ACCURATE KEPLER EQUATION SOLVER WITHOUT TRANSCENDENTAL FUNCTION EVALUATIONS

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The goal for the solution of Kepler's equation is to determine the eccentric anomaly accurately, given the mean anomaly and eccentricity. This paper presents a new approach to solve this very well documented problem. Here we focus on demonstrating the procedure for both the elliptical and hyperbolic cases. It is found that by means of a series approximation, an angle identity, the application of Sturm's theorem and an iterative correction method, the need to evaluate transcendental functions or store tables is eliminated. A 15^{th}-order polynomial is developed through a series approximation of Kepler's equation. Sturm's theorem is used to prove that only one real roots exists for this polynomial for the given range of mean anomaly and eccentricity. An initial approximation for this root is found using a 3^{rd}-order polynomial. Then a one time generalized Newton-Raphson correction is applied to obtain accuracies to the level of around 10^{-15} rad for the elliptical case and 10^{-13} rad for the hyperbolic case, which is near machine precision.

INTRODUCTION

A common and possibly the most basic problem in orbital mechanics is to find the position of a body with respect to time. Many propagation approaches can be used to solve the orbital dynamics equations of motion. For example, directly numerically integrating these equations is commonly referred to Cowell's method [1]. The advantage of this approach is that it can be used to determine trajectories when non-conservatives forces are present. Momentum conserving integrators have been developed using geometric integration approaches when only conservative forces are considered [2]. Other approaches are summarized in various texts, such as Ref. [3].

For pure two-body Keplerian motion, a standard approach to determine the position and velocity of an object involves using classical orbital elements, which include dimensional elements, such as the semi-major axis (size of orbit), the eccentricity (shape of orbit) and the initial mean anomaly (related to time), and non-dimensional elements, such as the inclination, the right ascension of the ascending node and argument of periapsis. For elliptical orbits the position of a body can be described in polar coordinates using its magnitude and the true anomaly, which is the angle between the direction of periapsis and the current position of the body, as seen from the main focus of the ellipse. For circular orbits a constant interval change in the true anomaly yields a constant interval change in time. However, for non-circular orbits a constant change in the true anomaly does not produce a constant change in time. This is the reason that the mean anomaly, which is the mean motion times the time interval, is preferred over the true anomaly in orbital motion studies.

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Kepler was the first to study how the true anomaly is related to time. To investigate this relationship, he circumscribed a circle around the the ellipse and used an intermediate variable, called the eccentric anomaly, that can be directly related to the true anomaly using a closed-form analytical expression, derived through simple geometric relations. He then related the eccentric anomaly to the mean anomaly through his well-known approach, called *Kepler's equation*, which is a transcendental equation since it involves a sine function. Given the true anomaly and eccentricity, the inverse problem involves determining the eccentric anomaly. Unfortunately a closed-form solution to this problem does not exist. Kepler himself stated "I am sufficiently satisfied that it [Kepler's equation] cannot be solved a priori, on account of the different nature of the arc and the sine. But if I am mistaken, and any one shall point out the way to me, he will be in my eyes the great Apollonius."

Kepler's rather trivial looking equation has spawned a plethora of mathematical and geometric solutions over three centuries, which are summarized nicely by the treatise in Ref. [4]. Nonanalytic solutions, such as the solution by cycloid, provide useful geometric interpretations of Kepler's equation. Infinite series approaches, such as Bessel functions, have found uses in other applications, such as electromagnetic waves, heat conduction and vibration to name a few. A simple Taylor series approach works well for low eccentricities but quickly becomes unstable for high eccentricities. Other approaches rely on transcendental function evaluations, which are more computationally expensive than computing powers of variables.

In 1853, Schubert posed the question of how one might choose an initial guess for the eccentric anomaly so that one iteration of Newton's method produces an approximate solution of desired accuracy for all mean anomalies and eccentricities [5]. Colwell states on page 106 of Ref. [4] that in his view the solution given by Mikkola [6] seems "closest to the ideal solution of Schubert's 1853 question" of all results seen. Mikkola used an auxiliary variable to rewrite Kepler's equation in arcsine form. A cubic approximation is then employed for the arcsine term to derive a cubic polynomial equation and then a correction factor is applied to obtain a maximum absolute relative error in the eccentric anomaly no greater than 0.002 radians. A fourth-order Newton correction is then applied to Kepler's equation obtain accuracies on the order of 10^{-15} rad, but this requires an evaluation of transcendental functions.

More modern solutions are given by Markley [7] in 1996 and Fukushima [8] in 1997. Markley uses a 3^{rd}-order Padé approximation of the sine function into Kepler's equation directly, which leads to a solution that is seven times more accurate than Mikkola's approach. Reference [9] uses a sequential solution, based on Ref. [10], to provide better accuracy over Markley's approach in some regions. Fukushima uses a grid of 128 equally spaced points to give a starting value with error no larger than 0.025, and then uses a generalized Newton-Raphson correction without transcendental function evaluations to achieve accuracies on the order of 10^{-15} rad. But, this approach requires storing tables. Turner [11] obtains a solution by defining four solution sub-domains and concatenating three computational methods in each sub-domain. Data storage requirements are much less than Markley's approach and accuracies around the order of 10^{-10} rad for most regions are obtained, although some regions are orders of magnitude worse. Reference [12] uses a Bézier curve approach to solve Kepler's equation. The approach requires no pre-computed data and no iterative processes. Accuracies on the order of 10^{-15} rad are obtained up to eccentricity of 0.99, but the approach requires a second-order Halley correction which involves computing transcendental functions.

