Fisher Information Based Analysis of Deterministic Relative Attitude Observability in Planar Vehicle Formations

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This paper examines the problem of relative attitude observability for formations of three, four and five vehicle formations. The analysis is rooted in within the principles of maximum likelihood estimation insomuch as the metric for observability is the resulting Fisher information matrix for the small attitude errors. For formations of three and four vehicles it is shown that the relative attitudes are not observable by proving that the Fisher information matrix is singular for these cases. For formations of five vehicles, the observability relies on the number of LOS vector pairs available within the formation. It is shown that a minimum number of 8 LOS vector pairs is necessary for the attitudes to be observable, with all configurations of less LOS vector pairs yielding a singular Fisher information matrix.

I. Introduction

Attitude determination involves the calculation of the orientation between two orthogonal reference frames. While attitude determination doesn't fall under any one specific discipline, it has been the space community that has had the most interest in the attitude determination problem. The ability to successfully accomplish a spacecraft's mission is often directly related to the ability to determine its attitude to within some tolerable level. Without accurate and dependable attitude knowledge a spacecraft risks losing power, communications links and the ability to perform maneuvers, for example.

A particularly interesting and important division of attitude determination is that of relative attitude determination. While all attitude determination is in some sense relative, here relative is defined as being between two vehicles, i.e. the relative attitude between two spacecraft is the orientation of one spacecraft's body coordinate frame with respect to the other spacecraft's body coordinate frame rather then with respect to some inertial or intermediate coordinate frame. The need for accurate relative attitude information is essential for a variety of applications with formation flying and rendezvous being just a few examples.

Relative attitude determination can be performed using both filtering and single point methods. These methods, in some way, involve comparison of multiple line-of-sight (LOS) vector observations expressed in one coordinate frame to those same vectors expressed in another coordinate frame. Filtering methods combine these LOS observations with the system dynamics in order to estimate the relative attitudes in real-time. Extended Kalman filters have been successfully used in estimating the relative position and attitude

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within formations of aircraft[1] and spacecraft.[2] Single point methods utilize only current LOS observations in order to construct a geometric solution which does not rely on knowledge of the system dynamics or prior knowledge of the system and/or observations. A purely deterministic solution to the problem of relative attitude determination within a formation of three spacecraft has been shown.[3] A similar problem is investigated in Ref. [4] where the relative attitude between two vehicles looking at a common object is determined.

The solutions methods presented in Refs. [3] and [4] build off the previous work of Shuster[5] which determines the attitude by calculating an axis of rotation and corresponding angle. While calculation of the attitude with these techniques is rather straightforward, one drawback is that the resulting covariance expressions for the attitude estimates can be quite cumbersome to calculate. Because of this, extension to formations larger than 3 vehicles is quite tedious. This work presents an alternative method to calculate the covariance matrix associated with single point relative attitude determination problem. The proposed solution is based on the principles of maximum likelihood attitude estimation[6] and expresses the covariance matrix as the inverse of the associated Fisher information matrix. The resulting Fisher information matrix is calculated and can be easily altered depending what information is available at a given instant. Also, through analysis of the Fisher information matrix for a particular formation it can be deduced whether relative attitudes within the formation are observable with single point methods or whether filtering methods are necessary to observe the relative attitudes.

The organization of this paper is as follows. First the measurement model and maximum likelihood for attitude estimation are reviewed to serve as a foundation for all the analysis to follow. Next, the Fisher information matrix for a formation of three planar vehicles is derived and the results are compared with existing results in Ref. [3]. Planar formations of four vehicles are then examined and it is shown that the Fisher information matrix is singular which shows that the relative attitudes within the formation are not observable. Cases when planar formations of five vehicles will also be examined.

II. Measurement Model

This work assumes planar formations of vehicles like that shown in Fig. 1 which depicts a chief and two deputy vehicles. Each vehicle is assumed to be equipped with some optical sensors which measure a line-of-sight (LOS) vector to the other vehicles within the formation. The LOS vectors (also shown in Fig. 1) are defined as follows: $\mathbf{b}^{\mathcal{X}}_{x/y}$ is a unit vector originating at emitter x and terminating at detector y, expressed in terms of coordinate system $\mathcal{X}$. Throughout this work it is assumed that the emitter and detector on each vehicle are co-located at the vehicle's center. In this regard each vehicle can be thought of much as being a point mass. This situation is a good approximation for most all spacecraft applications where the distance between vehicles is often orders of magnitude larger than the individual spacecraft themselves.

The detectors on-board each vehicle are assumed to be focal-plane detectors (FPD). The measurements from the FPD have the following form:

$$\tilde{\mathbf{b}} = \mathbf{b} + \mathbf{v} \tag{1}$$

where $\tilde{\mathbf{b}}$ is an observation of the true LOS $\mathbf{b}$ and $\mathbf{v}$ is zero-mean Gaussian white noise with covariance R, i.e. $\mathbf{v} \sim \mathcal{N}(\mathbf{v};\ \mathbf{0},\ R)$. The covariance of the measurement noise can be modeled using either the QUEST

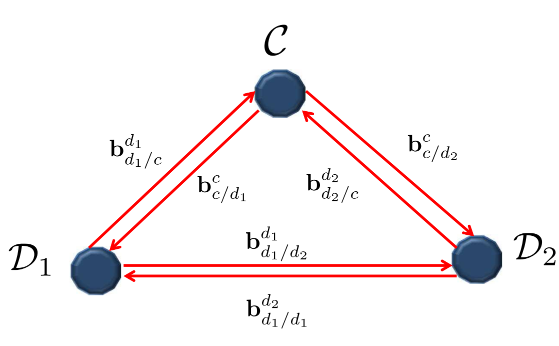


Figure 1. Three Vehicle Formation

measurement model[7] or the wide-field-of-view model,[8] whichever is appropriate. The results presented herein are not specific to a particular FPD covariance model and so the individual models are not discussed.

III. Maximum Likelihood Attitude Estimation

In many practical applications, the true LOS vector, $\mathbf{b}$ in Eq. (1), is not known, but rather given by a reference vector $\mathbf{r}$ expressed in a different coordinate system than $\mathbf{b}$. The relationship between $\mathbf{b}$ and $\mathbf{r}$ is given by

$$\mathbf{b}^{\mathcal{B}} = A_{\mathcal{R}}^{\mathcal{B}} \mathbf{r}^{\mathcal{R}} \tag{2}$$

where $A_{\mathcal{R}}^{\mathcal{B}}$ is an attitude or rotation matrix which maps coordinates coordinate system $\mathcal{R}$ to coordinate system $\mathcal{B}$. Making use of Eq. (2) along with Eq. (1) leads to the following likelihood function for $\tilde{\mathbf{b}}$:

$$L\left(\tilde{\mathbf{b}};\ A\right) \sim c \exp\left[-\frac{1}{2}\left(\tilde{\mathbf{b}} - A\mathbf{r}\right)^T R^{-1}\left(\tilde{\mathbf{b}} - A\mathbf{r}\right)\right] \tag{3}$$

where c is a normalization factor to ensure Eq. (3) is a proper probability density function (pdf) and $(\cdot)^T$ indicates the transpose. Note that the superscripts for the coordinate systems have been dropped momentarily for clarity. Given a sequence of n observations $\left\{\tilde{\mathbf{b}}_1,\ \tilde{\mathbf{b}}_2, \ldots,\ \tilde{\mathbf{b}}_n\right\}$ dependent on A, the joint likelihood function is given is given by

$$L(\tilde{\mathbf{b}}_1,\ \tilde{\mathbf{b}}_2, \ldots,\ \tilde{\mathbf{b}}_n;\ A) = \prod_{i=1}^{n} L\left(\tilde{\mathbf{b}}_i;\ A\right) = C \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\tilde{\mathbf{b}}_i - A\mathbf{r}_i\right)^T R_i^{-1}\left(\tilde{\mathbf{b}}_i - A\mathbf{r}_i\right)\right] \tag{4}$$

where $C = \prod_{i=1}^{n} c_i$. The maximum likelihood estimate of the attitude, $\hat{A}_{\text{ML}}$ is that which maximizes Eq. (4). Because of the monotonioc nature of the Gaussian distribution, this is equivalent to minimizing the negative log-likelihood function, $-\ln(L(\tilde{\mathbf{b}}_1,\ \tilde{\mathbf{b}}_2, \ldots,\ \tilde{\mathbf{b}}_n;\ A))$. Taking the negative of the natural log of Eq. (4) and retaining only terms dependent on $\hat{A}_{\text{ML}}$ leads to the following:

$$J(\hat{A}_{\text{ML}}) = \frac{1}{2}\sum_{i=1}^{n}\left(\tilde{\mathbf{b}}_i - A\mathbf{r}_i\right)^T R_i^{-1}\left(\tilde{\mathbf{b}}_i - A\mathbf{r}_i\right) \tag{5}$$

Shuster[6] presents a method to determine the maximum likelihood estimate of the attitude from Eq. (5). The present work focuses on analysis of the covariance of the maximum likelihood estimate. For estimation of an unbiased parameter, the negative log-likelihood function can be used to bound the covariance by means of the Cramér-Rao inequality[9]

$$P \geq F^{-1} \tag{6}$$

where P is the error covariance of the parameter being estimated and the Fisher information matrix, F is defined as the Hessian of the negative log-likelihood function

$$F = E\left\{\frac{\partial^2}{\partial \mathbf{x}\ \partial \mathbf{x}^T} J(\mathbf{x})\right\} \tag{7}$$

