Quaternion Data Fusion

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A numerical method for solving a class of constrained minimization problems encountered in quaternion data fusion is presented. The quaternion constraints are handled by the method of Lagrange multipliers. The number of the stationary points of the minimization problem is finite and all of them are found by solving via homotopy continuation a system of polynomial equations. The global minimum is the stationary point that minimizes the loss function of the minimization problem. A numerical example of two-quaternion data fusion is given to illustrate the viability of the method as a global minimization method for quaternion data fusion.

I. INTRODUCTION

Attitude quaternion [1] is the attitude parameterization of choice for spacecraft attitude estimation for several reasons: 1) it is free of singularities, 2) the attitude matrix is quadratic in the quaternion components, and 3) the kinematics equations is bilinear and an analytic solution exists for the propagation. However, the components of the attitude quaternion are not independent of each other and the norm of the attitude quaternion must be unity. This unity-norm constraint leads to problems for data fusion involving quaternions. The objective of data fusion is to find the optimal estimate from data of various sources.

Reference [2] addresses the problem of fusing or averaging a set of quaternions. The fused or averaged quaternion is defined as the optimal solution to a constrained minimization problem subject to one equality constraint (the quaternion constraint). The method of Lagrange multipliers is used to convert the constrained minimization problem to an unconstrained minimization problem. The Lagrange multiplier is the maximum eigenvalue of a 4×4 symmetric matrix composed from the quaternions and weights [3]. The optimal average quaternion is the eigenvector corresponding to the maximum eigenvalue [3].

Reference [3] addresses a more general data fusion problem in which the state vector of interest consists of one quaternion and a set of unconstrained parameters, for example, gyro biases. The data fusion problem is also formulated as a constrained minimization problem subject to one equality constraint. The Lagrange multiplier is now the maximum eigenvalue of an 8×8 asymmetric matrix or the maximum root of the 8^{th}-degree secular equation [3]. Given the Lagrange multiplier, the optimal state estimate is obtained by solving a linear system of equations [3].

This paper addresses an even more general data fusion problem in which the state vector includes two or more quaternions as well as a set of unconstrained parameters. Such a problem appears in formation flying involving multiple vehicles in which two or more

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relative attitudes need to be determined or fused [4]. The data fusion problem is again defined as a constrained minimization problem, but now subject to two or more equality constraints. The multiple quaternion constraints significantly increase the difficulty of the minimization problem. To our knowledge, no closed form solution exists for the minimization problem and few properties of the Lagrange multipliers are known. The latter makes it difficult to choose the initial guess of the Lagrange multipliers when solving the problem using an iterative method.

In this paper, a numerical method is presented for the constrained minimization problem subject to multiple quaternion constraints based on the solution of polynomial systems. Unlike the iterative gradient-based methods, which can only find a local minimum of the minimization problem, this method first finds all the stationary points and then selects the global minimum from them. In addition, the method provides insights to the properties, the number of local minima in particular, of the minimization problem.

The organization of the remainder of this paper is as follows. First, one-quaternion data fusion is reviewed. Next, the problem statement and formal solution of the multi-quaternion data fusion problem are given. Then, the numerical solution of the polynomial system is presented. Finally, a numerical example is given, followed by the conclusions.

II. One-Quaternion Data Fusion

The objective of data fusion is to fuse n estimates $\mathbf{x}_i$ of a state vector $\mathbf{x}$ to yield a single (better) estimate of the state vector. Throughout this paper, it is assumed that quaternion is part of the state vector and that the optimal estimate is the solution to a constrained minimization problem of which the loss function is quadratic in the state vector. While the solution to one-quaternion data fusion problem has been studied in [2, 3], the problems and solutions of one-quaternion data fusion are reviewed in this section for sake of completeness.

The vector and scalar parts of a quaternion are defined by $\mathbf{q} \triangleq \begin{bmatrix} \boldsymbol{\varrho}^T & q_4 \end{bmatrix}^T$, which are assumed to satisfy the unity-norm constraint $||\boldsymbol{\varrho}||^2 + q_4^2 = \mathbf{q}^T\mathbf{q} = 1$. The attitude matrix is related to the quaternion by

$$A(\mathbf{q}) = \Xi^T(\mathbf{q})\Psi(\mathbf{q}) = \left(q_4^2 - ||\boldsymbol{\varrho}||^2\right) I_{3\times3} + 2\,\boldsymbol{\varrho}\,\boldsymbol{\varrho}^T - 2\,q_4[\boldsymbol{\varrho}\times] \tag{1}$$

where $I_{3\times3}$ is a 3×3 identity matrix and $[\boldsymbol{\varrho}\times]$ is the cross-product matrix defined by

$$[\boldsymbol{\varrho}\times] \triangleq \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{2}$$

$$\Xi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix}, \Psi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{3}$$