In this paper, an accurate solution is obtained without using any transcendental function evaluations or lookup tables. The approach is based on extending Mikkola's concept by considering a higher-order polynomial approximation, whose one real root provides the accurate solution for the eccentric anomaly. An initial approximation to this polynomial is given by using a reduced third-order polynomial and then a generalized Newton-Raphson correction is used to obtain accuracies on the order of 10^{-15} rad for the elliptical case and 10^{-13} rad for the hyperbolic case, and with no transcendental function evaluations.

The organization of this paper proceeds as follows. First, Kepler's equation is reviewed followed by a review of Mikkola's approach. A review of a Sturm chain, which can be used to determine the number of real roots of a polynomial, is next provided. Then the new approach is derived, which involves finding the root of a 15$^{\text{th}}$-order polynomial. A Sturm chain is used to next show that only one real root exists for this polynomial. Finally, an accuracy plot is given over the entire range of mean anomalies and eccentricities. A similar procedure is then shown for the case of a hyperbolic trajectory.

MIKKOLA'S APPROACH

Kepler's equation for the elliptic case is given by

$$E - e\sin E = M, \quad (0 \le e \le 1,\ \ 0 \le M \le \pi) \tag{1}$$

Given the mean anomaly, M, and the eccentricity, e, the goal is to determine E accurately. A review Mikkola's approach to solve Kepler's equation is shown here. The series approximation for $\arcsin x$ is given by

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots \tag{2}$$

Also,the following multiple angle identity will be used [13]:

$$\begin{aligned} 2\,(-1)^{\frac{n-1}{2}}\sin n\theta &= (2\sin\theta)^n - n\,(2\sin\theta)^{n-2} + \frac{n(n-3)}{2!}(2\sin\theta)^{n-4} \\ &\quad - \frac{n(n-4)(n-5)}{3!}(2\sin\theta)^{n-6} + \frac{n(n-5)(n-6)(n-7)}{4!}(2\sin\theta)^{n-8} \\ &\quad - \cdots + 2n\,(-1)^{\frac{n-1}{2}}\sin\theta \end{aligned} \tag{3}$$

For $n = 3$, the following well-known triple angle identity is given:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{4}$$

Defining the auxiliary variable $x = \sin(E/3)$ and using Eq. (4), Kepler's equation becomes

$$3\arcsin x - e(3x - 4x^3) = M \tag{5}$$

Note that the range of x is $0 \le x \le \sin(\pi/3)$ since $0 \le E \le \pi$. Truncating the series in Eq. (2) to 3$^{\text{rd}}$-order, and substituting the result into Eq. (5) leads to an approximation of Kepler's equation in terms of a 3$^{\text{rd}}$-order polynomial:

$$\left(4e + \frac{1}{2}\right)x^3 + 3(1-e)\,x - M = 0 \tag{6}$$

By Descartes' sign rule, since $0 \le e \le 1$ and $0 \le M \le \pi$, then only one root of Eq. (6) is positive. The polynomial has the form

$$x^3 + a\,x + b = 0 \tag{7}$$

The positive root can be determined using Cardin's formula:

$$x = \left(-\frac{b}{2} + y\right)^{1/3} - \left(\frac{b}{2} + y\right)^{1/3} \tag{8}$$

where

$$y = \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \tag{9}$$

Mikkola then applied an empirical correction to improve the worst error associated with $M = \pi$, given by

$$w = x - \frac{0.078\,x^5}{1 + e} \tag{10}$$

Then, the approximate solution for E is given by

$$E = M + e(3w - 4w^3) \tag{11}$$

which has a maximum relative error of no greater than 0.002. Mikkola then applies a one-time generalized 4$^{\text{th}}$-order Newton-Raphson correction on $f(x) = x - e\sin x - M$ to achieve an accuracy of 10^{-15} rad. However, this correction requires evaluations of $\sin w$ and $\cos w$.

STURM CHAIN EXAMPLE

A Sturm chain will be used to later prove that only one real root of the to-be-developed 15$^{\text{th}}$-order polynomial to accurately solve Kepler's equation exists. Named after the French mathematician Jacques Sturm, a Sturm chain of a polynomial p is a sequence of polynomials associated to p and its derivative by a variant of the Euclidean algorithm for polynomials. The theorem yields the number of distinct real roots of a p located in an interval in terms of the number of changes in signs of the values of the sequence evaluated at the boundaries of a given interval.

