) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] d\tau + \int_s^t E\left[\widetilde{\mathbf{g}}(\tau) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{g}}(\tau) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] d\tau + \mathrm{Tr}\{P^* - P(t)\}$$

Now $dV(s,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ can be calculated as

$$\begin{aligned} dV(\cdot) &= E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\{dQ(t)\} \\ &+ E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T [\mathcal{A} - \mathcal{B}K_m]^T \mathcal{X} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + \\ &E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X}[\mathcal{A} - \mathcal{B}K_m] E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + \\ &E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\{dP(t)\} \\ &+ E\left[\widetilde{\mathbf{g}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{g}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt \\ &+ E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt \end{aligned}$$

Based on the inequality

$$\begin{aligned} &E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X}\mathcal{X} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] + \\ &E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] \ge \\ &E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] + \\ &E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{X} E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] \end{aligned}$$

an upper bound on $dV(\cdot)$ can be obtained as

$$\begin{aligned} dV(\cdot) &\le E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + \\ &E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \Big\{\mathcal{X}[\mathcal{A} - \mathcal{B}K_m] + [\mathcal{A} - \mathcal{B}K_m]^T \mathcal{X} + \mathcal{X}\mathcal{X}\Big\} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\{dQ(t) + dP(t)\} + \\ &E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[K_x(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt + \\ &E\left[\widetilde{\mathbf{g}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T E\left[\widetilde{\mathbf{g}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt \end{aligned}$$

Now employing the Cauchy-Schwarz's inequality gives

$$\begin{aligned} dV(\cdot) &\le E\left[\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\{dQ(t) + dP(t)\} + \\ &E\left[\widetilde{\mathbf{Y}}^T(t)K_x^T(t)K_x(t)\widetilde{\mathbf{Y}}(t) + \widetilde{\mathbf{g}}^T(t)\widetilde{\mathbf{g}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt \\ &- E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T \mathcal{M} E\left[\hat{\mathbf{X}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt \end{aligned}$$

Note

$$\widetilde{\mathbf{g}}^T(t)\widetilde{\mathbf{g}}(t) \le 2\mathbf{g}^T(\cdot)\mathbf{g}(\cdot) + 2\hat{\mathbf{g}}^T(t)\hat{\mathbf{g}}(t)$$

and

$$\mathbf{g}^T(\cdot)\mathbf{g}(\cdot) = |\mathbf{g}(\cdot)|^2 \le \kappa\left(1 + |\mathbf{x}|^2 + |\mathbf{u}|^2 + |\mathbf{w}|^2\right)$$

The last inequality follows from assumption 2. Thus, the following inequality holds:

$$\widetilde{\mathbf{g}}^T(t)\widetilde{\mathbf{g}}(t) \le 2\kappa\Big(1 + \mathbf{X}^T(t)\mathbf{X}(t) + \mathbf{u}^T(t)\mathbf{u}(t) + \mathbf{W}^T(t)\mathbf{W}(t)\Big) + 2\hat{\mathbf{g}}^T(t)\hat{\mathbf{g}}(t)$$

After substituting the control law given in (13), the above inequality can be rewritten as

$$\widetilde{\mathbf{g}}^T(t)\widetilde{\mathbf{g}}(t) \le 2\kappa\Big(1 + \mathbf{X}^T(t)\mathbf{X}(t) + \hat{\mathbf{Z}}^T(t)S^TS\hat{\mathbf{Z}}(t) + \mathbf{W}^T(t)\mathbf{W}(t)\Big) + 2\hat{\mathbf{g}}^T(t)\hat{\mathbf{g}}(t)$$