The simplest case of one-quaternion data fusion is now reviewed, where the i^{th} state estimate $\mathbf{x}_i = \mathbf{q}_i$. The loss function is chosen as [2]

$$J(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} \mathbf{q}^T\Xi(\mathbf{q}_i)\,\mathcal{W}_{qq_i}\Xi^T(\mathbf{q}_i)\mathbf{q} \tag{4}$$

where $\mathcal{W}_{qq_i}$ is a 3×3 positive definite weighting matrix. The three-dimensional vector $\Xi^T(\mathbf{q}_i)\mathbf{q}$ has been widely to measure the attitude error in spacecraft attitude estimation and the magnitude of $\Xi^T(\mathbf{q}_i)\mathbf{q}$ is the absolute value of the sine of the half-error angle [2]. The quaternion constraint $\mathbf{q}^T\mathbf{q} = 1$ is handled using the method of Lagrange multipliers, which gives the augmented loss function as

$$J(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} \mathbf{q}^T \Xi(\mathbf{q}_i)\, \mathcal{W}_{qq_i} \Xi^T(\mathbf{q}_i)\mathbf{q} + \frac{\lambda}{2}(\mathbf{q}^T\mathbf{q} - 1) \tag{5}$$

where λ is the Lagrange multiplier. The necessary conditions for minimization of Eq. (5) are

$$(\mathcal{F} + \lambda I_{4\times 4})\,\mathbf{q} = \mathbf{0} \tag{6a}$$

$$\mathbf{q}^T\mathbf{q} = 1 \tag{6b}$$

where

$$\mathcal{F} \triangleq \sum_{i=1}^{n} \Xi(\mathbf{q}_i)\, \mathcal{W}_{qq_i} \Xi^T(\mathbf{q}_i) \tag{7}$$

A vector satisfying the the necessary conditions is called a stationary point. Since the loss function for the quaternion satisfying Eq. (6) equals $-\lambda/2$, the optimal average quaternion is the eigenvector corresponding to the maximum eigenvalue of $-\mathcal{F}$.

When the i^{th} state estimate $\mathbf{x}_i$ is composed of a quaternion $\mathbf{q}_i$, which is subject to the constraint $\mathbf{q}_i^T\mathbf{q}_i = 1$, and other quantities $\mathbf{b}_i$, which are free of constraints, the loss function is given by

$$J(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} \Delta\mathbf{x}_i^T \mathcal{W}_i \Delta\mathbf{x}_i \tag{8}$$

with

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix} \tag{9a}$$

$$\Delta\mathbf{x}_i \triangleq \begin{bmatrix} \Xi^T(\mathbf{q}_i)\mathbf{q} \\ \\ \mathbf{b} - \mathbf{b}_i \end{bmatrix} \tag{9b}$$

$$\mathcal{W}_i \triangleq \begin{bmatrix} \mathcal{W}_{qq_i} & \mathcal{W}_{qb_i} \\ \\ \mathcal{W}_{qb_i}^T & \mathcal{W}_{bb_i} \end{bmatrix} \tag{9c}$$

where $\mathcal{W}_i$ is positive definite. The vector $\mathbf{b}$ can be of any dimension, denoted by n_b, and the same attitude error expression $\Xi^T(\mathbf{q}_i)\mathbf{q}$ has been used as in the previous case. Note that for nonzero $\mathcal{W}_{qb_i}$, $[\mathbf{q}^T, \mathbf{b}^T]^T$ and $[-\mathbf{q}^T, \mathbf{b}^T]^T$ yield different values of the loss function, although $\mathbf{q}$ and $-\mathbf{q}$ represent the same attitude.

The augmented loss function is now

$$J(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} \Delta\mathbf{x}_i^T \mathcal{W}_i \Delta\mathbf{x}_i + \frac{\lambda}{2}(\mathbf{q}^T\mathbf{q} - 1) \tag{10}$$