The approach is presented by example. In particular, a Sturm chain is demonstrated to prove the existence of only one real root for the 3$^{\text{rd}}$-order polynomial in Eq. (6) in the interval from 0 to $\pi/3$. We begin by defining the first element of the chain to be the polynomial itself:

$$f_0 = (4e + 0.5)x^3 + 3(1 - e)x - M \tag{12}$$

For the range of $0 \leq e \leq 1$ and $0 \leq M \leq 1$ there exists only one change in signs in this polynomial as x goes from 0 to $\pi/3$. The existence of only one sign change is independent of the values of eccentricity and mean anomaly; the latter would have to have a value larger than that of the rest of the equation for the order of the sign change to be altered, but not the existence of the sign change itself. That is, the change in signs may go from being negative-to-positive, to becoming positive-to-negative within the interval of interest. An issue arises when $e = 1$, which corresponds to the trivial solution $x = 0$ and will persist throughout the rest of the polynomials of this chain. The following element is f_0's derivative:

$$f_1 = (12e + 1.5)x^2 + 3(1 - e) \tag{13}$$

There are no sign changes in Eq. (13) over the interval of interest; any evaluation of f_1 will be in the positive range. To obtain the next element of the chain, the following operation is performed:

$$f_0 = q_0 f_1 - f_2, \quad \text{generalized as:} \quad f_{m-2} = q_{m-2} f_{m-1} - f_m \tag{14}$$

Figure 1. Surface plot of $-8e^3 + 24e^2 + (8M^2 - 24)e + M^2 + 8$ **in Eq.** (19)

where q_{m-2} represents the quotient of the division between f_{m-2} and f_{m-1}, and f_m represents the remainder. The operation may be represented as follows:

$$\frac{(4e + 0.5)x^3 + 3(1 - e)x - M}{(12e + 1.5)x^2 + 3(1 - e)} = \left(\frac{1}{3}e\right) x + \frac{-2(1 - e)x - M}{(12e + 1.5)x^2 + 3(1 - e)} \tag{15}$$

The next polynomial is given by

$$f_2 = -2(1 - e)x - M \tag{16}$$

This only has one possible sign change within the interval of interest, i.e., when the value of the mean anomaly, M, becomes larger than $-2(1 - e)$, the sign change in f_2 goes from being positive-to-negative into negative-to-positive.

Similarly, we proceed using polynomial division to obtain the following polynomial of the chain:

$$\begin{aligned}\frac{(12e + 1.5)x^2 + 3(1 - e)}{-2(1 - e)x - M} &= \left\{ \left[6 + \frac{13.5}{2(e - 1)}\right] x + \frac{1}{2} M \frac{6 + \dfrac{13.5}{2(e - 1)}}{1 - e} \right\} \\ &\quad + \left[\frac{-3(1 - e) + \dfrac{1}{2} M^2 \dfrac{6 + \dfrac{13.5}{2(e - 1)}}{1 - e}}{-2(1 - e)x - M} \right]\end{aligned} \tag{17}$$

$$f_3 = -3(1 - e) + \frac{1}{2} M^2 \frac{6 + \dfrac{13.5}{2(e - 1)}}{1 - e} \tag{18}$$

Equation (18) can be restructured as follows:

$$f_3 = -\frac{0.375(-8e^3 + 24e^2 + (8M^2 - 24)e + M^2 + 8)}{(e - 1)^2} \tag{19}$$

As the last element of a Sturm chain, this equation does not depend on x any longer. Not unlike the previous polynomials, a similar trivial solution occurs when $e = 1$; this result appears in Eq. (19) as a singularity in the denominator. Otherwise, examining the term inside the parenthesis of the numerator for any value of $0 \leq e \leq 1$ and $0 \leq M \leq \pi$ indicates that its value never ceases to be positive. This can easily be proven by consider the case where this term, $q \equiv -8e^3 + 24e^2 + (8M^2 - 24)e + M^2 + 8$, can possibly become negative. Clearly, this may occur at the boundary of $M = 0$ and when e is close to 1. Setting $M = 0$ and $e = 1 - \epsilon$, where ϵ is a small positive number, in q yields

$$\begin{aligned} q &= -8(1 - 3\epsilon + 3\epsilon^2 - \epsilon^3) + 24(1 - 2\epsilon + \epsilon^2) - 24(1 - \epsilon) + 8 \\ &= 8\epsilon^3 \end{aligned} \tag{20}$$

which is clearly always positive for positive ϵ. A surface plot of the function for $0 \leq e \leq 1$ and $0 \leq M \leq \pi$ is also shown in Figure 1. Therefore we conclude that there is no possible sign change that can occur in this last equation of the chain; any evaluation of f_3 will be negative.

Table 1 summarizes the findings above. It shows the possible sign change configurations that can be obtained from the Sturm chain f_0 to f_3 as eccentricity, e, and mean anomaly, M, vary from 0 to 1 independently. The difference between the sign chance counts at each boundary dictates the number of possible real roots in Eq. (6). It is thus shown that there is only one real root within this range, completing the desired proof.

Table 1. Number of Sign Changes at Interval Boundaries

x	0	$\pi/3$
f_0	−	+
f_1	+	+
f_2	+	−
f_3	−	−
#	2	1

NEW APPROACH

The new approach presented here requires no transcendental function evaluations or table generation. The approach is in essence an extension of Mikkola's method. Any order can be chosen, but a 15$^{\text{th}}$-order polynomial is chosen since it leads to solutions with errors on the order of 10^{-15} rad. Using Eq. (3), with $n = 15$, in Eq. (1) gives

$$\begin{aligned} 15 \arcsin x - e\,(15x - 560x^3 + 6048x^5 - 28800x^7 \\ + 70400x^9 - 92160x^{11} + 61440x^{13} - 16384x^{15}) = M \end{aligned} \tag{21}$$

where $x = \sin(E/15)$. Then, substituting Eq. (2), truncated to 15$^{\text{th}}$ order, into Eq. (21) gives the following polynomial:

$$\begin{aligned} &\left(\frac{3003}{14336} + 16384\,e\right)x^{15} + \left(\frac{3465}{13312} - 61440\,e\right)x^{13} + \left(\frac{945}{2816} + 92160\,e\right)x^{11} \\ &+ \left(\frac{175}{384} - 70400\,e\right)x^{9} + \left(\frac{75}{112} + 28800\,e\right)x^{7} + \left(\frac{9}{8} - 6048\,e\right)x^{5} \\ &+ \left(\frac{5}{2} + 560\,e\right)x^{3} + 15(1-e)x - M = 0 \end{aligned} \tag{22}$$

