Efficient Covariance Intersection of Attitude Estimates Using a Local Error Representation

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I. Introduction

As the complexity of spacecraft and space systems increases, there is an ever increasing need for more accurate and robust estimators to track the system. While the demand placed on an individual spacecraft is increasing, the spacecraft itself is being pushed towards smaller, independent pieces of hardware which may have limited power and/or computational abilities. One such initiative is often refereed to as Operational Responsive Space (ORS). The primary goal of ORS systems is acceleration of the mission conception-to-launch timeline of spacecraft by means of utilizing off-the-shelf type modular components for construction of the spacecraft. With such modular systems, customized interfacing and software necessary to direct all data to a central processor may not be possible given the accelerated timeline, the unfortunate result of which would be degradation of the estimates due to missing data.

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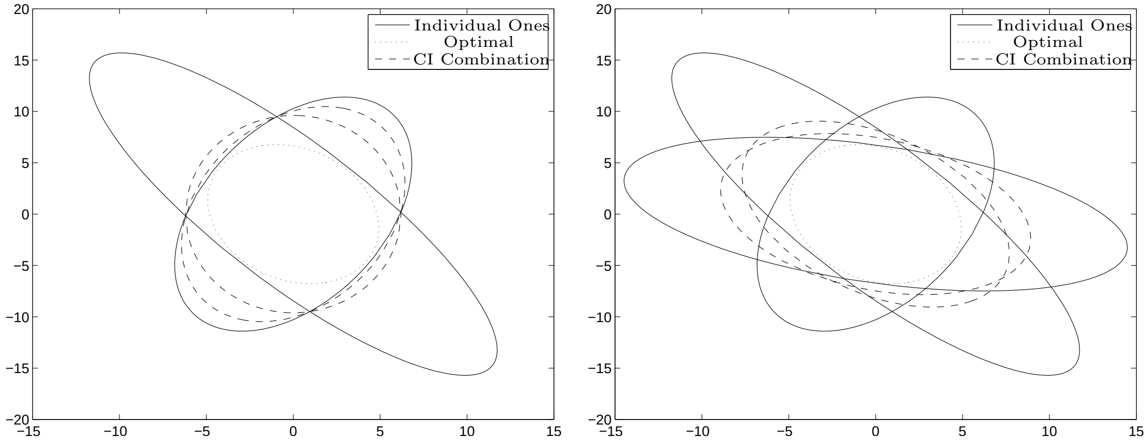
Another potentially hazardous situation arises when data is transmitted to some, but not all processors within a large, distributed system. Because all data cannot be transmitted to each processor, the resulting estimates are suboptimal in the sense that only a subset of data from the total data set can be reduced. In an ideal world, it would be possible to combine all of the suboptimal estimates in such a manner that the optimal solution, one which is obtained by processing all data, could be reconstructed. The main pitfall in this scenario is that, because multiple estimators will, in general, make use of the same piece of data, each individual estimate will be correlated with no means of determining the correlation. A simple yet elegant method to overcome this data fusion problem is the Covariance Intersection (CI) algorithm.[1,2] The CI algorithm overcomes the problem of unknown correlations by considering the intersection of error spaces which leads to a conservative, yet consistent fusion. Consistency occurs when the estimate converges in a probabilistic sense to the truth, for large samples. In addition, for a consistent fusion it must hold true that the difference between the fused estimate error covariance and true covariance is positive semi-definite.

One drawback of the CI algorithm is the implicit assumption that the data to be fused obtains a strictly additive error representation. This type of error representation only occurs when the data is not subject to any constraints. Crassidis *et al.*[3] have shown that the CI algorithm can be viewed as the solution to an optimization problem which minimizes the scalar- and matrix-weighted errors between the individual data sources and the fused estimate. This work uses the CI algorithm to fuse estimates from two extended Kalman filters for an attitude estimation problem. Using the quaternion as the attitude parametrization of choice, several solutions are presented which solve the CI equations while maintaining integrity of the quaternion norm constraint. For cases where the state vector to be fused includes multiple quaternions, the methods of Ref. [3] cannot be generalized to find the CI solution because the states must obey multiple quadratic equality constraints. One solution has been presented in Ref. [4] which uses an iterative Newton-Raphson approach to determine the optimal fused estimates.

