MEETING ORBIT DETERMINATION REQUIREMENTS FOR A SMALL SATELLITE MISSION

Adonis Pimienta-Peñalver,* Richard Linares,† and John L. Crassidis‡
University at Buffalo, State University of New York, Amherst, NY, 14260-4400

A study is conducted with the goal of utilizing estimation techniques on measurements obtained from various onboard resources such as a Sun sensor, a magnetometer and a commercial GPS unit in order to approximate the true trajectory of the vehicle in realtime and minimize the error associated with the process. This carries significant relevance to the field of orbit determination, where a small mission could operate a relatively cheap system for tracking purposes. This paper models a GPS sensor to become available only 5 minutes each day, which approximates a worst-case scenario where sparse pseudorange data is only possible due to suboptimal operating conditions for commercial GPS receivers taken to the space environment. Using a dynamic propagation model, which includes effects of Earth's gravity, $J2$ zonal harmonics, and atmospheric drag, a sequential filtering method is used in order to estimate the states (position and velocity) of the vehicle with respect to time. This study demonstrates the capability of this system to achieve an error of approximately 15 meters, which is greatly influenced by the inclusion of a low-cost GPS module as an alternative to high end units that may be unaffordable for low-budget small satellite missions.

INTRODUCTION

The issue of tracking a spacecraft takes high importance on missions where accurate position, velocity and timing data are needed. Normally, the orbit determination problem is approached by applying a ground-based tracking system such as radar or optical measurement station networks. However, the possibility of implementing an autonomous system of orbit determination, where the system relies on measurements available/generated on the spacecraft itself, has proven to be an attractive and popular solution to the problem. This is especially true for small satellite missions where power consumption, computing capacity and financial budget are limited. Hence, choosing an appropriate tracking method that falls within budget and fulfills the requirements becomes essential.

A relatively common solution for orbit determination in small satellites is the use of an onboard GPS module. This approach provides a semi-autonomous system for the spacecraft to determine its orbit from position and velocity data yielded by the GPS unit. The system isn't truly autonomous since the mission relies on the availability and quality of the signal received from the satellites of the GPS constellation. GPS units offer a low cost and relatively easy method of orbit determination by allowing the satellite to generate its own orbital elements using the GPS measurements. It takes

*Graduate Student, Department of Mechanical & Aerospace Engineering. Email: arp27@buffalo.edu, Student Member AAS.
†Graduate Student, Department of Mechanical & Aerospace Engineering. Email: linares2@buffalo.edu, Member AAS.
‡Professor, Department of Mechanical & Aerospace Engineering. Email: johnc@buffalo.edu, Member AAS.

advantage of minimal resources to yield realtime position and velocity data using the computing power of an onboard microprocessor.

Often, however, the overall cost of implementing a GPS-based orbit determination system using a space-rated GPS unit is too high for small satellite missions to consider. In these cases commercial-off-the-shelf (COTS) GPS modules, which consist of commercially available units for the general public use, are utilized in lieu of their space-rated counterparts with the goal of saving costs. COTS GPS units experience a certain deviation from proper functioning once they enter the space environment. This effect could be the result of several reasons:

1. There are legal restrictions that the U.S. military imposes on the capabilities of GPS units intended for commercial use. These could include an intentional degradation of the unit's capabilities once it reaches certain altitudes and/or speeds for non-U.S.-military users.

2. Because of the intrinsically high dynamics of satellites, the GPS signals may suffer more from Doppler shift as the received frequencies can change rapidly.

3. GPS satellites may only be visible for relatively short periods of time from low-Earth orbit (LEO). Therefore, the initial acquisition phase, in which a GPS unit waits to detect a viable satellite signal, may be too slow on a satellite because a GPS satellite may become occulted by the Earth before the acquisition phase is finished.

4. Lastly, space-based receivers need electronic components that are fit for the space environment. Solar radiation, a significant factor in space, can cause 'single event upset' or 'latch-up' to occur in digital electronics. Components must withstand launch vibration, high frequencies and greater temperature swings than those usually found on the Earth.

These factors place limits on a unit's ability to obtain viable signals from a sufficient number of GPS satellites. Once they reach a certain altitude above Earth these factors often make the yielded measurements sparse for a large part of the spacecraft's orbit.[1] This issue is visited by Ref. [2] as well.

Reference [2] demonstrates the observability and accuracy of an autonomous orbit determination system based on onboard magnetometer and Sun sensor data. This system bases its orbit determination capability on comparing magnetometer measurements of the Earth's magnetic field with a spherical harmonic model of the field, as proposed by Psiaki et al,[2] and using attitude data generated by an onboard Sun sensor as shown by Ref. [3]. Reference [4] demonstrates the accuracy and efficiency of a low-power autonomous orbit determination system based on measurements from a GPS receiver. Achieved accuracies for these studies range from 8 to 125 km using magnetometer data alone, 10 to 35 km using Sun-sensor data alone and 78 to 144 m for LEO using GPS measurements alone. The difference in accuracy between various methods is due to the uncertainty in the truth models each assumes, e.g., the Earth's magnetic field suffers from significant field model uncertainty. On the other hand, accurate orbital dynamics models exist, which decreases the uncertainty of the process that relies solely on GPS measurements.