Note that the range of x is now $0 \leq x \leq \sin(\pi/15)$. By using Sturm chains, it can be shown that only one real root exists for Eq. (22). The procedure follows exactly as the Sturm chains example for the the 3$^{\text{rd}}$-order polynomial shown in this paper but taken to 15$^{\text{th}}$-order. Unfortunately, the proof is too long to be put in this paper; see Ref. [14] for more details.

Figure 2. Error in Computed E

Finding this root can be computationally expensive. A simple and effective technique is employed here. First, the polynomial is truncated to 3rd-order:

$$\left(\frac{5}{2} + 560e\right) x^3 + 15(1-e)x - M = 0 \tag{23}$$

Again by Descartes' sign rule, since $0 \le e \le 1$ and $0 \le M \le \pi$, then only one root of Eq. (23) is positive, which can be determined using Eq. (8). A Sturm chain approach can be used to show that the polynomial only has one real root as well. A simpler approach is used here. First define the following:

$$a \equiv \frac{15(1-e)}{(5/2) + 560e} \tag{24a}$$

$$b \equiv -\frac{M}{(5/2) + 560e} \tag{24b}$$

The three roots are given by

$$x_1 = \alpha + \beta \tag{25a}$$

$$x_{2,3} = -\frac{1}{2}(\alpha + \beta) \pm \frac{j\sqrt{3}}{2}(\alpha - \beta) \tag{25b}$$

where

$$\alpha = \left(-\frac{b}{2} + y\right)^{1/3} \tag{26a}$$

$$\beta = -\left(\frac{b}{2} + y\right)^{1/3} \tag{26b}$$

where y is given by Eq. (9). Three possible cases exist:

- $y > 0$: There is one real root and two conjugate imaginary roots.
- $y = 0$: There are three real roots of which at least two are equal, given by
  - $b > 0$: $\quad x_1 = -2\sqrt{-a/3}, \quad x_{2,3} = \sqrt{-a/3}$
  - $b < 0$: $\quad x_1 = 2\sqrt{-a/3}, \quad x_{2,3} = -\sqrt{-a/3}$
  - $b = 0$: $\quad x_{1,2,3} = 0$
- $y < 0$: There are three real and unequal roots, given by

$$x_i = 2\sqrt{\frac{-a}{3}} \cos\left(\frac{\phi}{3} + \frac{2\pi(i-1)}{3}\right), \quad i = 1,\, 2,\, 3$$

where

$$\cos\phi = \begin{cases} -\sqrt{\dfrac{b^2/4}{-a^3/27}} & \text{if } b > 0 \\ \sqrt{\dfrac{b^2/4}{-a^3/27}} & \text{if } b < 0 \end{cases}$$

For the given range of eccentricity and mean anomaly, the case of $y > 0$ is always given producing only one real root, except when $e = 1$ and $M = 0$ which gives three roots at 0. This can also be verified using Descartes' sign rule. To find the number of negative roots, change the signs of the coefficients of the terms with odd exponents in Eq. (23), which gives

$$-\left(\frac{5}{2} + 560e\right) x^3 - 15(1-e)x - M = 0 \tag{27}$$

This polynomial has no sign changes which indicates that it has no negative roots. Thus, the number of complex roots is two.

To provide a refinement to this approximate solution, a number of methods can be employed involving either iterative or non-iterative approaches. We choose to employ a correction involving a generalized Newton-Raphson correction. The following variables are first defined:

$$u_1 = -\frac{f}{f^{(1)}} \tag{28a}$$

$$u_i = -\frac{f}{f^{(1)} + \displaystyle\sum_{j=2}^{i} \frac{1}{j!} f^{(j)} u_{i-1}^{j-1}}, \quad i > 1 \tag{28b}$$

where f is the function given by Eq. (22) and $f^{(j)}$ denotes the j^{th} derivative of f with respect to x. The correction is given by

$$x \to x + u_i \tag{29}$$

where $\to$ denotes replacement. One approach to determine a solution involves a one-time correction with u_{15}, which gives an accurate solution to within a precision on the order of 10^{-15} rad for any value of M and e.* Other approaches involve iterating four times using u_1 (i.e., the first-order Newton-Raphson correction); iterating three times using u_2; or iterating two times using u_3. Unfortunately, to achieve a high precision with only one correction, u_{15} must be used (i.e., using u_i with $3 \leq i \leq 14$ requires two iterations).