Now an upper bound on $dV(s,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ can be obtained as

$$dV(\cdot) \le E\left[\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t) \mid \mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right] dt - \mathrm{Tr}\{dQ(t) + dP(t)\} +$$

$$\Big\{E\left[\widetilde{\mathbf{Y}}^T(t)K_x^T(t)K_x(t)\widetilde{\mathbf{Y}}(t)+\mathbf{W}^T(t)\mathbf{W}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]-$$
$$E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T\mathcal{M}E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]+2E\left[\hat{\mathbf{g}}^T(t)\hat{\mathbf{g}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]$$
$$+2\kappa\left(1+E\left[\mathbf{X}^T(t)\mathbf{X}(t)+\hat{\mathbf{Z}}^T(t)S^TS\hat{\mathbf{Z}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\right)\Big\}dt$$

Since $\mathbf{Y}(t)=\mathbf{X}(t)+\mathbf{V}(t)$, the following inequality holds:

$$E\left[\mathbf{X}^T(t)\mathbf{X}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq E\left[\mathbf{Y}^T(t)\mathbf{Y}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right].$$

Now based on assumption 1, after substituting (21) and (23), an upper bound on $dV(s,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ can be obtained as

$$dV(\cdot)\leq\Big\{E\left[\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t)+2\hat{\mathbf{g}}^T(t)\hat{\mathbf{g}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]+$$
$$E\left[\widetilde{\mathbf{Y}}^T(t)K_x^T(t)K_x(t)\widetilde{\mathbf{Y}}(t)+2\kappa\mathbf{Y}^T(t)\mathbf{Y}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]-$$
$$E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T\mathcal{M}E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]+2\kappa(1+w_{\max})+$$
$$2\kappa E\left[\hat{\mathbf{Z}}^T(t)S^TS\hat{\mathbf{Z}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]-\mathrm{Tr}\{A_QQ(t)+Q(t)A_Q^T+$$
$$2\hat{\mathbf{g}}(t)\hat{\mathbf{g}}^T(t)+2\kappa S\hat{\mathbf{Z}}(t)\hat{\mathbf{Z}}^T(t)S^T\}-\mathrm{Tr}\{\frac{1}{r}\{\gamma\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t)+$$
$$2\kappa(1+w_{\max})+2\kappa\mathbf{Y}^T(t)\mathbf{Y}(t)\}I_{r\times r}\}-\mathrm{Tr}\{FP(t)+$$
$$P(t)F^T-P(t)H^TR^{-1}HP(t)+LQ(t)L^T\}\Big\}dt$$

Note A_Q is selected such that

$$-\mathrm{Tr}\left\{A_QQ(t)+Q(t)A_Q^T\right\}-\mathrm{Tr}\left\{LQ(t)L^T\right\}\leq 0,$$

and

$$\widetilde{\mathbf{Y}}^T(t)K_x^T(t)K_x(t)\widetilde{\mathbf{Y}}(t)\leq\parallel K_x^T(t)K_x(t)\parallel\widetilde{\mathbf{Y}}^T(t)\widetilde{\mathbf{Y}}(t).$$

Combining the similar terms yields

$$E\left[dV(\cdot)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq -E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T\mathcal{M}E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt+$$
$$E\left[\left(1+\parallel K_x^T(t)K_x(t)\parallel-\gamma\right)\mathrm{Tr}\left\{\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\right\}\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt-$$
$$E\left[\mathrm{Tr}\left\{FP(t)+P(t)F^T-P(t)H^TR^{-1}HP(t)\right\}\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt$$

Let $\gamma=\gamma_1+\gamma_2$, where γ_1 is selected such that

$$\gamma_1\geq 1+\parallel K_x^T(t)K_x(t)\parallel \tag{24}$$

Note

$$E\left[\widetilde{\mathbf{Y}}(t)\widetilde{\mathbf{Y}}^T(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\geq\bar{R}$$

Now an upper bound for $E\left[dV(s,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]$ can be obtained as

$$E\left[dV(\cdot)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq -E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T\mathcal{M}E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt+$$
$$E\left[\mathrm{Tr}\left\{FP(t)+P(t)F^T-P(t)H^TR^{-1}HP(t)\right\}\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt-$$
$$E\left[\gamma_2\mathrm{Tr}\left\{\bar{R}\right\}\right]dt$$