As discussed in Ref. [6], the Fisher matrix for Eq. (5) is not well defined when the attitude matrix is parametrized in terms of the quaternion[10] because the quaternion components are not independent of one another. For this reason the Fisher information matrix is instead defined in terms of the small roll, pitch and yaw Euler angles, $\boldsymbol{\delta\alpha}$, relating the maximum likelihood attitude estimate to the true attitude, A, via

$$\hat{A}_{\text{ML}} = \mathrm{e}^{-[\boldsymbol{\delta\alpha}\times]} A \tag{8}$$

where

$$\mathrm{e}^{-[\boldsymbol{\delta\alpha}\times]} = I_3 - \frac{\sin\left(|\boldsymbol{\delta\alpha}|\right)}{|\boldsymbol{\delta\alpha}|}\left[\boldsymbol{\delta\alpha}\times\right] + \frac{1 - \cos\left(|\boldsymbol{\delta\alpha}|\right)}{|\boldsymbol{\delta\alpha}|^2}\left[\boldsymbol{\delta\alpha}\times\right]^2 \tag{9}$$

and the matrix $[\mathbf{a}\times]$ is the cross product matrix defined such that $\mathbf{a} \times \mathbf{b} \equiv [\mathbf{a}\times]\,\mathbf{b}$ for any 3-vectors $\mathbf{a}$ and $\mathbf{b}$[9] and I_3 is the 3×3 identity matrix. When $|\boldsymbol{\delta\alpha}|$ is small, Eq. (9) can be approximated by

$$\mathrm{e}^{-[\boldsymbol{\delta\alpha}\times]} = I_3 - [\boldsymbol{\delta\alpha}\times] \tag{10}$$

which leads to the often seen first-order approximation to the attitude matrix

$$\hat{A}_{\mathrm{ML}} = (I_3 - [\boldsymbol{\delta\alpha}\times])\, A \tag{11}$$

After substituting Eq. (11) into Eq. (5), and taking two derivatives leads to

$$F = E\left\{\frac{\partial^2}{\partial\boldsymbol{\delta\alpha}\,\partial\boldsymbol{\delta\alpha}^T} J(\boldsymbol{\delta\alpha})\right\} = -\sum_{i=1}^{n} [\mathbf{b}_i\times]\, R_i^{-1}\, [\mathbf{b}_i\times] \tag{12}$$

where $\mathbf{b}_i$ is the true realization of the LOS observation $\tilde{\mathbf{b}}_i$. For the case of isotropic noise, $R_i = \sigma_i^2 I_3$, Eq. (12) reduces to the well known QUEST covariance[6,11]

$$P_{\mathrm{QUEST}}^{-1} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2}\left(I_3 - \mathbf{b}_i\mathbf{b}_i^T\right) \tag{13}$$

Because the true LOS vectors $\mathbf{b}_i$ are unknown, the measured values $\tilde{\mathbf{b}}_i$ can be used in calculation of the covariance while only introducing second order errors.[6]

IV. Three Vehicle Formations

The concepts and principles introduced in the previous section are now applied to the problem of relative attitude estimation in a formation of three planar vehicles, see Fig. 1. While there are three vehicles within the formation, there exists only two independent relative attitudes. These attitudes are given as $A_{d_1}^c$ and $A_{d_2}^c$, which are the attitudes mapping the deputy 1 frame to the chief frame, and that mapping the deputy 2 to chief frame, respectively. Note the attitude mapping the deputy 1 to deputy 2 frame can be constructed from the other two attitudes as $A_{d_1}^{d_2} = (A_{d_2}^c)^{-1} A_{d_1}^c$. Because the attitude matrix is proper orthogonal we have $A^{-1} = A^T$, it follows that $A_{d_1}^{d_2} = (A_{d_2}^c)^T A_{d_1}^c = A_c^{d_2} A_{d_1}^c$.

A. Fisher Information Matrix Derivation

The maximum likelihood estimates of the attitudes $A_{d_1}^c$ and $A_{d_2}^c$ can be found by generalizing the negative log-likelihood function from Eq. (5). Retaining only terms dependent on the unknown attitudes yields[a]

$$\begin{aligned} J(\hat{A}_{d_1}^c,\ \hat{A}_{d_2}^c) &= \frac{1}{2}\left(\tilde{\mathbf{b}}_{c/d_1}^c - \hat{A}_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1}\right)^T R_{c/d_1}^{c^{-1}} \left(\tilde{\mathbf{b}}_{c/d_1}^c - \hat{A}_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1}\right) \\ &+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{c/d_2}^c - \hat{A}_{d_2}^c \tilde{\mathbf{b}}_{c/d_2}^{d_2}\right)^T R_{c/d_2}^{c^{-1}} \left(\tilde{\mathbf{b}}_{c/d_2}^c - \hat{A}_{d_2}^c \tilde{\mathbf{b}}_{c/d_2}^{d_2}\right) \\ &+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{d_2/d_1}^{d_2} - \hat{A}_c^{d_2}\hat{A}_{d_1}^c \tilde{\mathbf{b}}_{d_2/d_1}^{d_1}\right)^T R_{d_2/d_1}^{d_2^{-1}} \left(\tilde{\mathbf{b}}_{d_2/d_1}^{d_2} - \hat{A}_c^{d_2}\hat{A}_{d_1}^c \tilde{\mathbf{b}}_{d_2/d_1}^{d_1}\right) \\ &\equiv \frac{1}{2}J_{c/d_1} + \frac{1}{2}J_{c/d_2} + \frac{1}{2}J_{d_2/d_1} \end{aligned} \tag{14}$$

where $A_{d_1}^{d_2} = A_c^{d_2} A_{d_1}^c$ has been used. Note that the subscript "ML" has been dropped from the attitudes for clarity. First-order approximations to the relative attitude matrices follow directly from Eq. (11):

$$\hat{A}_{d_1}^c = \left(I_3 - \left[\boldsymbol{\delta\alpha}_{d_1}^c\times\right]\right) A_{d_1}^c \tag{15a}$$

$$\hat{A}_{d_2}^c = \left(I_3 - \left[\boldsymbol{\delta\alpha}_{d_2}^c\times\right]\right) A_{d_2}^c \tag{15b}$$

Substituting Eq. (15a) into the first term of J_{c/d_1} from Eq. (14) leads to

$$\begin{aligned} \tilde{\mathbf{b}}_{c/d_1}^c - \hat{A}_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1} &= \tilde{\mathbf{b}}_{c/d_1}^c - \left(A_{d_1}^c - \left[\boldsymbol{\delta\alpha}_{d_1}^c\times\right] A_{d_1}^c\right)\tilde{\mathbf{b}}_{c/d_1}^{d_1} \\ &= \tilde{\mathbf{b}}_{c/d_1}^c - A_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1} + \left[\boldsymbol{\delta\alpha}_{d_1}^c\times\right] A_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1} \\ &= \Delta\tilde{\mathbf{b}}_{c/d_1}^c - \left[\tilde{\mathbf{b}}_{c/d_1}^c\times\right]\boldsymbol{\delta\alpha}_{d_1}^c \end{aligned} \tag{16}$$

[a]It is important to note that Eq. (14) implies that the LOS vector pairs between vehicles are parallel. Specifically this implies that the emitter and receiver are co-located on the vehicle.[3] While not strictly true in general, this assumption becomes valid as the distance between vehicles becomes much larger than the individual vehicle sizes.

where $\Delta\tilde{\mathbf{b}}^c_{c/d_1} \equiv \tilde{\mathbf{b}}^c_{c/d_1} - A^c_{d_1}\tilde{\mathbf{b}}^{d_1}_{c/d_1}$ and we have made use of the relation $[\mathbf{a}\times]\,\mathbf{b} = -[\mathbf{b}\times]\,\mathbf{a}$. Note that $\Delta\tilde{\mathbf{b}}^c_{c/d_1}$ is not identically zero because the observations are made from different sensors. It is easy to show that $\Delta\tilde{\mathbf{b}}^c_{c/d_1}$ has the following statistics:

$$E\left\{\Delta\tilde{\mathbf{b}}^c_{c/d_1}\right\} = \mathbf{0} \tag{17a}$$

$$E\left\{\Delta\tilde{\mathbf{b}}^c_{c/d_1}\Delta\tilde{\mathbf{b}}^{c^T}_{c/d_1}\right\} = R^c_{c/d_1} + A^c_{d_1}R^{d_1}_{c/d_1}A^{d_1}_c \tag{17b}$$

Substituting Eq. (16) into J_{c/d_1} yields

$$\begin{aligned} J_{c/d_1} &= \left(\Delta\tilde{\mathbf{b}}^c_{c/d_1} - \left[\tilde{\mathbf{b}}^c_{c/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1}\right)^T R^{c^{-1}}_{c/d_1}\left(\Delta\tilde{\mathbf{b}}^c_{c/d_1} - \left[\tilde{\mathbf{b}}^c_{c/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1}\right) \\ &= \Delta\tilde{\mathbf{b}}^{c^T}_{c/d_1}R^{c^{-1}}_{c/d_1}\Delta\tilde{\mathbf{b}}^c_{c/d_1} - 2\Delta\tilde{\mathbf{b}}^{c^T}_{c/d_1}R^{c^{-1}}_{c/d_1}\left[\tilde{\mathbf{b}}^c_{c/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} - \boldsymbol{\delta\alpha}^{c^T}_{d_1}\left[\mathbf{b}^c_{c/d_1}\times\right]R^{c^{-1}}_{c/d_1}\left[\mathbf{b}^c_{c/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} \end{aligned} \tag{18}$$