The goal of this study is to use estimation techniques on data obtained from various onboard resources such as Sun sensor, magnetometers and COTS GPS units in order to approximate and minimize the estimation error of the true trajectory of the vehicle. An alternate variation of the system which will be based on magnetometer and Sun sensor data will be utilized when GPS data isn't available. This problem is actually quite common and widely known as the dead-reckoning problem; it has counterparts in several other fields such as robotics, automotive and marine navigation. Dead reckoning is the technique of calculating the states of a process by means of estimation based on a previous state. The technique is affected greatly by the uncertainties and/or errors which can be accrued over time from the estimation routine. The present work extends on results achieved by previous studies to include an initial batch of GPS tracking data. This approach is particularly useful for many small satellite missions which, in fact, already include Sun sensors and magnetometers for attitude determination and control purposes, and low-cost GPS modules for orbit determination purposes. This represents a significant cut in monetary and computational costs for missions already limited in those departments.

The organization of this paper is as follows. First, the sensor model formulations utilized to simulate the outputs from a 3-axis magnetometer, a Sun sensor and a commercial-grade GPS are shown. Next, the orbital dynamics model formulation is discussed. This will be followed by a discussion of the simulation scheme, leading into an analysis of the simulation results. Finally conclusions are drawn, where the major results and considerations associated with the system are be condensed.

SENSOR MODELS

Because of the nonlinear nature of the models described in the following subsection, the problem is especially suited for nonlinear estimation techniques. This study uses an unscented Kalman filter. The basic estimation problem is formulated as using a continuous-time dynamics model and a discrete-time measurement model:[5]

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) + G(t)\mathbf{w}(t) \tag{1a}$$

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k,\, t_k) + \mathbf{v}_k \tag{1b}$$

The variable $\mathbf{x}$ represents the current state vector, while $\mathbf{f}$ represents the system model which depends on the current state, a forcing input $\mathbf{u}$ and time. The term $\mathbf{w}$ represents a zero-mean Gaussian noise process of the form: $\mathbf{w}(t) \sim \mathcal{N}(\mathbf{0}, Q(t))$. The term $\mathbf{y}_k$ represents the measurement vector and $\mathbf{v}_k$ represents a zero mean Gaussian measurement noise process of the form: $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k)$.

The model utilized to solve for the position (x, y, z) and clock bias τ from GPS measurements is given by the following system where ρ represents the GPS pseudorange measurement, τ is the clock bias and ν represents a zero-mean Gaussian measurement noise process:

$$\tilde{\rho} = \sqrt{x^2 + y^2 + z^2} + \tau + \nu \tag{2}$$

The state vector is given by

$$\mathbf{x} = \begin{bmatrix} x & y & z & \dfrac{dx}{dt} & \dfrac{dy}{dt} & \dfrac{dz}{dt} & \tau \end{bmatrix}^T_{\text{ECI}} \tag{3}$$

This study models the GPS measurements after a common off-the-shelf unit capable of providing position measurements down to an accuracy of 10 meters. Although this level will surely deviate in space due to factors put forth in the previous section, pseudorange data are provided every five minutes at the beginning of each day in order to represent a possible worse case scenario when GPS measurements become very sparse in the space environment. This measurement error is composed of several constituents: ionospheric delay, GPS satellite ephemeris and clock errors, and receiver generated noise. Multi-path error, although significant in terrestrial pseudo-range measurements, is not a significant error source for spacecraft.[4] The filter assumes that the pseudorange measurement error is a discrete-time white-noise process with zero mean and a standard deviation of 10 meters.

Three-axis magnetometers (TAMs) measure the local magnetic field ($\mathbf{b}_{\text{body}}$) in terms of the body coordinates of the spacecraft according to

$$\mathbf{b} = A\mathbf{b}_{\text{ECI}} \tag{4}$$

where the term $\mathbf{b}_{\text{ECI}}$ represents the local magnetic field in terms of Earth-centric inertial (ECI) components and the term A corresponds to the spacecraft's attitude matrix. Since this paper focuses on orbit determination, it is assumed *a priori* that the full attitude matrix is obtained from a precise independent sensor such as a star tracker. The TAM model including the observed attitude matrix is given by

$$\tilde{\mathbf{b}} = \tilde{A}\mathbf{b}_{\text{ECI}} + \mathbf{v}_{\text{b}} \tag{5}$$

where $\tilde{A}$ now denotes the attitude matrix given by the independent sensor that carries some error and $\mathbf{v}_{\text{b}}$ is the TAM error. The observed attitude is related to the true attitude by using a sequential rotation sequence with an error attitude, denoted by (δA):

$$\tilde{A} = \delta A\, A \tag{6}$$

Substituting Eq. (6) into Eq. (5) gives

$$\tilde{\mathbf{b}} = \delta A\, A\mathbf{b}_{\text{ECI}} + \mathbf{v}_{\text{b}} \tag{7}$$