In order to improve the worst error associated with $M = \pi$, the following empirical correction has been determined:

$$w = x - \frac{0.01171875\, x^{17}}{1+e} \tag{30}$$

Then, the solution for E is given by

$$\begin{aligned} E = M + e(&-16384\, w^{15} + 61440\, w^{13} - 92160\, w^{11} \\ &+ 70400\, w^9 - 28800\, w^7 + 6048\, w^5 - 560\, w^3 + 15\, w) \end{aligned} \tag{31}$$

Equation (49) gives an accuracy an the order of 10^{-15} rad for all valid ranges of M and e, as shown in Figure 2.

Algorithm Summary

The overall algorithm is now summarized. Given e and M the first step involves determining the real root of Eq. (23), denoted by $\bar{x}$. First define the following variables:

$$c_3 = \frac{5}{2} + 560e \tag{32a}$$

*Using the command "root" in MATLAB gives a precision on the order of 10^{-14} rad.

$$a = \frac{15(1-e)}{c_3} \tag{32b}$$

$$b = -\frac{M}{c_3} \tag{32c}$$

$$y = \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \tag{32d}$$

Then $\bar{x}$ is given by

$$\bar{x} = \left(-\frac{b}{2} + y\right)^{1/3} - \left(\frac{b}{2} + y\right)^{1/3} \tag{33}$$

Next define the following coefficients:

$$c_{15} = \frac{3003}{14336} + 16384\,e, \quad c_{13} = \frac{3465}{13312} - 61440\,e, \quad c_{11} = \frac{945}{2816} + 92160\,e \tag{34a}$$

$$c_9 = \frac{175}{384} - 70400\,e, \quad c_7 = \frac{75}{112} + 28800\,e, \quad c_5 = \frac{9}{8} - 6048\,e \tag{34b}$$

The following powers of $\bar{x}$ are defined to later save on computations:

$$\bar{x}_2 = \bar{x}^2, \quad \bar{x}_3 = \bar{x}_2\,\bar{x}, \quad \bar{x}_4 = \bar{x}_3\,\bar{x}, \quad \bar{x}_5 = \bar{x}_4\,\bar{x} \tag{35a}$$

$$\bar{x}_6 = \bar{x}_5\,\bar{x}, \quad \bar{x}_7 = \bar{x}_6\,\bar{x}, \quad \bar{x}_8 = \bar{x}_7\,\bar{x}, \quad \bar{x}_9 = \bar{x}_8\,\bar{x} \tag{35b}$$

$$\bar{x}_{10} = \bar{x}_9\,\bar{x}, \quad \bar{x}_{11} = \bar{x}_{10}\,\bar{x}, \quad \bar{x}_{12} = \bar{x}_{11}\,\bar{x} \tag{35c}$$

$$\bar{x}_{13} = \bar{x}_{12}\,\bar{x}, \quad \bar{x}_{14} = \bar{x}_{13}\,\bar{x}, \quad \bar{x}_{15} = \bar{x}_{14}\,\bar{x} \tag{35d}$$

Then the function f, which is given by Eq. (22), is defined along with its derivatives:

$$f = c_{15}\,\bar{x}_{15} + c_{13}\,\bar{x}_{13} + c_{11}\,\bar{x}_{11} + c_9\,\bar{x}_9 + c_7\,\bar{x}_7 + c_5\,\bar{x}_5 + c_3\,\bar{x}_3 + 15(1-e)\bar{x} - M \tag{36a}$$

$$f^{(1)} = 15\,c_{15}\,\bar{x}_{14} + 13\,c_{13}\,\bar{x}_{12} + 11\,c_{11}\,\bar{x}_{10} + 9\,c_9\,\bar{x}_8 + 7\,c_7\,\bar{x}_6 + 5\,c_5\,\bar{x}_4 + 3\,c_3\,\bar{x}_2 + 15(1-e) \tag{36b}$$

$$f^{(2)} = 210\,c_{15}\,\bar{x}_{13} + 156\,c_{13}\,\bar{x}_{11} + 110\,c_{11}\bar{x}_9 + 72\,c_9\,\bar{x}_7 + 42\,c_7\,\bar{x}_5 + 20\,c_5\,\bar{x}_3 + 6\,c_3\,\bar{x} \tag{36c}$$

$$f^{(3)} = 2730\,c_{15}\,\bar{x}_{12} + 1716\,c_{13}\,\bar{x}_{10} + 990\,c_{11}\,\bar{x}_8 + 504\,c_9\,\bar{x}_6 + 210\,c_7\,\bar{x}_4 + 60\,c_5\,\bar{x}_2 + 6\,c_3 \tag{36d}$$

$$f^{(4)} = 32760\,c_{15}\,\bar{x}_{11} + 17160\,c_{13}\bar{x}_9 + 7920\,c_{11}\,\bar{x}_7 + 3024\,c_9\,\bar{x}_5 + 840\,c_7\,\bar{x}_3 + 120\,c_5\,\bar{x} \tag{36e}$$

$$f^{(5)} = 360\,360\,c_{15}\,\bar{x}_{10} + 154\,440\,c_{13}\,\bar{x}_8 + 55440\,c_{11}\,\bar{x}_6 + 15120\,c_9\,\bar{x}_4 + 2520\,c_7\,\bar{x}_2 + 120\,c_5 \tag{36f}$$

$$f^{(6)} = 3\,603\,600\,c_{15}\,\bar{x}_9 + 1\,235\,520\,c_{13}\,\bar{x}_7 + 332\,640\,c_{11}\,\bar{x}_5 + 60480\,c_9\,\bar{x}_3 + 5040\,c_7\,\bar{x} \tag{36g}$$