Using an identical procedure to that above for J_{c/d_2} results in

$$J_{c/d_2} = \Delta\tilde{\mathbf{b}}^{c^T}_{c/d_2}R^{c^{-1}}_{c/d_2}\Delta\tilde{\mathbf{b}}^c_{c/d_2} - 2\Delta\tilde{\mathbf{b}}^{c^T}_{c/d_2}R^{c^{-1}}_{c/d_2}\left[\tilde{\mathbf{b}}^c_{c/d_2}\times\right]\boldsymbol{\delta\alpha}^c_{d_2} - \boldsymbol{\delta\alpha}^{c^T}_{d_2}\left[\mathbf{b}^c_{c/d_2}\times\right]R^{c^{-1}}_{c/d_2}\left[\mathbf{b}^c_{c/d_2}\times\right]\boldsymbol{\delta\alpha}^c_{d_2} \tag{19}$$

with

$$\Delta\tilde{\mathbf{b}}^c_{c/d_2} \equiv \tilde{\mathbf{b}}^c_{c/d_2} - A^c_{d_2}\tilde{\mathbf{b}}^{d_2}_{c/d_2} \tag{20a}$$

$$E\left\{\Delta\tilde{\mathbf{b}}^c_{c/d_2}\right\} = \mathbf{0} \tag{20b}$$

$$E\left\{\Delta\tilde{\mathbf{b}}^c_{c/d_2}\Delta\tilde{\mathbf{b}}^{c^T}_{c/d_2}\right\} = R^c_{c/d_2} + A^c_{d_2}R^{d_1}_{c/d_2}A^{d_2}_c \tag{20c}$$

Evaluation of the term J_{d_2/d_1} is more complex due to the coupling of the attitudes:

$$\begin{aligned} \hat{A}^{d_2}_c\hat{A}^c_{d_1} &= A^{d_2}_c\left(I_3 + \left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]\right)\left(I_3 - \left[\boldsymbol{\delta\alpha}^c_{d_1}\times\right]\right)A^c_{d_1} \\ &= A^{d_2}_{d_1} - A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_1}\times\right]A^c_{d_1} + A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]A^c_{d_1} - A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]\left[\boldsymbol{\delta\alpha}^c_{d_1}\times\right]A^c_{d_1} \end{aligned} \tag{21}$$

Substituting Eq. (21) into J_{d_2/d_1} and simplifying yields

$$\begin{aligned} J_{d_2/d_1} = {} & \Delta\tilde{\mathbf{b}}^{d_2^T}_{d_2/d_1}R^{d_2^{-1}}_{d_2/d_1}\Delta\tilde{\mathbf{b}}^{d_2^T}_{d_2/d_1} - 2\Delta\tilde{\mathbf{b}}^{d_2^T}_{d_2/d_1}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} + 2\Delta\tilde{\mathbf{b}}^{d_2^T}_{d_2/d_1}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_2} \\ & - 2\Delta\tilde{\mathbf{b}}^{d_2^T}_{d_2/d_1}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} - \boldsymbol{\delta\alpha}^{c^T}_{d_1}\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]A^c_{d_2}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} \\ & + 2\boldsymbol{\delta\alpha}^{c^T}_{d_1}\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]A^c_{d_2}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_2} - 2\boldsymbol{\delta\alpha}^{c^T}_{d_1}\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]A^c_{d_2}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} \\ & - \boldsymbol{\delta\alpha}^{c^T}_{d_2}\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]A^c_{d_2}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_2} + 2\boldsymbol{\delta\alpha}^{c^T}_{d_2}\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]A^c_{d_2}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} \\ & + \boldsymbol{\delta\alpha}^{c^T}_{d_1}\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]A^c_{d_2}R^{d_2^{-1}}_{d_2/d_1}A^{d_2}_c\left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right]\left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right]\boldsymbol{\delta\alpha}^c_{d_1} \end{aligned} \tag{22}$$

with

$$\Delta\tilde{\mathbf{b}}^{d_2}_{d_2/d_1} \equiv \tilde{\mathbf{b}}^{d_2}_{d_2/d_1} - A^{d_2}_{d_1}\tilde{\mathbf{b}}^{d_1}_{d_2/d_1} \tag{23a}$$

$$E\left\{\Delta\tilde{\mathbf{b}}^{d_2}_{d_2/d_1}\right\} = \mathbf{0} \tag{23b}$$

$$E\left\{\Delta\tilde{\mathbf{b}}^{d_2}_{d_2/d_1}\Delta\tilde{\mathbf{b}}^{c^T}_{d_2/d_1}\right\} = R^{d_2}_{d_2/d_1} + A^{d_2}_{d_1}R^{d_1}_{d_2/d_1}A^{d_1}_{d_2} \tag{23c}$$

Substituting Eqs. (18), (19) and (22) into Eq. (14) and collecting terms results in

$$\begin{aligned} J(\hat{A}^c_{d_1},\ \hat{A}^c_{d_2}) = {} & -\frac{1}{2}\boldsymbol{\delta\alpha}^{c^T}_{d_1}\left(\tilde{W}^c_{c/d_1} + \tilde{W}^c_{d_2/d_1}\right)\boldsymbol{\delta\alpha}^c_{d_1} - \frac{1}{2}\boldsymbol{\delta\alpha}^{c^T}_{d_2}\left(\tilde{W}^c_{c/d_2} + \tilde{W}^c_{d_2/d_1}\right)\boldsymbol{\delta\alpha}^c_{d_2} + \boldsymbol{\delta\alpha}^{c^T}_{d_1}\tilde{W}^c_{d_2/d_1}\boldsymbol{\delta\alpha}^c_{d_2} \\ & + \text{H.O.T.} + f(\Delta\tilde{\mathbf{b}}^c_{c/d_1},\ \Delta\tilde{\mathbf{b}}^c_{c/d_2},\ \Delta\tilde{\mathbf{b}}^{d_2}_{d_2/d_1}) \end{aligned} \tag{24}$$

where the following matrices have been defined to simplify notation:

$$\tilde{W}^c_{c/d_1} \equiv \left[\tilde{\mathbf{b}}^c_{c/d_1}\times\right] R^{c^{-1}}_{c/d_1} \left[\tilde{\mathbf{b}}^c_{c/d_1}\times\right] \tag{25a}$$

$$\tilde{W}^c_{c/d_2} \equiv \left[\tilde{\mathbf{b}}^c_{c/d_2}\times\right] R^{c^{-1}}_{c/d_2} \left[\tilde{\mathbf{b}}^c_{c/d_2}\times\right] \tag{25b}$$

$$\tilde{W}^c_{d_2/d_1} \equiv \left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\tilde{\mathbf{b}}^c_{d_2/d_1}\times\right] \tag{25c}$$

The higher-order terms (H.O.T.) are those which are third and fourth order in the small attitude angle errors. In the subsequent analysis these terms will be neglected. For reference the H.O.T. are found to be

$$\begin{aligned} \text{H.O.T.} = & \left(\boldsymbol{\delta\alpha}^{c^T}_{d_2} - \boldsymbol{\delta\alpha}^{c^T}_{d_1}\right) \left[\mathbf{b}^c_{d_2/d_1}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right] \left[\mathbf{b}^c_{d_2/d_1}\times\right] \boldsymbol{\delta\alpha}^c_{d_1} \\ & + \frac{1}{2}\boldsymbol{\delta\alpha}^{c^T}_{d_1} \left[\mathbf{b}^c_{d_2/d_1}\times\right] \left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\boldsymbol{\delta\alpha}^c_{d_2}\times\right] \left[\mathbf{b}^c_{d_2/d_1}\times\right] \boldsymbol{\delta\alpha}^c_{d_1} \end{aligned} \tag{26}$$

The terms in $f(\Delta\tilde{\mathbf{b}}^c_{c/d_1},\ \Delta\tilde{\mathbf{b}}^c_{c/d_2},\ \Delta\tilde{\mathbf{b}}^{d_2}_{d_2/d_1})$ are all those dependent on the $\Delta\tilde{\mathbf{b}}$'s. When calculating the Fisher information matrix with Eq. (7) these terms all disappear because they are either not a function of the small angle attitude errors or the $\Delta\tilde{\mathbf{b}}$ term appears linearly and being zero-mean the contribution vanishes during the expectation.

Substituting Eq. (24) into Eq. (7) with $\boldsymbol{\delta\alpha}^T \equiv \begin{bmatrix}\boldsymbol{\delta\alpha}^{c^T}_{d_1} & \boldsymbol{\delta\alpha}^{c^T}_{d_2}\end{bmatrix}$ results in

$$F \equiv \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix} = \begin{bmatrix} -\left(W^c_{c/d_1} + W^c_{d_2/d_1}\right) & W^c_{d_2/d_1} \\ W^c_{d_2/d_1} & -\left(W^c_{c/d_2} - W^c_{d_2/d_1}\right) \end{bmatrix} \tag{27}$$

where the W matrices are exactly those defined in Eq. (25) except that the true values of the vectors have been used.