Finally, note that

$$E\left[dV(\cdot)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]\leq -E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]^T\mathcal{M}E\left[\hat{\mathbf{X}}(t)\mid\mathcal{F}_s^{\widetilde{\mathbf{Y}}}\right]dt$$

if γ_2 is selected such that

$$\gamma_2\geq\mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t)-2F_mP(t)\right\}\mathrm{Tr}\left\{\bar{R}\right\}^{-1} \tag{25}$$

Thus selecting γ_1 and γ_2 according to (24) and (25) yields $V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ is a nonnegative $\mathcal{F}_s^{\widetilde{\mathbf{Y}}}$-supermartingale [10], [11] and the nonnegative supermartingale probability inequality yields [12]

$$\mathbb{P}\Big(\sup_{t\geq t_0}V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)\geq\lambda\Big)\leq$$
$$\frac{\hat{\mathbf{X}}^T(t_0)\mathcal{X}\hat{\mathbf{X}}(t_0)+\mathrm{Tr}\Big\{Q^*-Q(t_0)\Big\}+\mathrm{Tr}\Big\{P^*-P(t_0)\Big\}}{\lambda}$$

where $\lambda>0$ is any positive constant. Thus, selecting sufficiently large λ yields

$$\mathbb{P}\Big(\sup_{t\geq t_0}V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)<\infty\Big)=1$$

That is, $V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ is almost surely bounded. Assuming $\widetilde{\mathbf{Y}}(t_0)$ is precisely known yields

$$E\left[\widetilde{\mathbf{Y}}(t)\mid\mathcal{F}_{t_0}^{\widetilde{\mathbf{Y}}}\right]=E\left[\widetilde{\mathbf{Y}}(t)\right]$$

Now note $V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)$ is defined as

$$V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)=\int_{t_0}^{t}E\left[\widetilde{\mathbf{Y}}(\tau)\right]^TE\left[\widetilde{\mathbf{Y}}(\tau)\right]d\tau+$$
$$E\left[\hat{\mathbf{X}}(t)\right]^T\mathcal{X}E\left[\hat{\mathbf{X}}(t)\right]+\int_{t_0}^{t}E\left[\widetilde{\mathbf{g}}(\tau)\right]^TE\left[\widetilde{\mathbf{g}}(\tau)\right]d\tau+$$
$$\mathrm{Tr}\Big\{Q^*-Q(t)\Big\}+\mathrm{Tr}\Big\{P^*-P(t)\Big\}$$

and

$$E\left[dV(s,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)\right]\leq -E\left[\hat{\mathbf{X}}(t)\right]^T\mathcal{M}E\left[\hat{\mathbf{X}}(t)\right]dt$$

Therefore,

$$V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)\in L_\infty\quad\mathbb{P}\text{ almost surely (a.s.).}$$
$$\Longrightarrow E\left[\widetilde{\mathbf{Y}}(t)\right]\in L_2,\ E\left[\widetilde{\mathbf{g}}(t)\right]\in L_2,\ Q(t)\in L_\infty,$$
$$P(t)\in L_\infty\quad\mathbb{P}\text{ almost surely.}$$

and

$$E\left[V(t_0,t,\hat{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{g}},P,Q)\right]<\infty\ \Longrightarrow E\left[\hat{\mathbf{X}}(t)\right]\in L_2\cap L_\infty$$

Since $P(t)$ is $a.s.$ bounded, the estimator gain, $K(t)=P(t)H^TR^{-1}$, is also $a.s.$ bounded. Thus, there exists a k^* such that

$$\mathbb{P}\left(\sup_{t\geq t_0}\parallel K(t)\parallel>k^*\right)=0$$

Note that

$$E\left[\widetilde{\mathbf{Y}}(t)\right]\in L_2\Longrightarrow E\left[\widetilde{\mathbf{X}}(t)\right]\in L_2$$