$$f^{(7)} = 32\,432\,400\,c_{15}\,\bar{x}_8 + 8\,648\,640\,c_{13}\bar{x}_6 + 1\,663\,200\,c_{11}\,\bar{x}_4 + 181\,440\,c_9\,\bar{x}_2 + 5040\,c_7 \tag{36h}$$

$$f^{(8)} = 259\,459\,200\,c_{15}\,\bar{x}_7 + 5\,1891\,840\,c_{13}\,\bar{x}_5 + 6\,652\,800\,c_{11}\,\bar{x}_3 + 362\,880\,c_9\,\bar{x} \tag{36i}$$

$$f^{(9)} = 1.8162144 \times 10^9\,c_{15}\,\bar{x}_6 + 259\,459\,200\,c_{13}\,\bar{x}_4 + 19\,958\,400\,c_{11}\,\bar{x}_2 + 362\,880\,c_9 \tag{36j}$$

$$f^{(10)} = 1.08972864 \times 10^{10}\,c_{15}\,\bar{x}_5 + 1.0378368 \times 10^9\,c_{13}\,\bar{x}_3 + 39\,916\,800\,c_{11}\,\bar{x} \tag{36k}$$

$$f^{(11)} = 5.4486432 \times 10^{10}\,c_{15}\,\bar{x}_4 + 3.1135104 \times 10^9\,c_{13}\,\bar{x}_2 + 39\,916\,800\,c_{11} \tag{36l}$$

$$f^{(12)} = 2.17945728 \times 10^{11}\,c_{15}\,\bar{x}_3 + 6.2270208 \times 10^9\,c_{13}\,\bar{x} \tag{36m}$$

$$f^{(13)} = 6.53837184 \times 10^{11}\,c_{15}\,\bar{x}_2 + 6.2270208 \times 10^9\,c_{13} \tag{36n}$$

$$f^{(14)} = 1.307674368 \times 10^{13}\, c_{15}\, \bar{x} \tag{36o}$$

$$f^{(15)} = 1.307674368 \times 10^{13}\, c_{15} \tag{36p}$$

We now define the following factorial coefficients:

$$g_1 = \frac{1}{2}, \quad g_2 = \frac{1}{6}, \quad g_3 = \frac{1}{24}, \quad g_4 = \frac{1}{120} \tag{37a}$$

$$g_5 = \frac{1}{720}, \quad g_6 = \frac{1}{5040}, \quad g_7 = \frac{1}{40320}, \quad g_8 = \frac{1}{362\,880} \tag{37b}$$

$$g_9 = \frac{1}{3\,628\,800}, \quad g_{10} = \frac{1}{39\,916\,800}, \quad g_{11} = \frac{1}{479\,001\,600} \tag{37c}$$

$$g_{12} = \frac{1}{6.2270208 \times 10^{9}}, \quad g_{13} = \frac{1}{8.71782912 \times 10^{10}}, \quad g_{14} = \frac{1}{1.307674368 \times 10^{12}} \tag{37d}$$

The u_i values in Eq. (28b) are given by

$$u_1 = -\frac{f}{f^{(1)}} \tag{38a}$$

$$h_2 = f^{(1)} + g_1\, u_1\, f^{(2)}$$

$$u_2 = -\frac{f}{h_2} \tag{38b}$$

$$h_3 = f^{(1)} + g_1\, u_2\, f^{(2)} + g_2\, u_2^2\, f^{(3)}$$

$$u_3 = -\frac{f}{h_3} \tag{38c}$$

$$h_4 = f^{(1)} + g_1\, u_3\, f^{(2)} + g_2\, u_3^2\, f^{(3)} + g_3\, u_3^3\, f^{(4)}$$

$$u_4 = -\frac{f}{h_4} \tag{38d}$$

$$h_5 = f^{(1)} + g_1\, u_4\, f^{(2)} + g_2\, u_4^2\, f^{(3)} + g_3\, u_4^3\, f^{(4)} + g_4\, u_4^4\, f^{(5)}$$

$$u_5 = -\frac{f}{h_5} \tag{38e}$$

$$h_6 = f^{(1)} + g_1\, u_5\, f^{(2)} + g_2\, u_5^2\, f^{(3)} + g_3\, u_5^3\, f^{(4)} + g_4\, u_5^4\, f^{(5)} + g_5\, u_5^5\, f^{(6)}$$

$$u_6 = -\frac{f}{h_6} \tag{38f}$$

$$h_7 = f^{(1)} + g_1\, u_6\, f^{(2)} + g_2\, u_6^2\, f^{(3)} + g_3\, u_6^3\, f^{(4)} + g_4\, u_6^4\, f^{(5)} + g_5\, u_6^5\, f^{(6)} + g_6\, u_6^6\, f^{(7)}$$

$$u_7 = -\frac{f}{h_7} \tag{38g}$$

$$h_8 = f^{(1)} + g_1\, u_7\, f^{(2)} + g_2\, u_7^2\, f^{(3)} + g_3\, u_7^3\, f^{(4)} + g_4\, u_7^4\, f^{(5)} + g_5\, u_7^5\, f^{(6)} + g_6\, u_7^6\, f^{(7)} + g_7\, u_7^7\, f^{(8)}$$