B. Singularity of the Fisher Information Matrix

Reference [3] has shown that when all observations lie within a single plane, the resulting covariance is singular leading to the relative attitudes being unobservable. In what follows it is shown that the Fisher information matrix in Eq. (27) is singular for the case of planar vectors and the null vector of the Fisher information matrix is determined.

Begin by noting that the vector $\mathbf{b}^c_{d_2/d_1}$ can be written as a linear combination of the vectors $\mathbf{b}^c_{c/d_1}$ and $\mathbf{b}^c_{c/d_2}$

$$\mathbf{b}^c_{d_2/d_1} = \alpha\mathbf{b}^c_{c/d_1} - \beta\mathbf{b}^c_{c/d_2} \tag{28}$$

for arbitrary α and β in $[-1,\ 1]$. Substituting Eq. (28) into Eq. (25c) results in

$$\begin{aligned} W^c_{d_2/d_1} = & \alpha^2 \left[\mathbf{b}^c_{c/d_1}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\mathbf{b}^c_{c/d_1}\times\right] - \alpha\beta \left[\mathbf{b}^c_{c/d_1}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\mathbf{b}^c_{c/d_2}\times\right] \\ & - \alpha\beta \left[\mathbf{b}^c_{c/d_2}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\mathbf{b}^c_{c/d_1}\times\right] + \beta^2 \left[\mathbf{b}^c_{c/d_2}\times\right] A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c \left[\mathbf{b}^c_{c/d_2}\times\right] \end{aligned} \tag{29}$$

For notational simplicity, we make the following variable changes:

$$\begin{aligned} \mathbf{b}^c_{c/d_1} &\to \mathbf{r} \\ \mathbf{b}^c_{c/d_2} &\to \mathbf{b} \\ R^{c^{-1}}_{c/d_1} &\to R_{c1} \\ R^{c^{-1}}_{c/d_2} &\to R_{c2} \\ A^c_{d_2} R^{d_2^{-1}}_{d_2/d_1} A^{d_2}_c &\to R_{21} \end{aligned} \tag{30}$$

Using these definitions, the block entries in Eq. (27) become

$$F_{11} = -[\mathbf{r}\times]\left(R_{c1} + \alpha^2 R_{21}\right)[\mathbf{r}\times] - \beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times] + \alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times] + \alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times] \tag{31a}$$

$$F_{12} = F_{12}^T = \alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times] + \beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times] - \alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times] - \alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times] \tag{31b}$$

$$F_{22} = -[\mathbf{b}\times]\left(R_{c2} + \beta^2 R_{21}\right)[\mathbf{b}\times] - \alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times] + \alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times] + \alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times] \tag{31c}$$

The Fisher information matrix can be shown to be singular by proving the existence of a vector within the null space of F, that is, a vector $\mathbf{z}$ which satisfies $F\mathbf{z} = \mathbf{0}$. Partitioning the vector $\mathbf{z}$ as $\mathbf{z}^T = \begin{bmatrix}\mathbf{z}_1^T & \mathbf{z}_2^T\end{bmatrix}$, where $\mathbf{z}_1$ and $\mathbf{z}_2$ are each 3-vectors and carrying out the block multiplication of $F\mathbf{z}$ leads to

$$F_{11}\mathbf{z}_1 + F_{12}\mathbf{z}_2 = \mathbf{0} \tag{32a}$$

$$F_{12}\mathbf{z}_1 + F_{22}\mathbf{z}_2 = \mathbf{0} \tag{32b}$$

The null vector is anticipated to lie in the plane of the formation. Let the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ be linear combinations of $\mathbf{r}$ and $\mathbf{b}$ as

$$\mathbf{z}_1 = k_1\mathbf{r} + k_2\mathbf{b} \tag{33a}$$

$$\mathbf{z}_2 = m_1\mathbf{r} + m_2\mathbf{b} \tag{33b}$$

where k_1, k_2, m_1 and m_2 are some unknown but to be determined constants. Substituting Eqs. (31) and (33) into Eq. (32a) and simplifying the resulting expression yields

$$\begin{aligned} F_{11}\mathbf{z}_1 + F_{12}\mathbf{z}_2 = {} & k_1\alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - k_1\beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - k_2[\mathbf{r}\times]\left(R_{c1} + \alpha^2 R_{21}\right)[\mathbf{r}\times]\mathbf{b} + k_2\alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \\ & - m_1\alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times]\mathbf{r} + m_1\beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} + m_2\alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \\ & - m_2\alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \end{aligned} \tag{34}$$

Similarly Eq. (32b) becomes

$$\begin{aligned} F_{12}\mathbf{z}_1 + F_{22}\mathbf{z}_2 = {} & k_1\beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - k_1\alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times]\mathbf{r} + k_2\alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{b} - k_2\alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \\ & - m_1[\mathbf{b}\times]\left(R_{c2} + \beta^2 R_{21}\right)[\mathbf{b}\times]\mathbf{r} + m_1\alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - m_2\alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \\ & + m_2\alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \end{aligned} \tag{35}$$

The constants are now to be determined. Noting that the constants are arbitrary, by selecting $k_2 = m_1 = 0$ greatly simplifies Eqs. (34) and (35) to

$$F_{11}\mathbf{z}_1 + F_{12}\mathbf{z}_2 = k_1\alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - k_1\beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} + m_2\alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{b} - m_2\alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \tag{36a}$$

$$F_{12}\mathbf{z}_1 + F_{22}\mathbf{z}_2 = k_1\beta^2[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - k_1\alpha\beta[\mathbf{r}\times]R_{21}[\mathbf{b}\times]\mathbf{r} - m_2\alpha^2[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{b} + m_2\alpha\beta[\mathbf{b}\times]R_{21}[\mathbf{r}\times]\mathbf{b} \tag{36b}$$

Using $[\mathbf{a}\times]\,\mathbf{b} = -\,[\mathbf{b}\times]\,\mathbf{a}$ and grouping like terms leads to

$$F_{11}\mathbf{z}_1 + F_{12}\mathbf{z}_2 = \left(m_2\alpha^2 - k_1\alpha\beta\right)[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{b} + \left(m_2\alpha\beta - k_1\beta^2\right)[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} \tag{37a}$$

$$F_{12}\mathbf{z}_1 + F_{22}\mathbf{z}_2 = \left(k_1\beta^2 - m_2\alpha\beta\right)[\mathbf{b}\times]R_{21}[\mathbf{b}\times]\mathbf{r} + \left(k_1\alpha\beta - m_2\alpha^2\right)[\mathbf{r}\times]R_{21}[\mathbf{r}\times]\mathbf{r}\mathbf{b} \tag{37b}$$

By selecting the constants as $k_1 = \alpha$ and $m_2 = \beta$, it is easy to show that Eqs. (37) reduce to zero. The null vector is then given by

$$\mathbf{z} = \text{null}(F) = \begin{bmatrix}\alpha\mathbf{r} \\ \beta\mathbf{b}\end{bmatrix} = \begin{bmatrix}\alpha\mathbf{b}^c_{c/d_1} \\ \beta\mathbf{b}^c_{c/d_2}\end{bmatrix} \tag{38}$$

By proving the existence of a null vector, the Fisher information matrix is proven singular resulting in an unobservable system.

C. Discussion of Three Vehicle Singularity

The preceding discussion proved that the relative attitudes between three vehicles in a planar formation are unobservable which are consistent with those shown in Ref. [3]. The singularity of the Fisher information matrix is now discussed as a function of the amount of information available within the formation.

Recall from Fig. 1 that within the formation there exists three-pairs of matching LOS vectors, namely $\mathbf{b}^c_{c/d_1} - \mathbf{b}^{d_1}_{d_1/c}$, $\mathbf{b}^c_{c/d_2} - \mathbf{b}^{d_2}_{d_2/c}$ and $\mathbf{b}^{d_2}_{d_2/d_1} - \mathbf{b}^{d_1}_{d_1/d_2}$. While each LOS does not carry information on its own, each pair contains two pieces of information. This information is in the form of an azimuth and elevation angle between the two vehicles. Thus a single LOS pair contains information about two of the three angles

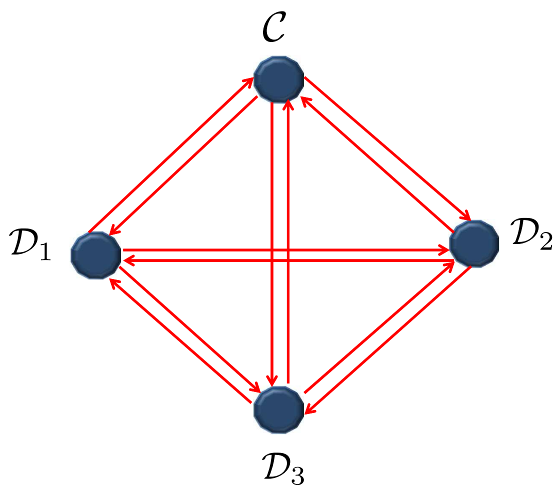


Figure 2. Four Vehicle Formation

necessary to quantify the relative attitude between the two vehicles. The missing piece of information corresponds to angular rotation about the LOS vector between the two vehicles. Considering all three LOS pairs within the formation it is clear that six pieces of information are available to determine the relative attitudes. With each relative attitude being comprised of three unknowns (the roll, pitch and yaw angles), it would appear that enough information is known to uniquely determine the attitudes. However, the fact that the LOS vectors between the vehicles form a triangle imposes an additional constraint on the system, meaning that one more piece of information is necessary to determine the attitudes. The triangle constraint is also equivalent to the constraint that all the LOS pairs lie within a common plane.[4] With the constraint taking up one of the pieces of information available we find for the attitudes to be observable requires seven pieces of information while the formation supplies only six leaving a deficit of one piece of information. This deficit is apparent through numerical analysis of Eq. (27) which shows that the Fisher information matrix is rank one deficient.