Since small errors are assumed then the attitude error matrix can be approximated by

$$\delta A = I - [\mathbf{v}_{\text{A}}\times] \tag{8}$$

where $\mathbf{v}_{\text{A}}$ is a zero-mean Gaussian noise process with covariance R_a and $[\mathbf{v}_{\text{A}}\times]$ is the cross product matrix. This represents the covariance matrix associated with the independent attitude sensor. This error contributes to the additive noise that affects the accuracy of the known attitude matrix. The total error accumulated both from the independent sensor $(\mathbf{v}_{\text{A}})$ and from the magnetometer itself $(\mathbf{v}_{\text{b}})$ is required. Substituting Eq. (8) into Eq. (7) gives

$$\tilde{\mathbf{b}} = (A - [\mathbf{v}_{\text{A}}\times]A)\mathbf{b}_{\text{ECI}} + \mathbf{v}_{\text{b}} \tag{9}$$

Simplifying gives

$$\tilde{\mathbf{b}} = A\mathbf{b}_{\text{ECI}} + [A\mathbf{b}_{\text{ECI}}\times])\mathbf{v}_{\text{A}} + \mathbf{v}_{\text{b}} \tag{10}$$

Since $\mathbf{v}_{\text{A}}$ and $\mathbf{v}_{\text{b}}$ are uncorrelated then the covariance for $\tilde{\mathbf{b}}$ is given by

$$\text{cov}(\tilde{\mathbf{b}}) = R_{\text{b}} + [A\mathbf{b}_{\text{ECI}}\times]R_{\text{a}}[A\mathbf{b}_{\text{ECI}}\times]^T \tag{11}$$

The term $A\mathbf{b}_{\text{ECI}}$ can be replaced by $\tilde{\mathbf{b}}$ in practice, which leads to second-order error effects that can be neglected. The covariance matrix is thus represented by

$$\text{cov}(\tilde{\mathbf{b}}) = R_{\text{b}} + [\tilde{\mathbf{b}}\times]R_{\text{a}}[\tilde{\mathbf{b}}\times]^{\text{T}} \tag{12}$$

Note that the local magnetic field in ECI coordinates is a function of the ECI position of the spacecraft. The estimation process will entail obtaining spatial position data from measured TAM outputs modeled by Eq. (13). Although primarily an attitude determination device, TAM measurements can be applied for orbit determination purposes. The TAM measurement process consists of measuring the local components of the Earth's magnetic field. Using *a priori* knowledge of attitude, the unit normal vectors corresponding to the spacecraft's attitude can then be compared to the magnetic field unit vectors. The device may also be used as an orbit determination tool. This stems from the fact that the Earth's magnetic field levels, although subject to certain time-dependant variation, are specific to a position in space. This is the basic premise utilized in this paper to determine spatial position from TAM measurements. A more rigorous explanation and formulation of this topic is presented in Ref. [4]. The formulation of the local components of the Earth's magnetic field model from a known position follows the procedure set out in Refs. [6] and [7].

The basic Sun-sensor measurement equation to be incorporated into the estimation process is given by

$$\mathbf{s} = A\mathbf{s}_{\text{ECI}} \tag{13}$$

Also more commonly associated with attitude determination tasks, a Sun sensor measures the direction of the normal vector from the spacecraft's position which is directed towards the Sun. This is then used to calculate the attitude of the spacecraft by finding the difference between the measured reference and a previously known attitude. In our case, since the normal of the direction to the Sun is specific to a position in space, it allows us to use this coupling in order to predict where the vehicle is situated in space at that point in time. This process requires an orbit propagator in order to feed position data into a Sun model which will yield the position of the Sun in ECI coordinates. Similar to the TAM case, the estimation process entails simulating spatial position data from measured Sun sensor outputs. The Sun model measurement equation follows the more rigorously-developed procedure set out in Ref. [8].

In this paper, position is propagated forward in time using a dynamic gravitational model of the motion of a body orbiting in space in accordance with basic Newtonian formulations. Since the fidelity of the estimator depends heavily on the accuracy of the measurement model, this dynamic model is enhanced by adding forcing effects that can be accounted for in space, such as atmospheric drag and $J2$ zonal harmonics.