The main difficulty in solving the quaternion CI problem lies in maintaining the quaternion norm constraint. One direct method for avoiding this problem is to utilize a three-dimensional attitude parametrization such as the Rodrigues parameters,[5] or Modified Rodrigues Parameters (MRPs)[6,7] which do not include the constraint. However, all three-dimensional parameterizations are singular at specific attitudes further complicating the problem. A unique solution to this problem uses the three-dimensional parametrizations only in a local error representation which allows the quaternion to remain the global attitude parametrization. This approach is especially prevalent in nonlinear estimation techniques such as unscented[8–10] and particle filtering.[11]

This Note presents an approach to overcome the difficulties in the CI fusion of attitude



(a) Fusion of Two Estimates (b) Fusion of Three Estimates

Figure 1. Geometric Interpretation of CI Algorithm

estimates by using a local MRP approach which circumvents the quaternion norm constraint while maintaining a singular free attitude representation. The organization of this Note is as follows. First, a compact review of the CI algorithm is given. Next, MRPs are introduced and an appropriate error representation is developed. Next the CI problem is solved using the local error representation. A simulation is presented which demonstrates that the proposed algorithm is numerically better conditioned than existing algorithms.

II. Covariance Intersection

The Covariance Intersection (CI) algorithm is a data fusion algorithm which allows for consistent fusion in the presence of unknown correlations. The CI algorithm is rooted in the principles of Gaussian intersection.[1] An appreciation for the CI algorithm can be gained by considering fusion of a simple two-dimensional state vector estimated from two different sources. In $\mathbb{R}^2$ the estimate covariance can be represented by ellipses as shown in Fig. 1(a). When the correlations between the individual estimates are known exactly, the optimal covariance can be exactly reconstructed. The resulting ellipse will lie completely within the intersection of the individual estimates covariance ellipses. When the correlations are unknown and the CI algorithm is used, the resulting fused covariance passes through the four points of intersection of the individual covariance ellipses as seen in Fig. 1(a). When more than two estimates are fused, one finds that, rather than passing through the points of intersection of the individual covariance ellipses, the CI combination simply contains the (set) intersection of all of the individual covariance ellipses, i.e. $P_{\text{CI}} \supseteq \bigcap_i P_i$. This can be seen in Fig. 1(b).

Correlated estimates arise in various space applications. Consider the example of a simplified spacecraft attitude determination system seen in Fig. 2. In this example the spacecraft is equipped with two star trackers, each obtaining individual star measurements. Each star tracker has its own associated extended Kalman filter which processes the star measurements. However, a common gyro is used by both filters rendering the estimates correlated. With use of the CI algorithm, the two estimates can be fused to obtain a better representation of the state vector $\mathbf{x}_{\text{CI}}$ and its covariance P_{CI}.

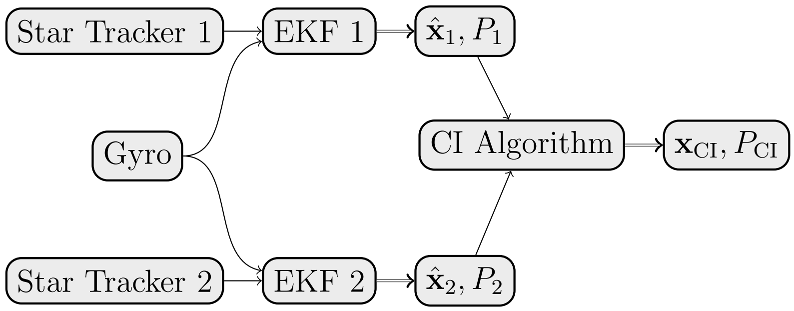


Figure 2. Example of Correlated Estimates

The CI algorithm is formally stated as: a consistent estimate $\mathbf{x}_{\text{CI}}$ and associated covariance P_{CI} can be constructed by fusing consistent estimates $\{\hat{\mathbf{x}}_i,\ P_i\},\ i = 1, \ldots, n$ by[12,13]