The necessary conditions for minimization of Eq. (10) are

$$\begin{bmatrix} \mathcal{B}_{qq} + \lambda I_{4\times 4} & \mathcal{B}_{qb} \\ \\ \mathcal{B}_{qb}^T & \mathcal{B}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \\ \mathbf{d} \end{bmatrix} \tag{11a}$$

$$\mathbf{q}^T\mathbf{q} = 1 \tag{11b}$$

where

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{qq} & \mathcal{B}_{qb} \\ \\ \mathcal{B}_{qb}^T & \mathcal{B}_{bb} \end{bmatrix} \triangleq \begin{bmatrix} \sum_{i=1}^n \Xi(\mathbf{q}_i)\mathcal{W}_{qq_i}\Xi^T(\mathbf{q}_i) & \sum_{i=1}^n \Xi(\mathbf{q}_i)\mathcal{W}_{qb_i} \\ \\ \sum_{i=1}^n \mathcal{W}_{qb_i}^T\Xi^T(\mathbf{q}_i) & \sum_{i=1}^n \mathcal{W}_{bb_i} \end{bmatrix} \tag{12a}$$

$$\mathbf{c} \triangleq \sum_{i=1}^n \Xi(\mathbf{q}_i)\mathcal{W}_{qb_i}\mathbf{b}_i \tag{12b}$$

$$\mathbf{d} \triangleq \sum_{i=1}^n \mathcal{W}_{bb_i}\mathbf{b}_i \tag{12c}$$

Solving the second subequation of Eq. (11a) leads to

$$\mathbf{b} = \mathcal{B}_{bb}^{-1}(\mathbf{d} - \mathcal{B}_{qb}^T\mathbf{q}) \tag{13}$$

With $\mathbf{b}$ eliminated using Eq. (13), Eq. (11) reduces to

$$(\mathcal{G} + \lambda I_{4\times4})\mathbf{q} = \mathbf{g} \tag{14a}$$

$$\mathbf{q}^T\mathbf{q} = 1 \tag{14b}$$

where

$$\mathcal{G} \triangleq \mathcal{B}_{qq} - \mathcal{B}_{qb}\mathcal{B}_{bb}^{-1}\mathcal{B}_{qb}^T \tag{15a}$$

$$\mathbf{g} \triangleq \mathbf{c} - \mathcal{B}_{qb}\mathcal{B}_{bb}^{-1}\mathbf{d} \tag{15b}$$

Equation (14) is the key equation to solve in one-quaternion data fusion. Given $\mathbf{x}_i$ and $\mathcal{W}_i$, the one-quaternion data fusion procedure is as follows:

1. Compute $\mathcal{B}_{qq}$, $\mathcal{B}_{qb}$, $\mathcal{B}_{bb}$, $\mathbf{c}$, $\mathbf{d}$ using Eq. (12)
2. Compute $\mathcal{G}$ and $\mathbf{g}$ using Eq. (15)
3. Solve Eq. (14) for the optimal $\mathbf{q}^*$ and λ^*
4. Compute the optimal $\mathbf{b}^*$ using Eq. (13)

Note that for nonzero $\mathbf{g}$, $\mathbf{q}^*$ and $-\mathbf{q}^*$ cannot both be solutions of Eq. (14). The optimal λ^* is known to be the maximum real eigenvalue of an 8×8 asymmetric matrix or the maximum root of the 8^{th}-degree secular equation [3]. Given the optimal λ^*, the optimal $\mathbf{q}^*$ can be obtained by solving $(\mathcal{G} + \lambda^* I_{4\times4})\mathbf{q}^* = \mathbf{g}$ [3].

III. Multi-Quaternion Data Fusion

In multi-quaternion data fusion, the state vector is assumed to consist of m quaternions $\mathbf{q}^{(j)}$, $j = 1, \ldots, m$, with $\mathbf{q}^{(j)T}\mathbf{q}^{(j)} = 1$, and a set of unconstrained quantities $\mathbf{b}$. Define

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{Q} \\ \\ \mathbf{b} \end{bmatrix}, \mathbf{x}_i \triangleq \begin{bmatrix} \mathbf{Q}_i \\ \\ \mathbf{b}_i \end{bmatrix}, \mathbf{Q} \triangleq \begin{bmatrix} \mathbf{q}^{(1)} \\ \vdots \\ \mathbf{q}^{(m)} \end{bmatrix}, \mathbf{Q}_i \triangleq \begin{bmatrix} \mathbf{q}_i^{(1)} \\ \vdots \\ \mathbf{q}_i^{(m)} \end{bmatrix}, \Sigma(\mathbf{Q}_i) \triangleq \begin{bmatrix} \Xi\left(\mathbf{q}_i^{(1)}\right) & & \\ & \ddots & \\ & & \Xi\left(\mathbf{q}_i^{(m)}\right) \end{bmatrix} \tag{16}$$

Note that $\Sigma(\mathbf{Q}_i)$ is a block-diagonal matrix. The loss function is of the form of Eq. (8), with

$$\Delta\mathbf{x}_i \triangleq \begin{bmatrix} \Sigma^T(\mathbf{Q}_i)\mathbf{Q} \\ \\ \mathbf{b}-\mathbf{b}_i \end{bmatrix} \tag{17a}$$

$$\mathcal{W}_i \triangleq \begin{bmatrix} \mathcal{W}_{QQ_i} & \mathcal{W}_{Qb_i} \\ \\ \mathcal{W}_{Qb_i}^T & \mathcal{W}_{bb_i} \end{bmatrix} \tag{17b}$$