V. Four Vehicle Formations

Next the case of a formation of of four vehicles is discussed. The geometry of the formation can be found in Fig. 2. In addition to the chief, deputy 1 and deputy 2, a third deputy (d_3) is introduced as seen. The independent relative attitudes within the formation are given by $A^c_{d_1}$, $A^c_{d_2}$ and $A^c_{d_3}$, while the remaining attitudes $A^{d_2}_{d_1}$, $A^{d_3}_{d_1}$ and $A^{d_3}_{d_2}$ can be constructed as was previously shown. We again note that all vectors defined in Fig. 2 lie within a plane.[b] It is also assumed that no three of the vehicles are colinear.

[b]Note the LOS vector definitions are not shown for clarity but obey the same naming convention as described in Sect. II.

A. Fisher Information Matrix Derivation

Following the same principles laid forth in the Section IV, the attitude dependent portion of the negative log-likelihood function, assuming a fully connected formation (i.e. no missing LOS information), is given by

$$
\begin{aligned}
J(\hat{A}_c^{d_1},\ \hat{A}_c^{d_2},\ \hat{A}_c^{d_3}) =& \frac{1}{2}\left(\tilde{\mathbf{b}}_{c/d_1}^c - \hat{A}_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1}\right)^T R_{c/d_1}^{c^{-1}} \left(\tilde{\mathbf{b}}_{c/d_1}^c - \hat{A}_{d_1}^c \tilde{\mathbf{b}}_{c/d_1}^{d_1}\right) \\
&+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{c/d_2}^c - \hat{A}_{d_2}^c \tilde{\mathbf{b}}_{c/d_2}^{d_2}\right)^T R_{c/d_2}^{c^{-1}} \left(\tilde{\mathbf{b}}_{c/d_2}^c - \hat{A}_{d_2}^c \tilde{\mathbf{b}}_{c/d_2}^{d_2}\right) \\
&+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{c/d_3}^c - \hat{A}_{d_3}^c \tilde{\mathbf{b}}_{c/d_3}^{d_3}\right)^T R_{c/d_3}^{c^{-1}} \left(\tilde{\mathbf{b}}_{c/d_3}^c - \hat{A}_{d_3}^c \tilde{\mathbf{b}}_{c/d_3}^{d_3}\right) \\
&+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{d_2/d_1}^{d_2} - \hat{A}_c^{d_2}\hat{A}_{d_1}^c \tilde{\mathbf{b}}_{d_2/d_1}^{d_1}\right)^T R_{d_2/d_1}^{d_2^{-1}} \left(\tilde{\mathbf{b}}_{d_2/d_1}^{d_2} - \hat{A}_c^{d_2}\hat{A}_{d_1}^c \tilde{\mathbf{b}}_{d_2/d_1}^{d_1}\right) \\
&+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{d_3/d_1}^{d_3} - \hat{A}_c^{d_3}\hat{A}_{d_1}^c \tilde{\mathbf{b}}_{d_3/d_1}^{d_1}\right)^T R_{d_3/d_1}^{d_3^{-1}} \left(\tilde{\mathbf{b}}_{d_3/d_1}^{d_3} - \hat{A}_c^{d_3}\hat{A}_{d_1}^c \tilde{\mathbf{b}}_{d_3/d_1}^{d_1}\right) \\
&+ \frac{1}{2}\left(\tilde{\mathbf{b}}_{d_3/d_2}^{d_3} - \hat{A}_c^{d_3}\hat{A}_{d_2}^c \tilde{\mathbf{b}}_{d_3/d_2}^{d_2}\right)^T R_{d_3/d_2}^{d_3^{-1}} \left(\tilde{\mathbf{b}}_{d_3/d_2}^{d_3} - \hat{A}_c^{d_3}\hat{A}_{d_2}^c \tilde{\mathbf{b}}_{d_3/d_2}^{d_2}\right)
\end{aligned} \tag{39}
$$

Following the same steps as in the three spacecraft analysis, Eq. (39) can be simplified to

$$
\begin{aligned}
J =& -\frac{1}{2}\boldsymbol{\delta\alpha}_{d_1}^{c^T} \tilde{W}_{c/d_1}^c \boldsymbol{\delta\alpha}_{d_1}^c \\
&-\frac{1}{2}\boldsymbol{\delta\alpha}_{d_2}^{c^T} \tilde{W}_{c/d_2}^c \boldsymbol{\delta\alpha}_{d_2}^c \\
&-\frac{1}{2}\boldsymbol{\delta\alpha}_{d_3}^{c^T} \tilde{W}_{c/d_3}^c \boldsymbol{\delta\alpha}_{d_3}^c \\
&-\frac{1}{2}\boldsymbol{\delta\alpha}_{d_1}^{c^T} \tilde{W}_{d_2/d_1}^c \boldsymbol{\delta\alpha}_{d_1}^c - \frac{1}{2}\boldsymbol{\delta\alpha}_{d_2}^{c^T} \tilde{W}_{d_2/d_1}^c \boldsymbol{\delta\alpha}_{d_2}^c + \boldsymbol{\delta\alpha}_{d_1}^{c^T} \tilde{W}_{d_2/d_1}^c \boldsymbol{\delta\alpha}_{d_2}^c \\
&-\frac{1}{2}\boldsymbol{\delta\alpha}_{d_1}^{c^T} \tilde{W}_{d_3/d_1}^c \boldsymbol{\delta\alpha}_{d_1}^c - \frac{1}{2}\boldsymbol{\delta\alpha}_{d_3}^{c^T} \tilde{W}_{d_3/d_1}^c \boldsymbol{\delta\alpha}_{d_3}^c + \boldsymbol{\delta\alpha}_{d_1}^{c^T} \tilde{W}_{d_3/d_1}^c \boldsymbol{\delta\alpha}_{d_3}^c \\
&-\frac{1}{2}\boldsymbol{\delta\alpha}_{d_2}^{c^T} \tilde{W}_{d_3/d_2}^c \boldsymbol{\delta\alpha}_{d_2}^c - \frac{1}{2}\boldsymbol{\delta\alpha}_{d_3}^{c^T} \tilde{W}_{d_3/d_2}^c \boldsymbol{\delta\alpha}_{d_3}^c + \boldsymbol{\delta\alpha}_{d_2}^{c^T} \tilde{W}_{d_3/d_2}^c \boldsymbol{\delta\alpha}_{d_3}^c \\
&+ \text{H.O.T.} + f(\Delta\tilde{\mathbf{b}}_{c/d_1}^c,\ \Delta\tilde{\mathbf{b}}_{c/d_2}^c,\ \Delta\tilde{\mathbf{b}}_{c/d_3}^c,\ \Delta\tilde{\mathbf{b}}_{d_2/d_1}^{d_2},\ \Delta\tilde{\mathbf{b}}_{d_3/d_1}^{d_3},\ \Delta\tilde{\mathbf{b}}_{d_3/d_2}^{d_3})
\end{aligned} \tag{40}
$$

where

$$\tilde{W}_{c/d_1}^c \equiv \left[\tilde{\mathbf{b}}_{c/d_1}^c \times\right] R_{c/d_1}^{c^{-1}} \left[\tilde{\mathbf{b}}_{c/d_1}^c \times\right] \tag{41a}$$

$$\tilde{W}_{c/d_2}^c \equiv \left[\tilde{\mathbf{b}}_{c/d_2}^c \times\right] R_{c/d_2}^{c^{-1}} \left[\tilde{\mathbf{b}}_{c/d_2}^c \times\right] \tag{41b}$$

$$\tilde{W}_{c/d_3}^c \equiv \left[\tilde{\mathbf{b}}_{c/d_3}^c \times\right] R_{c/d_3}^{c^{-1}} \left[\tilde{\mathbf{b}}_{c/d_3}^c \times\right] \tag{41c}$$

$$\tilde{W}_{d_2/d_1}^c \equiv \left[\tilde{\mathbf{b}}_{d_2/d_1}^c \times\right] A_{d_2}^c R_{d_2/d_1}^{d_2^{-1}} A_c^{d_2} \left[\tilde{\mathbf{b}}_{d_2/d_1}^c \times\right] \tag{41d}$$

$$\tilde{W}_{d_3/d_1}^c \equiv \left[\tilde{\mathbf{b}}_{d_3/d_1}^c \times\right] A_{d_3}^c R_{d_3/d_1}^{d_3^{-1}} A_c^{d_3} \left[\tilde{\mathbf{b}}_{d_3/d_1}^c \times\right] \tag{41e}$$

$$\tilde{W}_{d_3/d_2}^c \equiv \left[\tilde{\mathbf{b}}_{d_3/d_2}^c \times\right] A_{d_3}^c R_{d_3/d_2}^{d_3^{-1}} A_c^{d_3} \left[\tilde{\mathbf{b}}_{d_3/d_2}^c \times\right] \tag{41f}$$