SYSTEM MODEL

The measurement output contains seven entries which correspond to pseudorange, magnetic field components and Sun sensor outputs. The measurement model is given by

$$\tilde{\mathbf{y}}_k \equiv \left[\tilde{\rho}_k \;\; \tilde{\mathbf{b}}_k^T \;\; \tilde{\mathbf{s}}_k^T\right]^T = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k \tag{14}$$

The dynamics model is given by

$$\dot{\mathbf{r}} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \mathbf{a}_{\text{drag}} + \mathbf{a}_{\text{J}_2} \tag{15}$$

where $\mathbf{r}$ is the spacecraft's ECI position, and $\mathbf{a}_{\text{drag}}$ and $\mathbf{a}_{\text{J}_2}$ correspond to atmospheric drag and Earth's zonal harmonics $J2$ perturbations to the motion of the vehicle, respectively. Also μ is the gravitational parameter of the Earth. The acceleration due to the J_2 effect is given by

$$\mathbf{a}_{J2} = \frac{3}{2}J_2\left(\frac{\mu}{r^2}\right)\left(\frac{R_\oplus}{r}\right)^2 \begin{bmatrix} \left(1 - 5\left(\frac{z}{r}\right)^2\right)\frac{x}{r} \\ \left(1 - 5\left(\frac{z}{r}\right)^2\right)\frac{y}{r} \\ \left(3 - 5\left(\frac{z}{r}\right)^2\right)\frac{z}{r} \end{bmatrix} \mathbf{r} \tag{16}$$

where J_2 is the coefficient for the second zonal harmonic and $R_\oplus$ is the mean equatorial radius of the Earth.

The following equations to model the two perturbations can be found in Ref. [8]. The symbols ρ , c_D, $\mathcal{A}$, m, and $\mathbf{v}_{\text{rel}}$ correspond to atmospheric density, coefficient of drag of the vehicle, cross-sectional area, the vehicle's mass, and the velocity of the vehicle relative to the rotating atmosphere frame. The quantities of atmospheric drag, mass and surface area are assigned common values for small satellites. The atmospheric density value is determined using an exponential atmospheric model which can be found in Ref. [8]:

$$\mathbf{a}_{\text{drag}} = -\frac{1}{2}\frac{c_D\mathcal{A}}{m}\rho\|\mathbf{v}_{\text{rel}}\|^2\left(\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|}\right) \tag{17}$$

The model is a simplified one since it only includes gravitational, drag and $J2$ effects. This simplification reduces the computational load of the filter, but it also impacts the filters performance. We attempt to minimize the computational load while still maintaining a certain level of estimation accuracy. Psiaki et al.[4] have shown that this modeling simplification can be tolerated in LEO orbits, though it causes issues as altitude increases where there are long data gaps during which no GPS satellites are visible. A more accurate gravity model could improve the performance, but could slow down the estimation process.

In addition to the well-known perturbation model shown in Eq. (15), three additional parameters are added to take account of the GPS clock bias (τ), the periods of time when GPS measurements are available (g), and the periods of time when the Earth eclipses the sunlight with respect to the vehicle (s). The latter two quantities, g and s, will switch between 1 for times when the measurement condition is true and 0 for times when the condition is false (no measurements from corresponding device).

SIMULATION SCHEME

The overall estimation process is shown in Figure 1. Initial conditions of position, time, and date are fed into the dynamic propagation model which accounts for gravitational, drag and $J2$ effects. The states yielded from the forward integration are then utilized to produce the simulated measurements from the respective sensor devices. In this case, the GPS measurements are modeled to only be observable for a mere 5 minutes once a day. The TAM sensor is modeled to be working continuously; while, naturally, the Sun sensor is modeled to generate measurements only when the vehicle is subject to illumination from the Sun during its orbit. Figure 4 shows an example of the availability of sensors. There is one sensor (TAM) active at all times, while the Sun and GPS sensors are only active periodically, the latter being operational for the first 5 minutes of each day. The estimator sequence is shown in Table 1.

ANALYSIS OF RESULTS

Figures 2 and 3 show the error simulation results generated using the estimation scheme shown in the previous section. Position estimates, which show convergence in roughly the span of an hour, demonstrate accuracies of approximately 15 m in worst-case scenarios. In contrast, results observed by Psiaki et al.[3] show position accuracies of 75 m in their best cases. These results show