Now consider the state estimation error dynamics given in (20). Based on assumption 5 and Lemma 5.5 in [6] it can be concluded that

$$E\left[\widetilde{\mathbf{X}}(t)\right]\in L_\infty\quad\text{and}\quad\lim_{t\to\infty}E\left[\widetilde{\mathbf{X}}(t)\right]=\mathbf{0}$$

Note that the nominal controller is selected so that the matrix $(\mathcal{A} - \mathcal{B}K_m)$ is Hurwitz. Since $K_x(t)$ is almost surely bounded and $E\left[\widetilde{\mathbf{Y}}(t)\right] \in L_2$, based on Lemma 5.5 in [6] it can be shown that

$$\lim_{t\to\infty} E\left[\hat{\mathbf{X}}(t)\right] = \mathbf{0}$$

Since $\widetilde{\mathbf{X}}(t)$ is defined as

$$\widetilde{\mathbf{X}}(t) = \mathbf{X}(t) - \hat{\mathbf{X}}(t)$$

it can be easily concluded that

$$E\left[\mathbf{X}(t)\right] \in L_2 \cap L_\infty \quad \text{and} \quad \lim_{t\to\infty} E\left[\mathbf{X}(t)\right] = \mathbf{0}$$

■

Since the nonlinear system considered here is state feedback, the internal stability of the controlled system can be obtained based on assumption 1. The proposed scheme can be easily extended to output feedback system if the nonlinear system considered is *zero-detectable* [13], [14] and the assumption 2 can be expressed in terms of the output instead of system states. Zero-detectability is a central property in the general theory of nonlinear stabilization on the basis of output measurements. For more information on zero-detectability, see [15], [14], [16] and [17].

IV. Numerical Simulation Results

For simulation purposes, consider a two-degree-of-freedom helicopter whose nonlinear equations of motion for the helicopter can be written as

$$(J_{eq,p} + m_{heli}l_{cm}^2)\ddot{\theta} = K_{pp}V_{m,p} + K_{py}V_{m,y} - B_p\dot{\theta} - m_{heli}gl_{cm}\cos(\theta) - m_{heli}l_{cm}^2\cos(\theta)\sin(\theta)\dot{\psi}^2 + W_1(t) \quad (26a)$$

$$(J_{eq,y} + m_{heli}l_{cm}^2\cos(\theta)^2)\ddot{\psi} = K_{yy}V_{m,y} + K_{yp}V_{m,p} - B_y\dot{\psi} + 2m_{heli}l_{cm}^2\cos(\theta)\sin(\theta)\dot{\psi}\dot{\theta} + W_2(t) \quad (26b)$$

The linear model is given as

$$(J_{eq,p} + m_{heli}l_{cm}^2)\ddot{\theta} = K_{pp}V_{m,p} + K_{py}V_{m,y} - B_p\dot{\theta} \quad (27a)$$

$$(J_{eq,y} + m_{heli}l_{cm}^2)\ddot{\psi} = K_{yy}V_{m,y} + K_{yp}V_{m,p} - B_y\dot{\psi} \quad (27b)$$

A detailed description of system parameters are given in Table I.

The control input to the system are the input voltages of the pitch and yaw motors, $V_{m,p}$ and $V_{m,y}$, respectively. Let $\mathbf{u} = [u_1 \quad u_2]^T = [V_{m,p} \quad V_{m,y}]^T$. Now the assumed model can be rewritten as

$$\ddot{\theta}_m = a_{1_m}\dot{\theta}_m + b_{1_m}u_1 + b_{2_m}u_2$$
$$\ddot{\psi}_m = a_{2_m}\dot{\psi}_m + b_{3_m}u_1 + b_{4_m}u_2$$

where the parameters, a_{1_m}, a_{2_m}, b_{1_m}, b_{2_m}, b_{3_m}, and b_{4_m} can be calculated using the assumed system parameters given in Table I as

$$a_{1_m} = \frac{-B_p}{(J_{eq,p} + m_{heli}l_{cm}^2)} \qquad a_{2_m} = \frac{-B_y}{(J_{eq,y} + m_{heli}l_{cm}^2)}$$