The Fisher information matrix for this formation is computed with $\boldsymbol{\delta\alpha}^T \equiv \begin{bmatrix} \boldsymbol{\delta\alpha}_{d_1}^{c^T} & \boldsymbol{\delta\alpha}_{d_2}^{c^T} & \boldsymbol{\delta\alpha}_{d_3}^{c^T} \end{bmatrix}$ and is given by

$$\begin{aligned} F &\equiv \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}^T & F_{22} & F_{23} \\ F_{13}^T & F_{23}^T & F_{33} \end{bmatrix} \\ &= \begin{bmatrix} -\left(W_{c/d_1}^c + W_{d_2/d_1}^c + W_{d_3/d_1}^c\right) & W_{d_2/d_1}^c & W_{d_3/d_1}^c \\ W_{d_2/d_1}^c & -\left(W_{c/d_2}^c + W_{d_2/d_1}^c + W_{d_3/d_2}^c\right) & W_{d_3/d_2}^c \\ W_{d_3/d_1}^c & W_{d_3/d_2}^c & -\left(W_{c/d_3}^c + W_{d_3/d_1}^c + W_{d_3/d_2}^c\right) \end{bmatrix} \end{aligned} \tag{42}$$

B. Singularity of the Fisher Information Matrix

While not intuitive the Fisher information matrix for the four vehicle formation is also singular, indicative that the relative attitudes are not observable. This section proves that the Fisher information matrix is again singular by developing an expression for a vector in the null space of F. The following variable changes are made in addition to those used in Eq. (30):

$$\begin{aligned} \mathbf{b}_{c/d_3}^c &\to \mathbf{s} \\ R_{c/d_3}^{c^{-1}} &\to R_{c3} \\ A_{d_3}^c R_{d_3/d_1}^{d_3^{-1}} A_c^{d_3} &\to R_{31} \\ A_{d_3}^c R_{d_3/d_2}^{d_3^{-1}} A_c^{d_3} &\to R_{32} \end{aligned} \tag{43}$$

The derivation begins by noting that the inter-deputy LOS vectors can be written as:

$$\mathbf{b}_{d_2/d_1}^c = k_{21}\left(\alpha \mathbf{b}_{c/d_1}^c - \beta \mathbf{b}_{c/d_2}^c\right) = k_{21}\left(\alpha \mathbf{r} - \beta \mathbf{b}\right) \tag{44a}$$

$$\mathbf{b}_{d_3/d_1}^c = k_{31}\left(\alpha \mathbf{b}_{c/d_1}^c - \gamma \mathbf{b}_{c/d_3}^c\right) = k_{31}\left(\alpha \mathbf{r} - \gamma \mathbf{s}\right) \tag{44b}$$

$$\mathbf{b}_{d_3/d_2}^c = k_{32}\left(\beta \mathbf{b}_{c/d_2}^c - \gamma \mathbf{b}_{c/d_3}^c\right) = k_{32}\left(\beta \mathbf{b} - \gamma \mathbf{s}\right) \tag{44c}$$

where α, β,γ, k_{21}, k_{31} and k_{32} are positive constants. (See Appendix A for proof) Using the definitions in Eqs. (30), (43) and (44) in Eq. (41) results in

$$W_{c/d_1}^c = [\mathbf{r}\times] R_{c1} [\mathbf{r}\times] \tag{45a}$$

$$W_{c/d_2}^c = [\mathbf{b}\times] R_{c2} [\mathbf{b}\times] \tag{45b}$$

$$W_{c/d_3}^c = [\mathbf{s}\times] R_{c3} [\mathbf{s}\times] \tag{45c}$$

$$W_{d_2/d_1}^c = k_{21}^2\left(\alpha^2 [\mathbf{r}\times] R_{21} [\mathbf{r}\times] + \beta^2 [\mathbf{b}\times] R_{21} [\mathbf{b}\times] - \alpha\beta [\mathbf{b}\times] R_{21} [\mathbf{r}\times] - \alpha\beta [\mathbf{r}\times] R_{21} [\mathbf{b}\times]\right) \tag{45d}$$

$$W_{d_3/d_1}^c = k_{31}^2\left(\alpha^2 [\mathbf{r}\times] R_{31} [\mathbf{r}\times] + \gamma^2 [\mathbf{s}\times] R_{31} [\mathbf{s}\times] - \alpha\gamma [\mathbf{r}\times] R_{31} [\mathbf{s}\times] - \alpha\gamma [\mathbf{s}\times] R_{31} [\mathbf{r}\times]\right) \tag{45e}$$

$$W_{d_3/d_2}^c = k_{32}^2\left(\beta^2 [\mathbf{b}\times] R_{32} [\mathbf{b}\times] + \gamma^2 [\mathbf{s}\times] R_{32} [\mathbf{s}\times] - \beta\gamma [\mathbf{b}\times] R_{32} [\mathbf{s}\times] - \beta\gamma [\mathbf{s}\times] R_{32} [\mathbf{b}\times]\right) \tag{45f}$$

Following the results of the three formation, it is assumed that a vector in the null space of F takes on the form

$$\mathbf{z} \equiv \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix} = \begin{bmatrix} m\mathbf{r} \\ p\mathbf{b} \\ q\mathbf{s} \end{bmatrix} \tag{46}$$

so that $F\mathbf{z} = \mathbf{0}$ results in the following equations:

$$mF_{11}\mathbf{r} + pF_{12}\mathbf{b} + qF_{13}\mathbf{s} = \mathbf{0} \tag{47a}$$

$$mF_{12}\mathbf{r} + pF_{22}\mathbf{b} + qF_{23}\mathbf{s} = \mathbf{0} \tag{47b}$$

$$mF_{13}\mathbf{r} + pF_{23}\mathbf{b} + qF_{33}\mathbf{s} = \mathbf{0} \tag{47c}$$

Upon substituting Eqs. (45) and (46) into Eq. (47a), simplifying and gathering the results leads to

$$k_{21}^2(\alpha\beta p - \beta^2 m)\,[\mathbf{b}\times]\,R_{21}\,[\mathbf{b}\times]\,\mathbf{r} + k_{21}^2(\alpha^2 p - \alpha\beta m)\,[\mathbf{r}\times]\,R_{21}\,[\mathbf{r}\times]\,\mathbf{b} + k_{31}^2(\alpha\gamma q - \gamma^2 m)\,[\mathbf{s}\times]\,R_{31}\,[\mathbf{s}\times]\,\mathbf{r} + k_{31}^2(\alpha^2 q - \alpha\gamma m)\,[\mathbf{r}\times]\,R_{31}\,[\mathbf{r}\times]\,\mathbf{s} = 0 \tag{48}$$

Similarly, the results for Eqs. (47b) and (47c) are given by

$$k_{21}^2(\beta^2 m - \alpha\beta p)\,[\mathbf{b}\times]\,R_{21}\,[\mathbf{b}\times]\,\mathbf{r} + k_{21}^2(\alpha\beta m - \alpha^2 p)\,[\mathbf{r}\times]\,R_{21}\,[\mathbf{r}\times]\,\mathbf{b} + k_{32}^2(\beta^2 q - \beta\gamma p)\,[\mathbf{b}\times]\,R_{32}\,[\mathbf{b}\times]\,\mathbf{s} + k_{32}^2(\beta\gamma q - \gamma^2 p)\,[\mathbf{s}\times]\,R_{32}\,[\mathbf{s}\times]\,\mathbf{b} = 0 \tag{49a}$$

$$k_{31}^2(\gamma^2 m - \alpha\gamma q)\,[\mathbf{s}\times]\,R_{31}\,[\mathbf{s}\times]\,\mathbf{r} + k_{31}^2(\alpha\gamma m - \alpha^2 q)\,[\mathbf{r}\times]\,R_{31}\,[\mathbf{r}\times]\,\mathbf{s} + k_{32}^2(\gamma^2 p - \beta\gamma q)\,[\mathbf{s}\times]\,R_{32}\,[\mathbf{s}\times]\,\mathbf{b} + k_{32}^2(\beta\gamma p - \beta^2 q)\,[\mathbf{b}\times]\,R_{32}\,[\mathbf{b}\times]\,\mathbf{s} = 0 \tag{49b}$$

In order to prove the existence of a null vector, all of the leading coefficients in Eqs. (48) - (49) need to equate to zero. Of the twelve coefficients, through cancelling it can be found that there are actually only three independent conditions which must be satisfied:

$$\alpha p = \beta m \tag{50a}$$
$$\alpha q = \gamma m \tag{50b}$$
$$\beta q = \gamma p \tag{50c}$$

A solution to which is given by $m = \alpha$, $p = \beta$ and $q = \gamma$ so that the null vector is given by

$$\mathbf{z} = \begin{bmatrix} \alpha\mathbf{r} \\ \beta\mathbf{b} \\ \gamma\mathbf{s} \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{b}_{c/d_1}^c \\ \beta\mathbf{b}_{c/d_2}^c \\ \gamma\mathbf{b}_{c/d_3}^c \end{bmatrix} \tag{51}$$

C. Discussion of Four Vehicle Singularity

The results of the previous section show that even with the additional LOS information from the fourth vehicle, the relative attitudes within the formation are still unobervable. At first the results are counter-intuitive, but further examination of the total information available to the system gives insight into why the attitude are unobservable. Begin by noting that for the four vehicle system there were three unknown relative attitudes constituting 9 unknowns. For the formation shown in Fig. 2 there are 6 LOS pairs yielding 12 pieces of information available to the system. With more information than unknowns it would appear that the attitudes would be over-determined. However, the LOS vectors not on the outer ring, i.e. $\mathbf{b}_{c/d_2}$ and $\mathbf{b}_{d_1/d_3}$ each lead to the construction of two triangles a piece. Namely, $\mathbf{b}_{c/d_2}$ constructs $\triangle\mathcal{C} - \mathcal{D}_1 - \mathcal{D}_2$ and $\triangle\mathcal{C} - \mathcal{D}_2 - \mathcal{D}_3$, leading to two constraints. With the four constraints taken into consideration there are now 13 pieces of information needed with only 12 pieces of information available. The result is that the system is rank one deficient which is apparent through numerical analysis of Eq. (42).