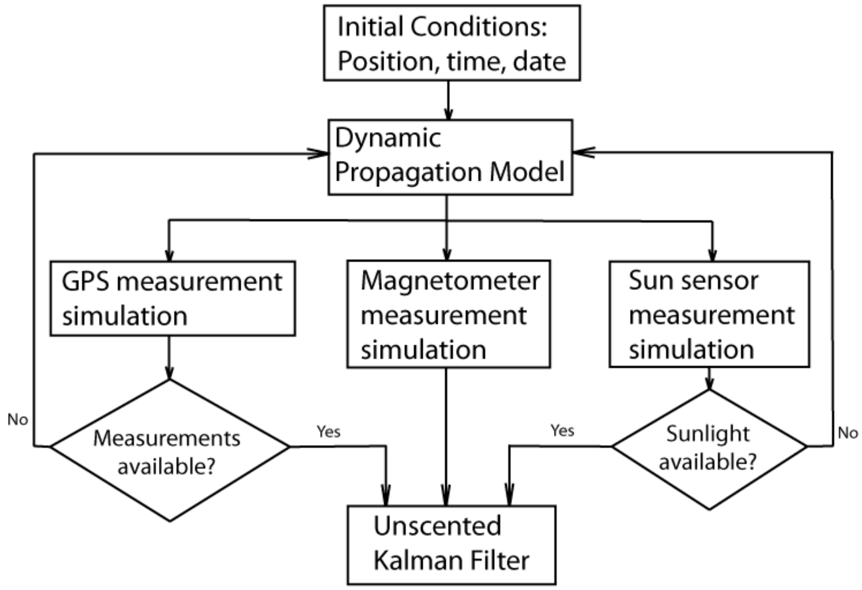

Figure 1. Estimation Process

Table 1. Unscented Kalman Filter Formulation

Initialize with $\hat{\mathbf{x}}_o = E\{\mathbf{x}_o\} \quad \mathbf{P}_o = E\{(\mathbf{x}_o - \hat{\mathbf{x}}_o)(\mathbf{x}_o - \hat{\mathbf{x}}_o)^T\}$
Calculate Sigma Points $\boldsymbol{\chi}_k = \left[\hat{\mathbf{x}}_k \quad \hat{\mathbf{x}}_k + \gamma\sqrt{P_{k-1}} \quad \hat{\mathbf{x}}_k - \gamma\sqrt{P_{k-1}}\right]$
Propagate Sigma Points $\boldsymbol{\chi}_k = \mathbf{f}\left[\boldsymbol{\chi}_{k-1}, t\right]$
Update $\hat{\mathbf{x}}_{k+1}^- = \sum_{i=0}^{2L} W_i^{\text{mean}} \boldsymbol{\chi}_{k+1}(i)$ $P_{k+1}^- = \sum_{i=0}^{2L} W_i^{\text{cov}} \left[\boldsymbol{\chi}_{k+1}(i) - \hat{\mathbf{x}}_{k+1}^-\right]\left[\boldsymbol{\chi}_{k+1}(i) - \hat{\mathbf{x}}_{k+1}^-\right]^T + Q_{k+1}$
Measurement update $\boldsymbol{\gamma}_k(i) = \mathbf{h}\left[\boldsymbol{\chi}_k(i), \hat{\mathbf{q}}_k^-\right]$ $\hat{\mathbf{y}}_k^- = \sum_{i=0}^{2L} W_i^{\text{mean}} \boldsymbol{\gamma}_k(i)$ $P_k^{yy} = \sum_{i=0}^{2L} W_i^{\text{cov}} \left[\boldsymbol{\gamma}_k(i) - \hat{\mathbf{y}}_k^-\right]\left[\boldsymbol{\gamma}_k(i) - \hat{\mathbf{y}}_k^-\right]^T$ $P_k^{vv} = P_k^{yy} + R_k$ $P_k^{xy} = \sum_{i=0}^{2L} W_i^{\text{cov}} \left[\boldsymbol{\chi}_k(i) - \hat{\mathbf{x}}_k^-\right]\left[\boldsymbol{\gamma}_k(i) - \hat{\mathbf{y}}_k^-\right]^T$ $K_k = P_k^{xy}(P_k^{vv})^{-1}$ $\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k\left[\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k\right]$ $P_k^+ = P_k^- - K_k P^{vv} K_k^T$

the advantage of utilizing the proposed trio of GPS, TAM and Sun sensors as a low-cost, viable alternative of orbit determination in small satellite missions. The above results are produced with