Augmenting the constraint function with the m quaternion constraints gives

$$J(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} \Delta\mathbf{x}_i^T \mathcal{W}_i \Delta\mathbf{x}_i + \frac{1}{2}\sum_{j=1}^{m} \lambda_j \left(\mathbf{q}^{(j)T}\mathbf{q}^{(j)} - 1\right) \tag{18}$$

with λ_j, $j = 1, \ldots, m$, the Lagrange multipliers.

The necessary conditions for minimization of Eq. (18) are

$$\begin{bmatrix} \mathcal{B}_{QQ}+\Lambda & \mathcal{B}_{Qb} \\ \\ \mathcal{B}_{Qb}^T & \mathcal{B}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{Q} \\ \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \\ \mathbf{d} \end{bmatrix} \tag{19a}$$

$$\mathbf{q}^{(j)T}\mathbf{q}^{(j)} = 1,\ j = 1, \ldots, m \tag{19b}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 I_{4\times 4} & & \\ & \ddots & \\ & & \lambda_m I_{4\times 4} \end{bmatrix} \tag{20}$$

$$\mathcal{B}_{QQ} \triangleq \sum_{i=1}^{n} \Sigma(\mathbf{Q}_i)\mathcal{W}_{QQ_i}\Sigma^T(\mathbf{Q}_i) \tag{21a}$$

$$\mathcal{B}_{Qb} \triangleq \sum_{i=1}^{n} \Sigma(\mathbf{Q}_i)\mathcal{W}_{Qb_i} \tag{21b}$$

$$\mathcal{B}_{bb} \triangleq \sum_{i=1}^{n} \mathcal{W}_{bb_i} \tag{21c}$$

$$\mathbf{C} \triangleq \sum_{i=1}^{n} \Sigma(\mathbf{Q}_i)\mathcal{W}_{Qb_i}\mathbf{b}_i \tag{21d}$$

$$\mathbf{d} \triangleq \sum_{i=1}^{n} \mathcal{W}_{bb_i}\mathbf{b}_i \tag{21e}$$

Solving the second subequation of Eq. (19a) leads to

$$\mathbf{b} = \mathcal{B}_{bb}^{-1}(\mathbf{d} - \mathcal{B}_{Qb}^T\mathbf{Q}) \tag{22}$$

With $\mathbf{b}$ eliminated using Eq. (22), Eq. (19a) reduces to

$$(\mathcal{H}+\Lambda)\mathbf{Q}=\mathbf{h} \tag{23a}$$

$$\mathbf{q}^{(j)T}\mathbf{q}^{(j)}=1,\ j=1,\ldots,m \tag{23b}$$

where

$$\mathcal{H}\triangleq\mathcal{B}_{QQ}-\mathcal{B}_{Qb}\mathcal{B}_{bb}^{-1}\mathcal{B}_{Qb}^{T} \tag{24a}$$

$$\mathbf{h}\triangleq\mathbf{C}-\mathcal{B}_{Qb}\mathcal{B}_{bb}^{-1}\mathbf{d} \tag{24b}$$

Equation (23) is the key equation to solve in multi-quaternion data fusion. Given $\mathbf{x}_i$ and $\mathcal{W}_i$, the multi-quaternion data fusion procedure is as follows:

1. Compute $\mathcal{B}_{QQ}$, $\mathcal{B}_{Qb}$, $\mathcal{B}_{bb}$, $\mathbf{C}$, $\mathbf{d}$ using Eq. (21)
2. Compute $\mathcal{H}$ and $\mathbf{h}$ using Eq. (24)
3. Solve Eq. (23) for the optimal $\mathbf{Q}^*$ and λ_j^*, $j=1,\ldots,m$
4. Compute the optimal $\mathbf{b}^*$ using Eq. (22)

Note that for nonzero $\mathbf{h}$, if $\mathbf{Q}^*$ is the solution of Eq. (23), its "conjugate" with one or more $\mathbf{q}^{(j)*}$ replaced by $-\mathbf{q}^{(j)*}$ is not a solution of Eq. (23). Equation (23) or the equivalent of it may be solved using an iterative gradient-based algorithm, but it cannot guarantee that $\mathbf{Q}^*$ and λ_j^* are globally optimal. Since little is known about λ_j^*, choosing an appropriate initial guess for the gradient-based algorithm is not easy.