TABLE I

Helicopter Model Parameters

System Param.	Description	Assum. Values	True Values	Units
B_p	Viscous damping (pitch)	0.8000	1	N/V
B_y	Viscous damping (yaw)	0.3180	0.3021	N/V
$J_{eq,p}$	Moment of inertia (yaw)	0.0384	0.0288	$Kg{\cdot}m^2$
$J_{eq,y}$	Moment of inertia (pitch)	0.0432	0.0496	$Kg{\cdot}m^2$
K_{pp}	Torque constant (pitch) from pitch motor/propeller	0.2041	0.2552	$N{\cdot}m/V$
K_{py}	Torque constant (pitch) from yaw motor/propeller	0.0068	0.0051	$N{\cdot}m/V$
K_{yp}	Torque constant (yaw) from pitch motor/propeller	0.0219	0.0252	$N{\cdot}m/V$
K_{yy}	Torque constant (yaw) from yaw motor/propeller	0.0720	0.0684	$N{\cdot}m/V$
m_{heli}	Total mass of the helicopter	1.3872	1.3872	Kg
l_{cm}	Location of center-of-mass	0.1857	0.1764	m

$$b_{1_m} = \frac{K_{pp}}{(J_{eq,p} + m_{heli}l_{cm}^2)} \qquad b_{2_m} = \frac{K_{py}}{(J_{eq,p} + m_{heli}l_{cm}^2)}$$
$$b_{3_m} = \frac{K_{yp}}{(J_{eq,y} + m_{heli}l_{cm}^2)} \qquad b_{4_m} = \frac{K_{yy}}{(J_{eq,y} + m_{heli}l_{cm}^2)}$$

Let $\mathbf{X}_1(t) = [\theta(t) \quad \psi(t)]^T$ and $\mathbf{X}_2(t) = [\dot{\theta}(t) \quad \dot{\psi}(t)]^T$. Now the state-space representation of the assumed linear model is

$$\dot{\mathbf{X}}_{1_m}(t) = A_1\mathbf{X}_{1_m}(t) + A_2\mathbf{X}_{2_m}(t)$$
$$\dot{\mathbf{X}}_{2_m}(t) = A_3\mathbf{X}_{1_m}(t) + A_{4_m}\mathbf{X}_{2_m}(t) + B_m\mathbf{u}(t)$$

where $A_1 = A_3 = 0_{2\times2}$, $A_2 = I_{2\times2}$, $A_{4_m} = \begin{bmatrix} a_{1_m} & 0 \\ 0 & a_{2_m} \end{bmatrix}$, and $B_m = \begin{bmatrix} b_{1_m} & b_{2_m} \\ b_{3_m} & b_{4_m} \end{bmatrix}$. The state-space representation of the nonlinear system in (26) is

$$\dot{\mathbf{X}}_1(t) = A_1\mathbf{X}_1(t) + A_2\mathbf{X}_2(t)$$
$$\dot{\mathbf{X}}_2(t) = A_3\mathbf{X}_1(t) + A_{4_m}\mathbf{X}_2(t) + B_m\mathbf{u}(t) + \mathbf{g}\left(\mathbf{X}(t), \mathbf{u}(t), \mathbf{W}(t)\right)$$

where the nonlinear term $\mathbf{g}(\cdot) \triangleq \left[g_1(\cdot) \quad g_2(\cdot)\right]^T : \Re^4 \times \Re^2 \times \Re^2 \mapsto \Re^2$ is defined as

$$g_1(\cdot) = \frac{1}{(J_{eq,p} + m_{heli}l_{cm}^2)}\Big\{K_{pp}u_1 + K_{py}u_2 - B_p\dot{\theta} - m_{heli}gl_{cm}\cos(\theta) - m_{heli}l_{cm}^2\cos(\theta)\sin(\theta)\dot{\psi}^2 + W_1(t)\Big\} - \Big\{a_{1_m}\dot{\theta} + b_{1_m}u_1 + b_{2_m}u_2\Big\}$$