$$u_8 = -\frac{f}{h_8} \tag{38h}$$

$$\begin{aligned} h_9 = {} & f^{(1)} + g_1\, u_8\, f^{(2)} + g_2\, u_8^2\, f^{(3)} + g_3\, u_8^3\, f^{(4)} + g_4\, u_8^4\, f^{(5)} + g_5\, u_8^5\, f^{(6)} + g_6\, u_8^6\, f^{(7)} \\ & + g_7\, u_8^7\, f^{(8)} + g_8\, u_8^8\, f^{(9)} \end{aligned}$$

$$u_9 = -\frac{f}{h_9} \tag{38i}$$

$$\begin{aligned} h_{10} = {} & f^{(1)} + g_1\, u_9\, f^{(2)} + g_2\, u_9^2\, f^{(3)} + g_3\, u_9^3\, f^{(4)} + g_4\, u_9^4\, f^{(5)} + g_5\, u_9^5\, f^{(6)} + g_6\, u_9^6\, f^{(7)} \\ & + g_7\, u_9^7\, f^{(8)} + g_8\, u_9^8\, f^{(9)} + g_9\, u_9^9\, f^{(10)} \end{aligned}$$

$$u_{10} = -\frac{f}{h_{10}} \tag{38j}$$

$$h_{11} = f^{(1)} + g_1\, u_{10}\, f^{(2)} + g_2\, u_{10}^2\, f^{(3)} + g_3\, u_{10}^3\, f^{(4)} + g_4\, u_{10}^4\, f^{(5)} + g_5\, u_{10}^5\, f^{(6)} + g_6\, u_{10}^6\, f^{(7)} + g_7\, u_{10}^7\, f^{(8)} + g_8\, u_{10}^8\, f^{(9)} + g_9\, u_{10}^9\, f^{(10)} + g_{10}\, u_{10}^{10}\, f^{(11)}$$

$$u_{11} = -\frac{f}{h_{11}} \tag{38k}$$

$$h_{12} = f^{(1)} + g_1\, u_{11}\, f^{(2)} + g_2\, u_{11}^2\, f^{(3)} + g_3\, u_{11}^3\, f^{(4)} + g_4\, u_{11}^4\, f^{(5)} + g_5\, u_{11}^5\, f^{(6)} + g_6\, u_{11}^6\, f^{(7)} + g_7\, u_{11}^7\, f^{(8)} + g_8\, u_{11}^8\, f^{(9)} + g_9\, u_{11}^9\, f^{(10)} + g_{10}\, u_{11}^{10}\, f^{(11)} + g_{11}\, u_{11}^{11}\, f^{(12)}$$

$$u_{12} = -\frac{f}{h_{12}} \tag{38l}$$

$$h_{13} = f^{(1)} + g_1\, u_{12}\, f^{(2)} + g_2\, u_{12}^2\, f^{(3)} + g_3\, u_{12}^3\, f^{(4)} + g_4\, u_{12}^4\, f^{(5)} + g_5\, u_{12}^5\, f^{(6)} + g_6\, u_{12}^6\, f^{(7)} + g_7\, u_{12}^7\, f^{(8)} + g_8\, u_{12}^8\, f^{(9)} + g_9\, u_{12}^9\, f^{(10)} + g_{10}\, u_{12}^{10}\, f^{(11)} + g_{11}\, u_{12}^{11}\, f^{(12)} + g_{12}\, u_{12}^{12}\, f^{(13)}$$

$$u_{13} = -\frac{f}{h_{13}} \tag{38m}$$

$$h_{14} = f^{(1)} + g_1\, u_{13}\, f^{(2)} + g_2\, u_{13}^2\, f^{(3)} + g_3\, u_{13}^3\, f^{(4)} + g_4\, u_{13}^4\, f^{(5)} + g_5\, u_{13}^5\, f^{(6)} + g_6\, u_{13}^6\, f^{(7)} + g_7\, u_{13}^7\, f^{(8)} + g_8\, u_{13}^8\, f^{(9)} + g_9\, u_{13}^9\, f^{(10)} + g_{10}\, u_{13}^{10}\, f^{(11)} + g_{11}\, u_{13}^{11}\, f^{(12)} + g_{12}\, u_{13}^{12}\, f^{(13)} + g_{13}\, u_{13}^{13}\, f^{(14)}$$

$$u_{14} = -\frac{f}{h_{14}} \tag{38n}$$

$$h_{15} = f^{(1)} + g_1\, u_{14}\, f^{(2)} + g_2\, u_{14}^2\, f^{(3)} + g_3\, u_{14}^3\, f^{(4)} + g_4\, u_{14}^4\, f^{(5)} + g_5\, u_{14}^5\, f^{(6)} + g_6\, u_{14}^6\, f^{(7)} + g_7\, u_{14}^7\, f^{(8)} + g_8\, u_{14}^8\, f^{(9)} + g_9\, u_{14}^9\, f^{(10)} + g_{10}\, u_{14}^{10}\, f^{(11)} + g_{11}\, u_{14}^{11}\, f^{(12)} + g_{12}\, u_{14}^{12}\, f^{(13)} + g_{13}\, u_{14}^{13}\, f^{(14)} + g_{14}\, u_{14}^{14}\, f^{(15)}$$

$$u_{15} = -\frac{f}{h_{15}} \tag{38o}$$

The solution for x in Eq. (22) is given by

$$x = \bar{x} + u_{15} \tag{39}$$

Then the solution for E is given by

$$w = x - \frac{0.01171875\, x^{17}}{1+e} \tag{40a}$$

$$E = M + e(-16384\, w^{15} + 61440\, w^{13} - 92160\, w^{11} + 70400\, w^9 - 28800\, w^7 + 6048\, w^5 - 560\, w^3 + 15\, w) \tag{40b}$$