D. Four Vehicle Formations with Missing Links

The preceding analysis has focused on the formation when all communications links are present. However, it may not be possible for all of the spacecraft to communicate with each other, be it because of a communications fault of bandwidth limitations. Consider the case where the deputy 1 - deputy 3 link is not present. It can be shown that the Fisher information matrix for this system is given by

$$F = \begin{bmatrix} -\left(W_{c/d_1}^c + W_{d_2/d_1}^c\right) & W_{d_2/d_1}^c & 0 \\ W_{d_2/d_1}^c & -\left(W_{c/d_2}^c - W_{d_2/d_1}^c + W_{d_3/d_2}^c\right) & W_{d_3/d_2}^c \\ 0 & W_{d_3/d_2}^c & -\left(W_{c/d_3}^c + W_{d_3/d_2}^c\right) \end{bmatrix} \tag{52}$$

which is equivalent to Eq. (42) with W_{d_3/d_1}^c set equal to zero thus accounting for the missing link. For this formation the relative attitudes remain unobservable and the Fisher information matrix is rank one deficient.

This is because there is 10 pieces of information available throught the five existing LOS pairs, nine unknown attitude parameters and two constraints in the form of $\triangle\mathcal{C}-\mathcal{D}_1-\mathcal{D}_2$ and $\triangle\mathcal{C}-\mathcal{D}_2-\mathcal{D}_3$ for a total of 11 pieces of information necessary. It can be shown, and has been verified numerically, that the null vector to Eq. (52) is also given by Eq. (51).

It can be shown that for all combinations of one missing link, the Fisher information matrix can be constructed by setting the missing link equal to zero in Eq. (42) and that the corresponding null vector is given by Eq. (51). For configurations where more than one link is missing the Fisher information matrix is also singular because there will be at most eight pieces of information and nine unknown attitude components.

VI. Five Vehicle Formation

The following section presents and discusses the preliminary findings associated with formations of five vehicles arranged as shown in Fig. 3. Inclusion of the new fourth deputy (d_4) means that a fourth relative attitude, $A^c_{d_4}$, is now present. As with all the preceding analysis, it is assumed that all vectors shown lie within a plane and that no three vehicles are colinear.

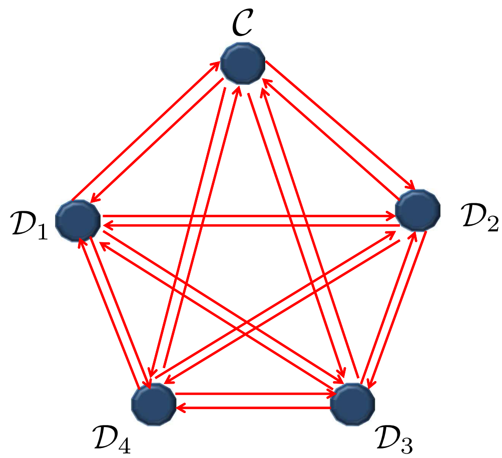


Figure 3. Five Vehicle Formations

A. Fisher Information Matrix Derivation

Derivation of the Fisher information matrix for the fully connected five vehicle formation follows the derivations of the preceding sections. Only the results here are presented for brevity:

$$\boldsymbol{\delta\alpha}^T \equiv \begin{bmatrix} \boldsymbol{\delta\alpha}^{c^T}_{d_1} & \boldsymbol{\delta\alpha}^{c^T}_{d_2} & \boldsymbol{\delta\alpha}^{c^T}_{d_3} & \boldsymbol{\delta\alpha}^{c^T}_{d_4} \end{bmatrix} \tag{53a}$$

$$F \equiv \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{12}^T & F_{22} & F_{23} & F_{24} \\ F_{13}^T & F_{23}^T & F_{33} & F_{34} \\ F_{14}^T & F_{24}^T & F_{34}^T & F_{44} \end{bmatrix} = \begin{bmatrix} F_{11} & W^c_{d_2/d_1} & W^c_{d_3/d_1} & W^c_{d_4/d_1} \\ W^c_{d_2/d_1} & F_{22} & W^c_{d_3/d_2} & W^c_{d_4/d_2} \\ W^c_{d_3/d_1} & W^c_{d_3/d_2} & F_{33} & W^c_{d_4/d_3} \\ W^c_{d_4/d_1} & W^c_{d_4/d_2} & W^c_{d_4/d_3} & F_{44} \end{bmatrix} \tag{53b}$$

where

$$F_{11} \equiv -\left(W^c_{c/d_1} + W^c_{d_2/d_1} + W^c_{d_3/d_1} + W^c_{d_4/d_1}\right) \tag{54a}$$

$$F_{22} \equiv -\left(W^c_{c/d_2} + W^c_{d_2/d_1} + W^c_{d_3/d_2} + W^c_{d_4/d_2}\right) \tag{54b}$$

$$F_{33} \equiv -\left(W^c_{c/d_3} + W^c_{d_3/d_1} + W^c_{d_3/d_2} + W^c_{d_4/d_3}\right) \tag{54c}$$

$$F_{44} \equiv -\left(W^c_{c/d_4} + W^c_{d_4/d_1} + W^c_{d_4/d_2} + W^c_{d_4/d_3}\right) \tag{54d}$$

Definitions of the matrices $W^c_{a/b}$ are given by Eq. (41) along with the new definitions

$$W^c_{c/d_4} \equiv \left[\mathbf{b}^c_{c/d_1}\times\right] R^{c^{-1}}_{c/d_1} \left[\mathbf{b}^c_{c/d_1}\times\right] \tag{55a}$$

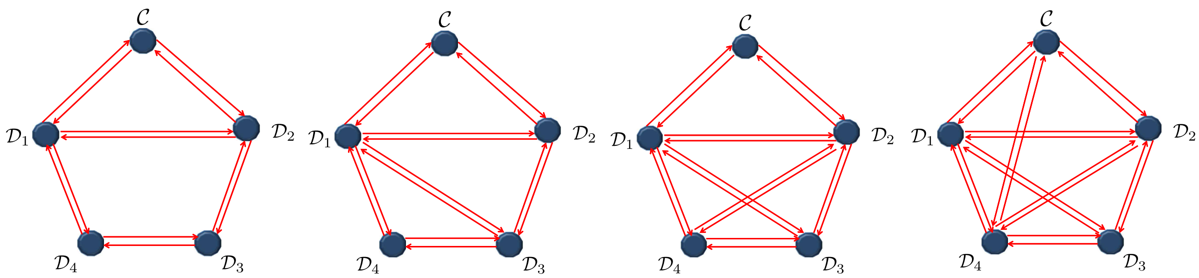
$$W^c_{d_4/d_1} \equiv \left[\mathbf{b}^c_{d_4/d_1}\times\right] A^c_{d_4} R^{d_4^{-1}}_{d_4/d_1} A^{d_4}_c \left[\mathbf{b}^c_{d_4/d_1}\times\right] \tag{55b}$$

$$W^c_{d_4/d_2} \equiv \left[\mathbf{b}^c_{d_4/d_2}\times\right] A^c_{d_4} R^{d_4^{-1}}_{d_4/d_2} A^{d_4}_c \left[\mathbf{b}^c_{d_4/d_2}\times\right] \tag{55c}$$

$$W^c_{d_4/d_3} \equiv \left[\mathbf{b}^c_{d_4/d_3}\times\right] A^c_{d_4} R^{d_4^{-1}}_{d_4/d_3} A^{d_4}_c \left[\mathbf{b}^c_{d_4/d_3}\times\right] \tag{55d}$$

B. Numerical Analysis of Fisher Information Matrix

This section examines the observability of the five vehicle formation numerically. Specifically, we look at the five vehicle formation with a varying number of LOS vectors in order see if the relative attitudes become observable; and if so, what is the minimum number of LOS vector pairs necessary for observability. In addition to the fully connected five vehicle formation shown in Fig. 3, the formations shown in Fig. 4 will also be considered.