Noting that Eq. (23) is a system of polynomial equations in $\mathbf{Q}$ and λ_j, we are motived to solve Eq. (23) using the homotopy continuation method [5, 6], which is capable of finding all the isolated solutions, real and complex, of the polynomial system. The globally optimal $\mathbf{Q}^*$ and λ^* are then provided by the real solution that minimizes $J(\mathbf{x})$. The detail of the numerical solution is given in the next section.

IV. Numerical Solution for Multi-Quaternion Data Fusion

Define the $5m$-dimensional vector

$$\mathbf{y}\triangleq\begin{bmatrix}\mathbf{Q}\\ \boldsymbol{\lambda}\end{bmatrix}=\begin{bmatrix}\mathbf{q}^{(1)}\\ \vdots\\ \mathbf{q}^{(m)}\\ \lambda_1\\ \vdots\\ \lambda_m\end{bmatrix} \tag{25}$$

where m is the number of quaternions in the state vector. The $5m$-dimensional vector $\mathbf{y}$ is required to satisfy the following $5m$ polynomial equations (also Eq. (23)):

$$(\mathcal{H}+\Lambda)\mathbf{Q}=\mathbf{h} \tag{26a}$$

$$\mathbf{q}^{(j)T}\mathbf{q}^{(j)}=1,\ j=1,\ldots,m \tag{26b}$$

In general, this polynomial system has a finite number of solutions, which can be all found via homotopy continuation [5,6]. The idea of homotopy continuation is to cast the target polynomial system in a parameterized family of systems, one of which (the start system) has known or easily found solutions [7,8]. After choosing this family, one chooses a path from the start system to the target system, constructs a homotopy between the two, and tracks the solution paths. The homotopy continuation method finds all the real and complex solutions to a polynomial system. Since only real solutions of the polynomial system are of interest, the complex solutions found by the solver are discarded.

For illustration purposes, an example of homotopy continuation of one equation in one unknown is given [7]. To solve

$$f(x) = x^5 + ax + b = 0 \tag{27}$$

where a and b are two constants, one may construct the continuation equation as

$$h(x,t) = x^5 + atx + [tb - (1-t)q^5], \quad 0 \leq t \leq 1 \tag{28}$$

where q is a complex constant. The target system (the original equation) corresponds to $t = 1$ and the start system corresponds to $t = 0$, given by

$$h(x,0) = x^5 - q^5 \tag{29}$$

The five complex solutions to the start system is obvious. The solution of $h(x,t)$ can be viewed as a function of t, denoted by $x(t)$, $0 \leq t \leq 1$. Geometrically, $x(t)$ are paths originating from $x(0)$ and ending at $x(1)$. With $x(0)$ given, $x(\Delta t)$, where Δt is a small step size, can be found using the Euler or Newton method. Step by step, the paths are tracked until $t = 1$.

Although the idea of homotopy continuation is simple, several important issues need to be handled with care. These include but are not limited to determination of the number of paths, path crossing, path divergence to infinity, and singular solutions [7]. A singular solution has a singular Jacobian matrix, defined as the derivatives of the equations with respect to the unknowns [7]. Thanks to homotopy continuation based solvers such as HOM4PS [6], Bertini [5], PHCpack [9], and HomLab [8], solving a system of polynomial equations via homotopy continuation is easy for a user, who only needs to provide a model description of the polynomial system that can be processed by the solver but does not need to provide a starter system.

By definition, the real solutions of the polynomial system in Eq. (26) are the stationary points of the constrained minimization problem. The local minimizers of the minimization problem are the stationary points with positive definite Hessian. Because the loss function is quadratic in the attitude quaternions, the Hessian without considering the m quaternion constraints is $(\mathcal{H}+\Lambda^*)$, where Λ^* is determined by the $\boldsymbol{\lambda}^*$ in $\mathbf{y}^*$. The dimension of $(\mathcal{H}+\Lambda^*)$ is $(4m) \times (4m)$. Since one attitude only has three degrees of freedom and m attitudes only have $3m$ degrees of freedom, a $(3m) \times (3m)$ Hessian is defined as

$$H^* = \mathcal{P}^{*T}(\mathcal{H} + \Lambda^*)\mathcal{P}^* \tag{30}$$

with the $(3m) \times (4m)$ $\mathcal{P}^*$ satisfying

$$\mathcal{P}^{*T}\mathcal{P}^* = I_{(3m)\times(3m)} \tag{31}$$

$$\mathcal{P}^{*T}\mathcal{C}^* = 0_{(3m)\times m} \tag{32}$$

where

$$\mathcal{C}^* = \begin{bmatrix} \mathbf{q}^{*(1)} & \cdots & \mathbf{0}_{4\times 1} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{4\times 1} & \cdots & \mathbf{q}^{*(m)} \end{bmatrix} \tag{33}$$

The columns of $\mathcal{C}^*$ correspond to the gradient of the quaternion constraints. The matrix $\mathcal{P}^*$ can be found using the QR decomposition of $\mathcal{C}^*$. A stationary point is a local minimizer if H^* is positive definite, which is equivalent to that the eigenvalues of $(\mathcal{H} + \Lambda^*)$ corresponding to the eigenvectors perpendicular to the columns of $\mathcal{C}^*$ are positive.