HYPERBOLIC CASE

Kepler's equation in hyperbolic form is given by:

$$e \sinh H - H = M, \quad (1 \le e,\ \ 0 \le M \le \pi) \tag{41}$$

where H represents the hyperbolic eccentric anomaly. Although of different behavior to its elliptical analogue, a similar solution approach remains valid for the hyperbolic case. The methodology is now described for the 15th-order case since, once again, it leads to solutions with the smallest error with a one-time correction.

The following multiple angle identity for hyperbolic sine is given:

$$\begin{aligned}2\sinh n\theta &= (2\sinh\theta)^n + n\,(2\sinh\theta)^{n-2} + \frac{n(n-3)}{2!}(2\sinh\theta)^{n-4} \\ &+ \frac{n(n-4)(n-5)}{3!}(2\sinh\theta)^{n-6} + \frac{n(n-5)(n-6)(n-7)}{4!}(2\sinh\theta)^{n-8} \\ &+ \cdots + 2n\,(-1)^{\frac{n-1}{2}}\sinh\theta\end{aligned} \tag{42}$$

Defining the auxiliary variable $x = \sinh(H/15)$ and using Eq. (42), with $n = 15$, in Eq. (41) gives

$$\begin{aligned}&-15\ln(x-\sqrt{1+x^2}) + e\,(15x + 560x^3 + 6048x^5 + 28800x^7 \\ &+ 70400x^9 + 92160x^{11} + 61440x^{13} + 16384x^{15}) = M\end{aligned} \tag{43}$$

Then, the following identity is substituted, with $n = 15$, for the logarithm term:

$$\ln(x-\sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots \tag{44}$$

The following 15th-order polynomial approximation is obtained for $0 \le x \le \sinh(\pi/15)$:

$$\begin{aligned}&\left(16384\,e + \frac{429}{2048}\right)x^{15} + \left(61440\,e - \frac{3465}{13312}\right)x^{13} + \left(92160\,e + \frac{945}{2816}\right)x^{11} \\ &+ \left(70400\,e - \frac{175}{384}\right)x^9 + \left(28800\,e + \frac{75}{112}\right)x^7 + \left(6048\,e - \frac{9}{8}\right)x^5 \\ &+ (560\,e + 5/2)\,x^3 + (15\,e - 15) - M = 0\end{aligned} \tag{45}$$

Much like its elliptical counterpart, it can be shown, through Sturm chains, that the polynomial in Eq. (45) only has one real root in the aforementioned interval. A full proof of the Sturm chain method is documented in Ref. [14].

To find the root, Eq. (45) is truncated to 3rd order, obtaining an expression similar to Eq. (23):

$$\left(560\,e + \frac{5}{2}\right)x^3 + 15\,(e-1) - M = 0 \tag{46}$$

By Descartes' sign rule, since $1 \le e$ and $0 \le M \le \pi$, then only one root of Eq. (46) is positive, which can be determined using Eq. (8). An approach similar to that used in the elliptical case is used. Refer to Eqs. (24), (25), and (26), with the only difference being that

$$a \equiv \frac{15(e-1)}{(5/2) + 560e} \tag{47}$$

Once again, for the given range of eccentricity and mean anomaly, the case of $y > 0$ persists, yielding only one real root, except when $e = 1$ and $M = 0$ which gives three roots at 0; this represents the trivial solution. To find the number of negative roots, change the signs of the coefficients of the terms with odd exponents in Eq. (46), this yields no sign changes which indicates that the polynomial has no negative roots. Thus, the three roots consist of one real positive root, and two complex roots.

Figure 3. Error in Computed H

To refine this approximation, a one-time Newton-Raphson correction is employed as defined in Eqs. (28) and (29). The correction algorithm is analogous to that one utilized on the elliptical case. In order to improve the worst error associated with $M = \pi$, the following empirical correction is determined:

$$w = x - \frac{0.01171875\, x^{17}}{(1 + 0.45x^2)(1 + 4x^2)e} \tag{48}$$

Then, the solution for H is given by

$$\begin{aligned} H = e(&16384\, w^{15} + 61440\, w^{13} + 92160\, w^{11} \\ &+ 70400\, w^9 + 28800\, w^7 + 6048\, w^5 + 560\, w^3 + 15\, w) \\ &- M \end{aligned} \tag{49}$$

Figure 3 shows an accuracy of 10^{-13} rad for the valid ranges of M and e.

CONCLUSIONS

This paper presented a new solution for Kepler's equation. The main focus of this approach is to determine the real root of a 15$^{\text{th}}$-order polynomial. Here a generalized Newton-Raphson solution was used to obtain this root, but other methods may be investigated that may provide the same accuracy with less computational effort. The main advantage of the new approach is that near machine precision accuracy is obtained without evaluating transcendental functions or using lookup tables. A computational study showed that the approach obtains accurate solutions for the eccentric anomaly even for the difficult case involving high eccentricities.

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