$$g_2(\cdot) = \frac{1}{(J_{eq,y} + m_{heli}l_{cm}^2\cos(\theta)^2)}\Big\{K_{yy}u_2 + K_{yp}u_2 - B_y\dot{\psi} + 2m_{heli}l_{cm}^2\cos(\theta)\sin(\theta)\dot{\psi}\dot{\theta} + W_2(t)\Big\} - \Big\{a_{2_m}\dot{\psi} + b_{3_m}u_1 + b_{4_m}u_2\Big\}$$

The external disturbance dynamics is selected as

$$d\mathbf{W}(t) = A_w\mathbf{W}(t)dt + d\boldsymbol{\mathcal{B}}(t) \quad \mathbf{W}(t_0) = \mathbf{0} \quad (28)$$

where

$$A_w = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}$$

and $E\left[\{\boldsymbol{\mathcal{B}}(\tau)-\boldsymbol{\mathcal{B}}(\zeta)\}\{\boldsymbol{\mathcal{B}}(\tau)-\boldsymbol{\mathcal{B}}(\zeta)\}^T\right]=10^{-2}I_{2\times 2}|\tau-\zeta|$. The nominal controller is a linear quadratic regulator, which minimizes the cost function

$$J=E\Big[\frac{1}{2}\int_0^{\infty}\big((\mathbf{X}_m(t)-\mathbf{x}_d)^T\mathcal{Q}_{\mathbf{x}}(\mathbf{X}_m(t)-\mathbf{x}_d)+ \\ \mathbf{u}^T(t)\mathcal{R}_{\mathbf{u}}\mathbf{u}(t)\big)dt\Big]$$

where $\mathbf{x}_d^T=[\theta_d \quad \psi_d \quad 0 \quad 0]$, θ_d and ψ_d are some desired final values of θ and ψ, respectively, and $\mathcal{Q}_{\mathbf{x}}$ and $\mathcal{R}_{\mathbf{u}}$ are two symmetric positive definite matrices. The nominal control that minimizes the above cost function is

$$\bar{\mathbf{u}}(t)=-K_m(\hat{\mathbf{X}}(t)-\mathbf{x}_d)$$

where K_m is the feedback gain that minimizes the cost function. Now the total control law can be written in terms of the estimated states and the estimated disturbance term as

$$\mathbf{u}(t)=\begin{bmatrix}-K_m & -B_m^{-1}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}(t)-\mathbf{x}_d\\ \hat{\mathbf{g}}(t)\end{bmatrix}=S\hat{\mathbf{Z}}(t)+K_m\mathbf{x}_d$$

TABLE II

NOMINAL CONTROLLER/ESTIMATOR MATRICES

LQR Weighting Matrices
$\mathcal{R}_{\mathbf{u}}=10\times I_{2\times 2}$, $\mathcal{Q}_{\mathbf{x}}=\begin{bmatrix}500\times I_{2\times 2} & 0_{2\times 2}\\ 0_{2\times 2} & 100\times I_{2\times 2}\end{bmatrix}$
Covariance Matrices
$R=\begin{bmatrix}10^{-3}\times I_{2\times 2} & 0_{2\times 2}\\ 0_{2\times 2} & 10^{-2}\times I_{2\times 2}\end{bmatrix}$, $Q(t)=\begin{bmatrix}q_1(t) & q_2(t)\\ q_3(t) & q_4(t)\end{bmatrix}$, $P(t_0)=\begin{bmatrix}1\times 10^{-3}\times I_{4\times 4} & 0_{4\times 2}\\ 0_{2\times 4} & I_{2\times 2}\end{bmatrix}$