A stationary point or local minimizer is the global minimizer if gives the global minimum of the loss function of multi-quaternion data fusion, whose data-dependent part is equivalent to

$$J_{eq}(\mathbf{Q}) = \frac{1}{2}\mathbf{Q}^T\mathcal{H}\mathbf{Q} - \mathbf{h}^T\mathbf{Q} \tag{34}$$

The loss functions J and J_{eq} have the same stationary points and minima. Evaluating the equivalent loss function at a local minimum and substituting Eq. (26) gives

$$\begin{aligned} J_{eq}(\mathbf{Q}^*) &= \frac{1}{2}\mathbf{Q}^{*T}\mathcal{H}\mathbf{Q}^* - \mathbf{h}^T\mathbf{Q}^* \\ &= \frac{1}{2}\mathbf{Q}^{*T}\mathcal{H}\mathbf{Q}^* - \mathbf{Q}^{*T}(\mathcal{H} + \Lambda^*)\mathbf{Q}^* \\ &= -\frac{1}{2}\mathbf{Q}^{*T}\mathcal{H}\mathbf{Q}^* - \sum_{j=1}^{m}\lambda_j^* \end{aligned} \tag{35}$$

An alternative form of J_{eq} is given by

$$J_{eq} = -\frac{1}{2}\mathbf{Q}^{*T}\mathbf{h} - \frac{1}{2}\sum_{j=1}^{m}\lambda_j^* \tag{36}$$

To find the globally optimal $\mathbf{Q}^*$ and λ_j^*, $j = 1, \ldots, m$, the following procedure is used:

1. Find all real solutions $\mathbf{y}^*$ of the polynomial system given by Eq. (26)
2. Extract $\mathbf{Q}^*$ and λ_j^*, $j = 1, \ldots, m$, from $\mathbf{y}^*$
3. Find the set of $\mathbf{Q}^*$ and λ_j^*, $j = 1, \ldots, m$, with the minimum value of the equivalent loss function given by Eq. (34)

V. Numerical Example

An example of fusing two estimates of a 11-dimensional state vector is presented. In a typical run, the numerical values of $\mathbf{x}_1$ and $\mathbf{x}_2$ are

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{q}_1^{(1)} \\ \mathbf{q}_1^{(2)} \\ \mathbf{b}_1 \end{bmatrix} = \begin{bmatrix} -0.000277382573244 \\ -0.000204231576920 \\ 0.000192024591025 \\ 0.999999922237461 \\ -0.000351386991909 \\ 0.000409974757656 \\ 0.500076203448043 \\ 0.865981234896502 \\ 2.498888167313556 \\ 1.419147803026932 \\ 1.515444247510682 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \mathbf{q}_2^{(1)} \\ \mathbf{q}_2^{(2)} \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 0.000460180571557 \\ -0.000137458850112 \\ 0.000082907410592 \\ 0.999999881232627 \\ 0.000201400729929 \\ 0.000114725959778 \\ 0.499722080513250 \\ 0.866185770215148 \\ 4.262501700209700 \\ 2.678350976663482 \\ -0.290141003402079 \end{bmatrix} \tag{37}$$

The 11-dimensional state vector consists of two four-dimensional quaternions and a three-dimensional bias vector (in deg/hr). The individual blocks of the weighting matrices for the two estimates as defined by Eq. (17b) are given by

$$\mathcal{W}_{QQ_1} =$$

$$\begin{bmatrix} 2.6028 & -1.1934 & -0.0338 & 0.0005 & -0.2992 & 0.4473 & -0.4107 & 0.2368 \\ -1.1934 & 2.6220 & 0.0618 & 0.0002 & 0.3700 & -0.3078 & 0.4449 & -0.2566 \\ -0.0338 & 0.0618 & 1.6481 & -0.0003 & 0.0046 & -0.0102 & 0.0071 & -0.0041 \\ 0.0005 & 0.0002 & -0.0003 & 0.0000 & -0.0000 & 0.0001 & -0.0000 & 0.0000 \\ -0.2992 & 0.3700 & 0.0046 & -0.0000 & 1.8740 & 0.3170 & 0.0723 & -0.0412 \\ 0.4473 & -0.3078 & -0.0102 & 0.0001 & 0.3170 & 1.4845 & -0.1206 & 0.0691 \\ -0.4107 & 0.4449 & 0.0071 & -0.0000 & 0.0723 & -0.1206 & 1.0865 & -0.6273 \\ 0.2368 & -0.2566 & -0.0041 & 0.0000 & -0.0412 & 0.0691 & -0.6273 & 0.3622 \end{bmatrix} \times 10^7 \tag{38}$$