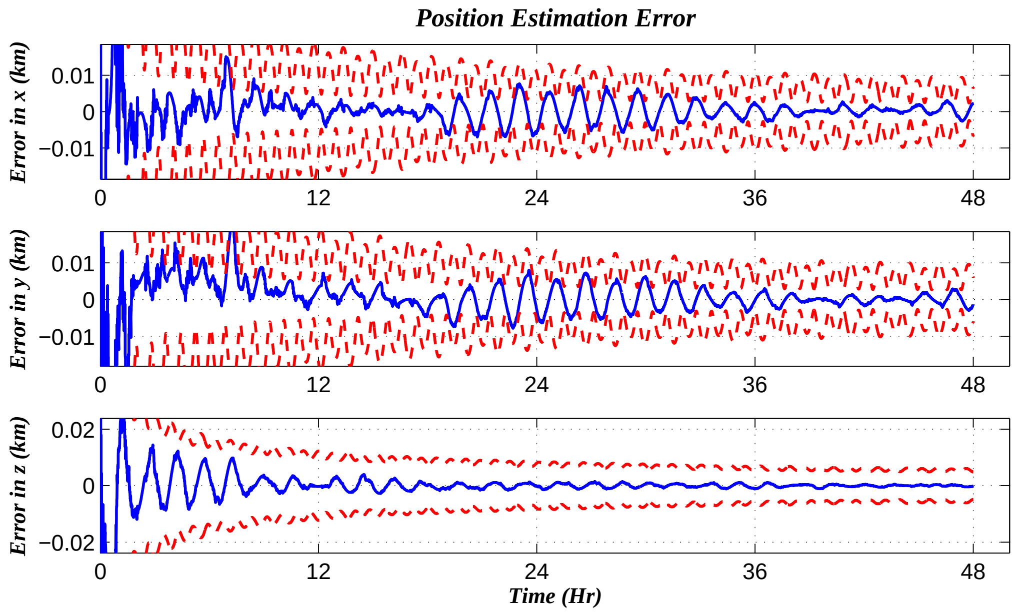

Figure 2. Position Estimation Errors

a TAM measurement error of 10 nT, Sun sensor accuracy of 0.005 deg, a reference orbit with 7,000 km, 0.011 eccentricity, and 10.5 deg of inclination. It is also seen in Figure 2 that at certain times the system begins to experience a gradual degradation of the estimation error. This effect is due to the prolonged absence of GPS data, which is in agreement with what is observed in Ref. [2], where an orbit determination system which relies on TAM and Sun sensor outputs achieves lower levels of accuracy. This follows intuition that after a long enough time, the estimates in our study would begin to degrade to those levels found in the aforementioned study. Once a new batch of GPS data is available, however, the estimates regain their previous levels of accuracy to their lowest values.

In order to demonstrate the advantage of using a commercial GPS unit for orbit determination purposes simulations are conducted without simulated GPS pseudorange data. The study shows that the accuracy associated with utilizing a GPS unit improves greatly, as the study shows a decrease of the 3σ error bounds from values on the kilometer to meter level accuracy. The convergence property of the orbit estimator degrades greatly as GPS pseudorange measurements vanish, i.e., estimates without GPS fail to ever remain within the kilometer-level error. Figures 5 and 6 illustrate this phenomenon.

A study is also performed to show the dependence on the orbital parameters of the vehicle. It is observed no major variation of the results with varied values of eccentricity or inclination; however, it is important to note that the estimation error in the y-ECI coordinate degrades slightly as the orbit of the vehicle becomes polar or nearly polar. This effect is expected because of the marked variation of Earth's magnetic fields over the polar regions.