Table II shows the nominal controller and estimator matrices. Since the measurement noise covariance, R, can be obtained from sensor calibration, the process noise matrix, Q, is treated as a tuning parameter. Based on the weighting matrices given in Table II, the feedback gain is calculated to be

$$K_m=\begin{bmatrix}7.0229 & 0.8239 & 1.6691 & 0.3310\\ -0.8239 & 7.0229 & -0.0830 & 2.4486\end{bmatrix}$$

For simulation purposes, the initial states are selected to be $[\theta_0 \quad \psi_0 \quad \dot{\theta}_0 \quad \dot{\psi}_0]^T=[-45^o \quad 0 \quad 0 \quad 0]^T$ and the desired states θ_d and ψ_d are selected to be 45^o and 30^o, respectively. For the implementation of the adaptive law, the following parameters are selected:

$$\kappa=10,\ w_{\max}=1,\ A_Q=-0.5\times I_{2\times 2} \quad \text{and}$$

$$\gamma=\parallel K(t)\parallel+\mathrm{Tr}\left\{P(t)H^TR^{-1}HP(t)-2F_mP(t)\right\}\mathrm{Tr}\left\{R\right\}^{-1}$$

Note that an upper bound on $w_{\max}$ can be obtained by solving the algebraic matrix Lyapunov equation:

$$A_w\bar{P}+\bar{P}A_w=-10^{-2}\times I_{2\times 2}$$

The desired response given in Fig. 1 is the linear system response to the nominal control when there is no model error and external disturbance.

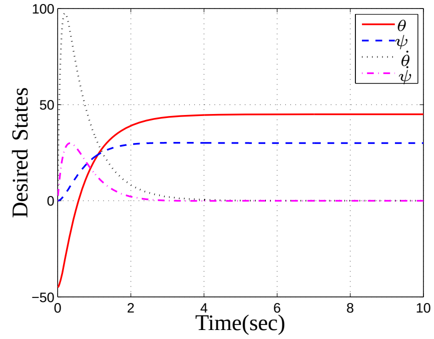
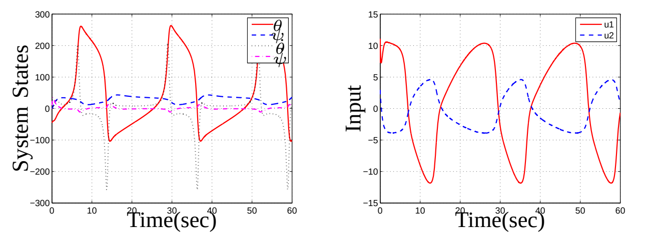


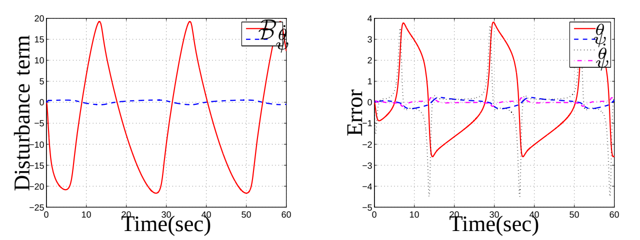
Fig. 1. Desired System Response

For the first simulation, a constant process noise covariance is selected as $Q=10^{-2}\times I_{2\times 2}$. Figures 2(a) and 2(b) show the system response and the disturbance accommodating control input obtained for the first simulation. Figures 3(a) and 3(b) contain the estimated disturbance



(a) (b)

Fig. 2. Nonlinear System States and Control Input: Without Adaptive Scheme & $Q=10^{-2}\times I_{2\times 2}$

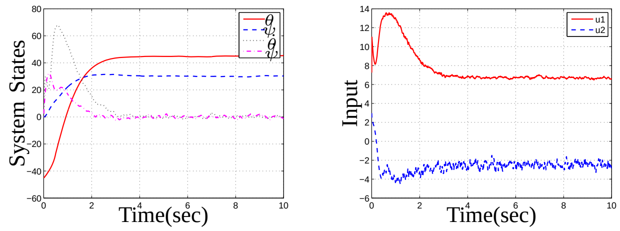


(a) (b)