$$\mathcal{W}_{QQ_2} =$$

$$\begin{bmatrix} 2.6470 & 0.2882 & -0.1381 & -0.0012 & 0.7717 & 0.0829 & 1.4650 & -0.8454 \\ 0.2882 & 1.7538 & -0.0554 & 0.0001 & 0.3028 & 0.0258 & 0.3218 & -0.1858 \\ -0.1381 & -0.0554 & 1.2859 & -0.0001 & -0.1252 & -0.0097 & -0.0964 & 0.0556 \\ -0.0012 & 0.0001 & -0.0001 & 0.0000 & -0.0003 & -0.0000 & -0.0006 & 0.0004 \\ 0.7717 & 0.3028 & -0.1252 & -0.0003 & 1.8273 & -0.0658 & 0.4699 & -0.2715 \\ 0.0829 & 0.0258 & -0.0097 & -0.0000 & -0.0658 & 1.2899 & -0.0125 & 0.0071 \\ 1.4650 & 0.3218 & -0.0964 & -0.0006 & 0.4699 & -0.0125 & 2.8015 & -1.6164 \\ -0.8454 & -0.1858 & 0.0556 & 0.0004 & -0.2715 & 0.0071 & -1.6164 & 0.9326 \end{bmatrix} \times 10^7 \tag{39}$$

$$\mathcal{W}_{Qb_1} = \begin{bmatrix} -87.1316 & 718.0253 & -7.1995 \\ 0.1239 & -17.8754 & -34.1459 \\ -8.9145 & 5.1567 & -718.6707 \\ -405.9057 & 29.0107 & -0.2878 \\ 189.8244 & -194.0680 & 237.5290 \\ -136.9964 & -141.7521 & 135.0680 \\ 2.7172 & -0.0123 & 25.9761 \\ -41.2554 & 36.2763 & -20.9183 \end{bmatrix} \tag{40}$$

$$\mathcal{W}_{Qb_2} = \begin{bmatrix} 40.2161 & 52.0082 & -26.7553 \\ -0.0091 & 151.4475 & 18.2024 \\ 360.3686 & -207.9424 & -460.4823 \\ -178.0111 & 73.2052 & 0.1814 \\ -394.2257 & -29.2972 & -255.2308 \\ 147.3439 & -663.7105 & -257.5989 \\ 106.0858 & 0.2612 & -571.5016 \\ -42.6669 & -377.3766 & 217.8556 \end{bmatrix} \tag{41}$$

$$\mathcal{W}_{bb_1} = \begin{bmatrix} 0.2034 & 0.0278 & 0.0002 \\ 0.0278 & 5.4026 & 0.0075 \\ 0.0002 & 0.0075 & 0.1230 \end{bmatrix} \tag{42}$$

$$\mathcal{W}_{bb_2} = \begin{bmatrix} 0.2462 & -0.0091 & -0.0131 \\ -0.0091 & 0.1299 & 0.0334 \\ -0.0131 & 0.0334 & 0.1574 \end{bmatrix} \tag{43}$$

From the estimates and weighting matrices, $\mathcal{H}$ and $\mathbf{h}$ are computed:

$$\mathcal{H} =$$

$$\begin{bmatrix} 4.9966 & -0.9375 & -0.1396 & -0.0006 & 0.3153 & 0.5017 & 0.9629 & -0.5558 \\ -0.9375 & 4.2338 & 0.0170 & 0.0003 & 0.6291 & -0.2846 & 0.6949 & -0.4009 \\ -0.1396 & 0.0170 & 2.9294 & -0.0004 & -0.0986 & -0.0165 & -0.0739 & 0.0426 \\ -0.0006 & 0.0003 & -0.0004 & 0.0000 & -0.0003 & 0.0000 & -0.0006 & 0.0004 \\ 0.3153 & 0.6291 & -0.0986 & -0.0003 & 3.5923 & 0.2349 & 0.4678 & -0.2697 \\ 0.5017 & -0.2846 & -0.0165 & 0.0000 & 0.2349 & 2.7710 & -0.1418 & 0.0812 \\ 0.9629 & 0.6949 & -0.0739 & -0.0006 & 0.4678 & -0.1418 & 3.8214 & -2.2053 \\ -0.5558 & -0.4009 & 0.0426 & 0.0004 & -0.2697 & 0.0812 & -2.2053 & 1.2726 \end{bmatrix} \times 10^7 \tag{44}$$

$$\mathbf{h} = \begin{bmatrix} 28.9572 \\ -510.8195 \\ -17.3103 \\ -0.1430 \\ 184.0905 \\ 16.0382 \\ 347.8748 \\ -200.7379 \end{bmatrix} \tag{45}$$