(a) 6 LOS Vector Pairs (b) 7 LOS Vector Pairs (c) 8 LOS Vector Pairs (d) 9 LOS Vector Pairs

Figure 4. Five Vehicle Formations with Varying Number of LOS Vector Pairs

Numeric simulations are carried out by representing the vehicles as point masses within the 2-D plane. LOS vectors are generated by assuming that the LOS vectors originate and terminate at the vehicle point. These conditions strictly enforce the planar and parallel beam conditions. For simplicity, the measurement noise covaraince of all LOS vectors is given by

$$R^c \equiv \sigma^2 I_3 \tag{56}$$

with $\sigma^2 = 17 \times 10^{-5}$. The true LOS vectors are used within the Fisher information matrix to access the observability. The Fisher information matrix for the formation of Fig. 3 is that given by Eq. (53b). For the formations of Fig. 4 the Fisher information matrix is given by Eq. (53b) but with the entries corresponding to the missing links set equal to zero, in the same fashion as was shown in Sect. V.D. Figure 5 shows the rank deficiency of the Fisher information matrix for the five vehicle configurations under consideration. What is found is that for the 6 LOS vector pair configuration the Fisher information matrix is rank 2 deficient, implying that there are two fewer pieces of information than unknowns. For this case it was expected that the Fisher information matrix would be singular owing the fact that there are 12 unknown attitude parameters to be determined, 12 pieces of information from the LOS vectors and one triangle formed. It is unclear at the moment where the additional piece of information is being lost. At the time this manuscript was prepared no attempt has been made to analytically verify the vectors within the null space of the Fisher information matrix.

For the case of 7 LOS vector pairs the Fisher information matrix is only rank 1 deficient which is consistent with our previous assumptions regarding the total amount of information held within the configuration. The 7 LOS vector pairs supply 14 pieces of information while the LOS configuration leads to the formation of three triangles. The resulting system appears to have one fewer piece of information than is needed to be observable. Again, no attempt has been made to analytically verify the null vector.

For the cases of 8, 9 and 10 LOS vector pairs it was found that the Fisher information matrix is full rank, implying that the relative attitudes were all observable. These results are interesting in the sense

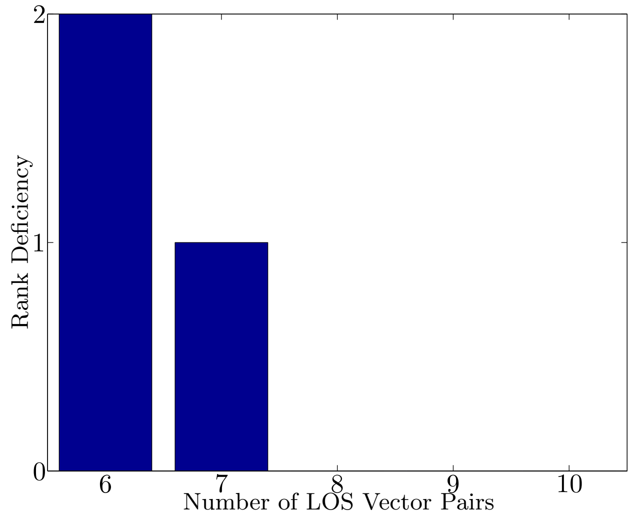


Figure 5. Rank Deficiency of the Fisher Information Matrix for Five Vehicle Formations

that they are not consistent with the previous findings regarding the total amount of information within the configurations. Consider the 8 LOS vector pair configuration. The 8 LOS vector pairs supply 16 pieces of information. The LOS vectors between deputies 1 and 2, deputies 1 and 3 and deputies 2 and 4 lead to the creation of 5 triangles. If all of the triangles used up one of the available pieces of information it would appear that the Fisher information matrix should be rank 1 deficient. Similar observations can be made about the 9 and 10 LOS pair configurations.

VII. Conclusion & Future Work

This paper presented current research efforts focused on studying the observability of relative attitudes within formations of planar vehicles. The observability analysis was conducted by means of examining the rank of the Fisher information matrix of the small attitude angle errors. For formations of three and four vehicles it was shown that the Fisher information matrix was rank deficient by showing the existence of a vector within the null space of the Fisher information matrix. Following in the steps of previously published works it was reasoned that the Fisher information matrix was singular because of LOS vectors within the formations formed planar triangles which absorbed pieces of information supplied by the LOS vectors.[4] For formations of five vehicles the results on the rank of the Fisher information matrix were not consistent with these earlier observations. It was found that for configurations of 6 and 7 LOS vector pairs the Fisher information matrix was rank deficient, while for configurations of 8, 9 and 10 LOS vector pairs the Fisher information matrix was full rank, implying that the relative attitudes were fully observable. It is unsure as to why these findings are not consistent with those from the three and four vehicle formations.

Future work in this area will focus on examining the five vehicle formations more closely in an effort to fully understand why, or why not, the relative attitudes are observable. Additionally, efforts into finding solutions for four and five vehicle relative attitudes. Observability analysis for multiple vehicle formations without planar vectors will also be examined as this is the most general case of formations and should be addressed.

A. Proof of Inter-Deputy LOS

This appendix proves the relationships in Eq. (44). Recall these relationships were given by

$$\mathbf{b}^c_{d_2/d_1} = k_{21}\left(\alpha \mathbf{b}^c_{c/d_1} - \beta \mathbf{b}^c_{c/d_2}\right) \tag{44a}$$

$$\mathbf{b}^c_{d_3/d_1} = k_{31}\left(\alpha \mathbf{b}^c_{c/d_1} - \gamma \mathbf{b}^c_{c/d_3}\right) \tag{44b}$$

$$\mathbf{b}^c_{d_3/d_2} = k_{32}\left(\beta \mathbf{b}^c_{c/d_2} - \gamma \mathbf{b}^c_{c/d_3}\right) \tag{44c}$$

where α, β, γ, k_{21}, k_{31} and k_{32} are positive constants. The LOS vectors between the vehicles within the formation are simply a unit vector representation of the position vector between vehicles within the formation. Let $\boldsymbol{\rho}^c_{c/d_1}$ be defined as the position vector from $\mathcal{C}$ to $\mathcal{D}_1$ expressed in terms of coordinate frame $\mathcal{C}$ with similar definitions for all the position vectors within the formation. The LOS vector $\mathbf{b}^c_{c/d_1}$, given in terms of $\boldsymbol{\rho}^c_{c/d_1}$ is

$$\mathbf{b}^c_{c/d_1} = \frac{\boldsymbol{\rho}^c_{c/d_1}}{\|\boldsymbol{\rho}^c_{c/d_1}\|} \tag{58}$$

Similarly, the LOS vector $\mathbf{b}^c_{d_2/d_1}$ can be written as

$$\mathbf{b}^c_{d_2/d_1} = \frac{\boldsymbol{\rho}^c_{d_2/d_1}}{\|\boldsymbol{\rho}^c_{d_2/d_1}\|} \tag{59}$$

Noting that the position vector $\boldsymbol{\rho}^c_{d_2/d_1}$ can be expressed as

$$\boldsymbol{\rho}^c_{d_2/d_1} = \boldsymbol{\rho}^c_{c/d_1} - \boldsymbol{\rho}^c_{c/d_2} \tag{60}$$

and substituting into Eq. (59) leads to

$$\mathbf{b}^c_{d_2/d_1} = \frac{\boldsymbol{\rho}^c_{c/d_1} - \boldsymbol{\rho}^c_{c/d_2}}{\|\boldsymbol{\rho}^c_{d_2/d_1}\|} \tag{61}$$

Rearranging Eq. (58) and substituting into Eq. (61) leads to

$$\mathbf{b}^c_{d_2/d_1} = \frac{\|\boldsymbol{\rho}^c_{c/d_1}\|\ \mathbf{b}^c_{c/d_1} - \|\boldsymbol{\rho}^c_{c/d_2}\|\ \mathbf{b}^c_{c/d_2}}{\|\boldsymbol{\rho}^c_{d_2/d_1}\|} \tag{62}$$

Similarly, for $\mathbf{b}^c_{d_3/d_1}$ and $\mathbf{b}^c_{d_3/d_2}$ can be expressed as

$$\mathbf{b}^c_{d_3/d_1} = \frac{\|\boldsymbol{\rho}^c_{c/d_1}\|\ \mathbf{b}^c_{c/d_1} - \|\boldsymbol{\rho}^c_{c/d_3}\|\ \mathbf{b}^c_{c/d_3}}{\|\boldsymbol{\rho}^c_{d_3/d_1}\|} \tag{63a}$$

$$\mathbf{b}^c_{d_3/d_2} = \frac{\|\boldsymbol{\rho}^c_{c/d_2}\|\ \mathbf{b}^c_{c/d_2} - \|\boldsymbol{\rho}^c_{c/d_3}\|\ \mathbf{b}^c_{c/d_3}}{\|\boldsymbol{\rho}^c_{d_3/d_2}\|} \tag{63b}$$

With the use of the following associations:

$$\begin{aligned} \alpha &= \|\boldsymbol{\rho}^c_{c/d_1}\| \qquad & k_{21} &= \|\boldsymbol{\rho}^c_{d_2/d_1}\| \\ \beta &= \|\boldsymbol{\rho}^c_{c/d_2}\| \qquad & k_{31} &= \|\boldsymbol{\rho}^c_{d_3/d_1}\| \\ \gamma &= \|\boldsymbol{\rho}^c_{c/d_3}\| \qquad & k_{32} &= \|\boldsymbol{\rho}^c_{d_3/d_2}\| \end{aligned}$$

it is clear that Eqs. (62) and (63) reduce to Eq. (44) which is the desired result. It should be noted that when given only the LOS vectors within the formation the constants cannot be uniquely determined (the problem is in fact over determined), although simply proving the relationships of Eq. (44) are sufficient for the context of this paper.

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