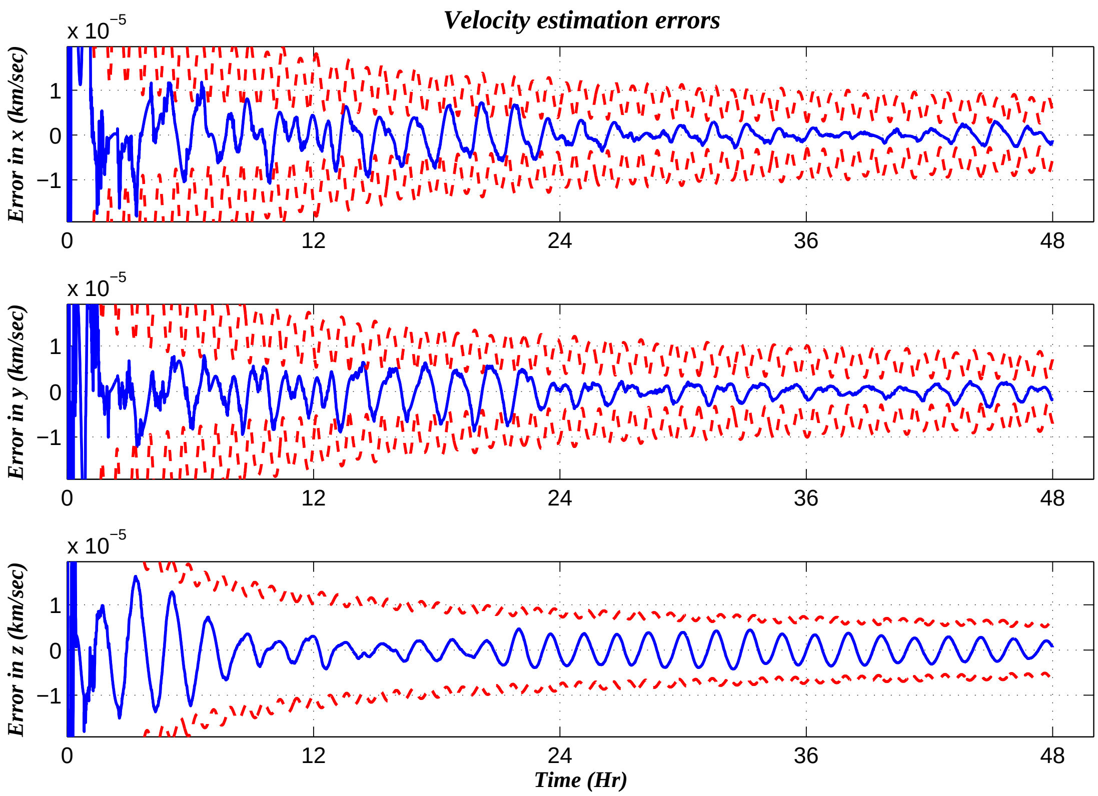

Figure 3. Velocity Estimation Errors

CONCLUSION

The system demonstrates the ability to converge to errors of approximately 15 meters. This outcome improves on results obtained in preceding publications and shows the practical advantage of using an orbit determination system which includes the three inexpensive and off-the-shelf sensors. At certain times a gradual degradation of the accuracy in the estimates is seen which is due to the prolonged absence of GPS data. Once a new batch of GPS data is available, however, the estimates regain their previous levels of accuracy to their lowest values. Convergence rates depend on the presence of GPS observations. Variations of the orbital parameters of the vehicle studied in LEO do not significantly alter the accuracy of the previously obtained results. However, it is understood that the effect of the uncertainties in the assumed models can result in varying degrees of estimation performance. In the future further studies will be conducted to refine the dynamic propagation model and sensor measurement models in order to increase robustness and accuracy while keeping in mind computational efficiency.