$$P_{\text{CI}}^{-1} = \sum_{i=1}^{n} \gamma_i P_i^{-1} \tag{1a}$$

$$\mathbf{x}_{\text{CI}} = P_{\text{CI}} \sum_{i=1}^{n} \gamma_i P_i^{-1} \hat{\mathbf{x}}_i \tag{1b}$$

where $\gamma_i \in [0,\ 1]$ is a scalar weight that satisfies $\sum_i \gamma_i = 1$. The conditions on the weights ensure that the CI solution maintains consistency. The weights are somewhat arbitrary but typically chosen to minimize the trace or determinant of the fused covariance matrix P_{CI}. Selection of different values of γ_i results in different CI combinations of the covariance as shown in Fig. 1. This means that estimate fusion with the CI algorithm is non-unique. Several metrics for determining the weights were explored in Ref. [3] and it was determined that selecting the weights to minimize the trace of P_{CI} produced the best results.

Reference [3] has shown that Eq. (1) is also the solution to the following optimization problem:

$$J(\mathbf{x}_{\text{CI}}) = \sum_{i=1}^{n} \gamma_i \Delta\mathbf{x}_i^T P_i^{-1} \Delta\mathbf{x}_i \tag{2}$$

where $\Delta\mathbf{x}_i \equiv \mathbf{x}_{\text{CI}} - \hat{\mathbf{x}}_i$ is the error between the individual estimate and the CI fused estimate.

III. Modified Rodrigues Parameters

While the quaternion has been the preferred attitude parametrization since Davenport's q-Method[14] was proposed as a solution to Wahba's problem,[15] the modified Rodrigues parameters (MRPs) have been gaining attention for use in spacecraft estimation and control. Schaub and Junkins[16] have shown that MRPs, along with Rodrigues Parameters, belong to a more general group called stereographic parameters. These parameters are the result of a stereographic projection of the quaternion onto a three-dimensional hyperplane. In particular, the Rodrigues parameters and MRPs are part of a subset of stereographic parameters which are referred to as symmetric. The general form of the symmetric stereographic parameters are defined in terms of the quaternion, $\mathbf{q}^T \equiv [\boldsymbol{\varrho}^T \quad q_4]$, as

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\varrho}}{q_4 - a} \tag{3}$$

where $a = \cos(\phi_s/2)$ is the projection point determined by the singular rotation, ϕ_s. Choosing the singularity to be at $\pm\pi$ leads to $a = 0$ which defines the classic Rodrigues parameters. The MRPs, $\mathbf{p}$, are singular at $\pm 2\pi$ which corresponds to $a = -1$. In particular, MRPs are given by

$$\mathbf{p} \equiv \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \frac{\boldsymbol{\varrho}}{1 + q_4} \tag{4}$$

The inverse transformation from MRPs to quaternion components are given by

$$\boldsymbol{\varrho} = \frac{2\mathbf{p}}{1 + \mathbf{p}^T\mathbf{p}} \tag{5a}$$

$$q_4 = \frac{1 - \mathbf{p}^T\mathbf{p}}{1 + \mathbf{p}^T\mathbf{p}} \tag{5b}$$

While MRPs represent an unconstrained parametrization, they do not obey a strict, additive error representation. The reason for this is that each attitude can be characterized by two distinct MRPs, one lying within the unit sphere and the other outside the unit sphere.[17] MRPs outside the unit sphere are typically referred to as the shadow set. Thus any strictly additive correction to an MRP could push the MRP from inside the unit sphere to the outside, potentially landing on an erroneous shadow MRP characterizing an incorrect attitude. To overcome this difficulty, consider a rotation of $\boldsymbol{\delta}\mathbf{p}$ about $\mathbf{p}_a$ which results in $\mathbf{p}_b$. Symbolically this is expressed as $\mathbf{p}_b = \boldsymbol{\delta}\mathbf{p} \odot \mathbf{p}_a$ where the MRP composition operator $\odot$ is defined as[7]