Fig. 3. Disturbance Term and State Error: Without Adaptive Scheme & $Q=10^{-2}\times I_{2\times 2}$

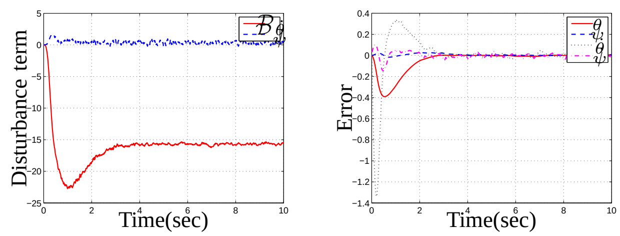
term and the error between the desired states and the true states corresponding to the first simulation. Note that the first simulation results given in Figs. 2 and 3 are much different from the desired results because of the low value of the process noise covariance matrix selected. The small process noise covariance reduces the disturbance accommodating control into the nominal state feedback controller designed for the linear model and the system nonlinearities degrade the performance, as shown in Figs. 2 and 3.

A second simulation is conducted using the proposed adaptive scheme, where the initial process noise covariance is selected to be $Q(t_0)=10^{-2}\times I_{2\times 2}$. Figures 4(a) and 4(b) show the system response and the control input obtained for the second simulation. Figures 5(a) and 5(b) contain



(a) (b)

Fig. 4. Nonlinear System States and Control Input: With Adaptive Scheme & $Q(t_0) = 10^{-2} \times I_{2\times 2}$



(a) (b)

Fig. 5. Disturbance Term and State Error: With Adaptive Scheme & $Q(t_0) = 10^{-2} \times I_{2\times 2}$

the estimated disturbance term and the error between the desired states and the true states corresponding to the second simulation. Note that the second simulation results given in Figs. 4 and 5 indicate that the adaptive scheme was able to recover the desired performance despite the low initial process noise covariance selected. Figure 6 shows the time varying process noise covariance obtained for the second simulation.

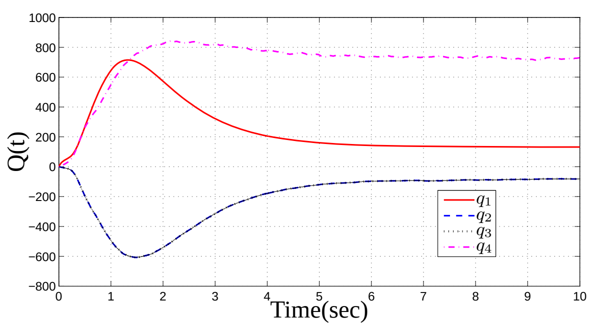


Fig. 6. Adaptive Process Noise Covariance: $Q(t_0) = 10^{-2} \times I_{2\times 2}$

V. Conclusion

This paper presents the formulation of an adaptive disturbance accommodating controller for nonlinear stochastic systems. The control scheme presented here utilizes an estimator to determine the corrections to the nominal control input required to minimize the adverse effects of both unknown system nonlinearities and external disturbances. The estimated states are used to develop a nominal control law while the estimated disturbance term is used to make necessary corrections to the nominal control input. Performance of the proposed approach depends on the accuracy of the assumed disturbance model. The process noise covariance used in the estimator is a quantitative measure on the accuracy of the assumed disturbance term dynamics. Selecting a very low process noise covariance value would reduce the disturbance accommodating control to just the nominal control, and the unknown system nonlinearities and external disturbances could severely degrade the controlled system performance. This paper focuses on developing a stochastic adaptive approach to update the process noise covariance online so that the closed-loop performance of the nonlinear plant is guaranteed. Based on the stochastic analysis, an adaptive law is developed for updating the selected process noise covariance online so that the controlled system is stable. The simulation results given here supports the theory presented.

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