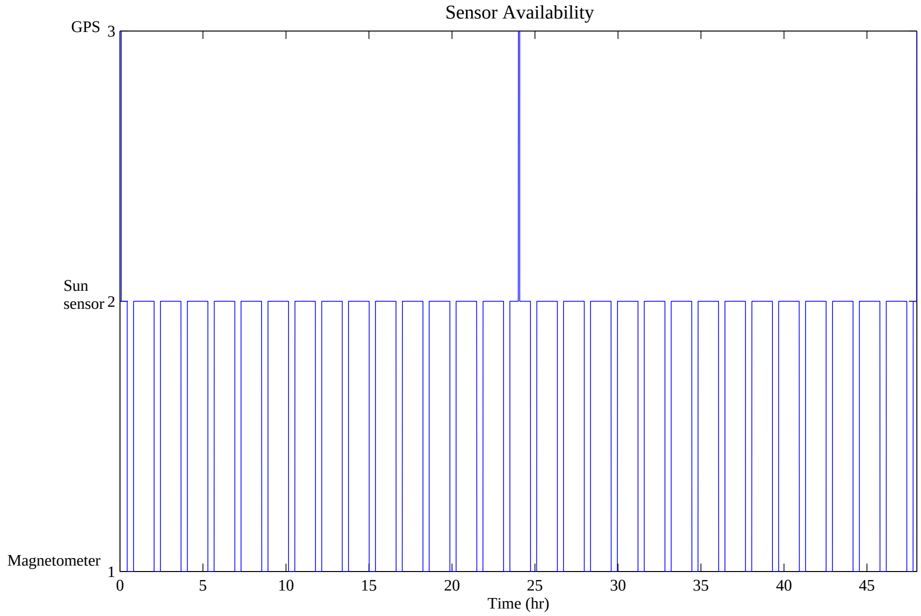

Figure 4. Available Sensors Over Time

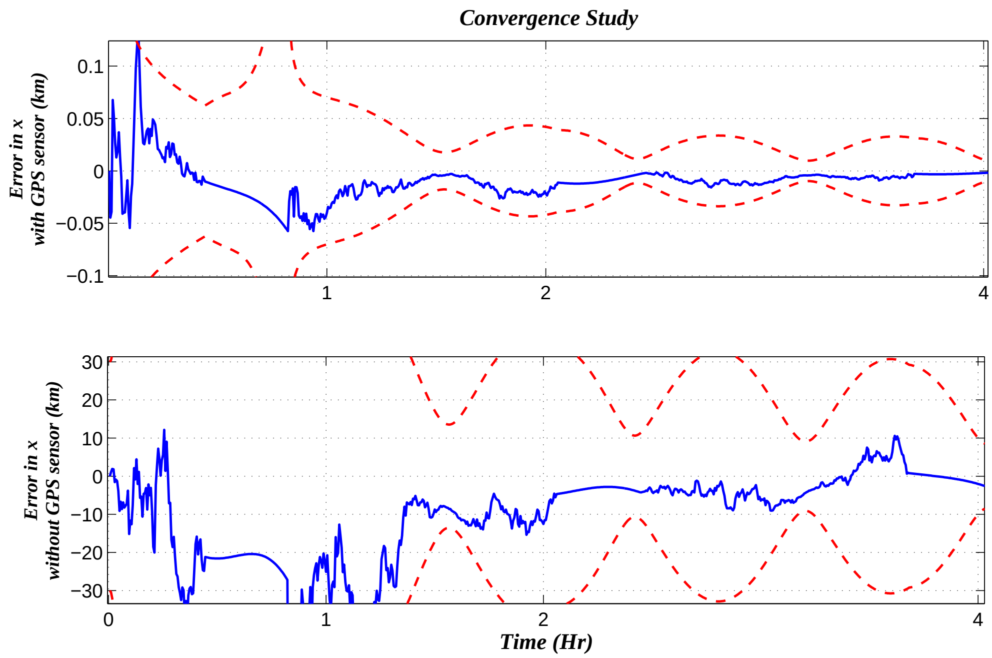

Figure 5. Convergence Comparison

REFERENCES

[1] Lightsey, E. G., Blackburn, G. C., and Simpson, J. E., "Going Up-A GPS Receiver Adapts to Space," *GPS World*, Vol. 11, No. 9, 2000, pp. 30–35.

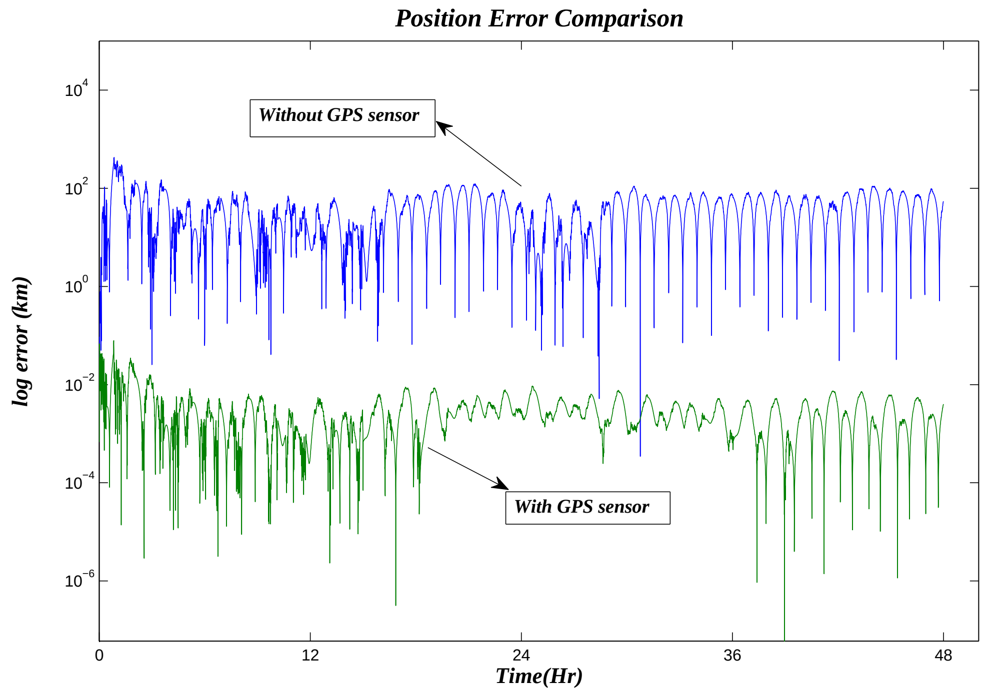

Figure 6. Error Comparison Between a Configuration With and Without GPS Measurements

[2] Psiaki, M. L., "Autonomous Orbit and Magnetic Field Determination Using Magnetometer and Star Sensor Data." *Journal of Guidance Control Dynamics*, Vol. 18, 1995, pp. 584–592.

[3] Babcock, E. P., "CubeSat Attitude Determination via Kalman Filtering of Magnetometer and Solar Cell Data," *25th AIAA/USU Conference on Small Satellites*, 2011.

[4] Martel, F., Pal, P. K., and Psiaki, M. L., "Three-Axis Attitude Determination via Kalman Filtering of Magnetometer Data," *Flight Mechanics/Estimation Theory Symposium 1988*, Vol. 1, 1988, pp. 344–367.

[5] Crassidis, J. L. and Junkins, J. L., *Optimal Estimation of Dynamic Systems*, Chapman & Hall, Boca Raton, FL, 2011.

[6] Mueller, A. C., "A Fast Recursive Algorithm for Calculating the Forces Due to the Geopotential," Tech. Rep. 75-FM-42, NASA JSC Internal Note, June 9 1975.

[7] Lundberg, J. B. and Schutz, B. E., "Recursion Formulas of Legendre Functions for Use with Nonsingular Geopotential Models," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 32-38, Jan.-Feb. 1998.

[8] Vallado, D. A., *Fundamentals of Astrodynamics and Applications*, Microcosm Press, Hawthorne, CA and Springer, New York, NY, 3rd ed., 2007.