Given $\mathcal{H}$ and $\mathbf{h}$, Eq. (26) is solved using the polyhedral homotopy continuation method of HOM4PS 2.0 [6], which is chosen for its high speed. On a Macintosh computer with a 3 GHz Intel Core 2 Duo processor, the real and complex solutions of the equations are found in less then 0.5 seconds.

For this specific run, 52 solutions are real and 12 solutions are complex. Note that because $\mathbf{h}$ is nonzero, $[\mathbf{q}^{(1)T}, \mathbf{q}^{(2)T}]^T$, $[-\mathbf{q}^{(1)T}, \mathbf{q}^{(2)T}]^T$, $[\mathbf{q}^{(1)T}, -\mathbf{q}^{(2)T}]^T$, and $[-\mathbf{q}^{(1)T}, -\mathbf{q}^{(2)T}]^T$ are not equivalent. If one of them solves Eq. (26), the other three do not. The number of real solutions depends on both $\mathcal{H}$ and $\mathbf{h}$. For different values of $\mathcal{H}$ and $\mathbf{h}$, it is observed 1) that the number of real solutions varies from 48 to 64 but is always even and 2) that the total number of solutions is always 64.

The local minimizers are found by checking the positive definiteness of the Hessians of the stationary points. The number of local minimizers (including the global minimizer) is four in all runs and does not depend on the values of $\mathcal{H}$ and $\mathbf{h}$. The global minimizer as well as the other three local minimizers for this specific run are given in Table 1. The data of the global minimizer are given in column 1. The data of the three local minimizers are in columns 2-4. The 8×8 matrices $(\mathcal{H} + \Lambda^*)$ corresponding to the four local minimizers have six large positive eigenvalues and two small eigenvalues. The two small eigenvalues

Table 1. Global and Local Minimizers

$\mathbf{Q}^*$	−0.000082243849120	0.000071570723061	−0.000086071032790	0.000075397906390
	0.000070491750852	−0.000103146337690	0.000077016986080	−0.000109671573462
	−0.000131920775874	0.000131512370531	−0.000143351102418	0.000142942697036
	−0.999999985431886	0.999999983471480	−0.999999983055311	0.999999980927343
	0.000017062159143	−0.000003434267469	−0.000141582193092	0.000155210084446
	−0.000187251357369	0.000187114821815	0.000255151457924	−0.000255287991806
	−0.499941870374986	0.499960737563446	0.499730976389634	−0.499712105366728
	−0.866058941927033	0.866048050555663	0.866180619783703	−0.866191504511458
$\boldsymbol{\lambda}^*$	−4.953412878868375	−5.016727156337563	−5.452070929467506	−5.515385202667380
	−4.015817997239147	−4.006374995954769	−4.505033071225871	−4.514476066555964
$\mathbf{b}^*$	3.776978219337864	3.184611496825840	3.660908235836134	3.300681480455019
	1.417500286986059	1.453817987354283	1.394096956033854	1.477221318092247
	1.442498556379314	−0.205862220232566	0.712077571903349	0.524558761516893
J_{eq}	**4.464238890975681**	4.518110168722604	4.953453955161509	5.026211225024237
J	**4.815951252844668**	4.869822531936476	5.305166316980205	5.377923588248336

of $(\mathcal{H}+\Lambda^*)$ are seven orders of magnitude less than the six large eigenvalues. Both small eigenvalues are positive for the global minimizer. For the other three local minimizers, one or two of the small eigenvalues are negative, which is likely caused by numerical errors. The eigenvalues of the 6×6 Hessians are positive for all four local minimizers (including the global minimizer).

VI. Conclusions

In quaternion data fusion, the best quaternion estimate is the solution to a minimization problem. The necessary condition for the local minima of the minimization problem is a system of polynomial equations with a finite number of solutions. All the real and complex solutions of the polynomial system can be found via homotopy continuation. The homotopy continuation based method does not depend on the initial guess and is guaranteed to find the global minimizer of the minimization problem, which is one of the real solutions of the polynomial system that yields the smallest value of the loss function.

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