$$\mathbf{p}_b = \frac{(1 - \|\mathbf{p}_a\|^2)\, \boldsymbol{\delta}\mathbf{p} + (1 - \|\boldsymbol{\delta}\mathbf{p}\|^2)\, \mathbf{p}_a - 2\, [\boldsymbol{\delta}\mathbf{p}\times]\, \mathbf{p}_a}{1 + \|\boldsymbol{\delta}\mathbf{p}\|^2 \|\mathbf{p}_a\|^2 - 2\boldsymbol{\delta}\mathbf{p}^T\mathbf{p}_a} \tag{6}$$

where $\|\cdot\|$ expresses the ℓ_2 norm of the argument and $[\mathbf{a}\times]$ is the skew-symmetric cross-product matrix defined so that $[\mathbf{a}\times]\mathbf{b} = \mathbf{a} \times \mathbf{b}$.[18] Assuming that $\boldsymbol{\delta}\mathbf{p}$ is small allows second order terms in $\boldsymbol{\delta}\mathbf{p}$ to be neglected, resulting in the following linear approximation:[19]

$$\begin{aligned}\mathbf{p}_b &\approx \left(1 + 2\boldsymbol{\delta}\mathbf{p}^T\mathbf{p}_a\right)\left[\left(1 - \mathbf{p}_a^T\mathbf{p}_a\right)\boldsymbol{\delta}\mathbf{p} + \mathbf{p}_a - 2\left[\boldsymbol{\delta}\mathbf{p}\times\right]\mathbf{p}_a\right] \\ &= \mathbf{p}_a + \left[\left(1 - \mathbf{p}_a^T\mathbf{p}_a\right)I_3 + 2\left[\mathbf{p}_a\times\right] + 2\mathbf{p}_a\mathbf{p}_a^T\right]\boldsymbol{\delta}\mathbf{p} \end{aligned} \tag{7}$$

The error MRP can then be expressed as

$$\boldsymbol{\delta}\mathbf{p} = B^{-1}(\mathbf{p}_a)\left(\mathbf{p}_b - \mathbf{p}_a\right) \tag{8}$$

where

$$B(\mathbf{p}) = \left(1 - \mathbf{p}^T\mathbf{p}\right)I_3 + 2\left[\mathbf{p}\times\right] + 2\mathbf{p}\mathbf{p}^T \tag{9}$$

Equation (8) can further be simplified through realization of the matrix inverse, $B^{-1}(\mathbf{p})$, which is given by[5, 20, 21]

$$B^{-1}(\mathbf{p}) = \left(\frac{1}{1 + \mathbf{p}^T\mathbf{p}}\right)^2 B^T(\mathbf{p}) \tag{10}$$

This leads to the following simple form for the error MRP:

$$\boldsymbol{\delta}\mathbf{p} = \left(\frac{1}{1 + \mathbf{p}_a^T\mathbf{p}_a}\right)^2 B^T(\mathbf{p}_a)\left(\mathbf{p}_b - \mathbf{p}_a\right) \tag{11}$$

Note that this form for the error MRP is free of matrix inverses requiring only simple matrix multiplication which is computationally efficient. The assumption of small $\boldsymbol{\delta}\mathbf{p}$ also leads to the association of $\boldsymbol{\delta}\mathbf{p}$ with the small quarter-angle attitude errors, $\boldsymbol{\delta}\mathbf{p} \approx \boldsymbol{\delta\alpha}/4$, where $\boldsymbol{\delta\alpha}$ are the small roll, pitch and yaw attitude angle errors.

IV. Modified Rodrigues Parameter Covariance Intersection

Equation (2) provides a general framework for data fusion using the CI algorithm. In this section, focus is placed on the problem of fusing state vectors which consist of quaternions and other unconstrained variables (such as gyro biases), i.e. $\mathbf{x}^T \equiv [\mathbf{q}_1^T \ \ \dots \ \ \mathbf{q}_N^T \ \ \boldsymbol{\beta}^T]$. The i^{th} error vector is comprised of the small attitude errors for each of the quaternions and the additive error between the unconstrained variables, $\Delta\mathbf{x}_i^T = [\boldsymbol{\delta\alpha}_{1,i}^T \ \ \dots \ \ \boldsymbol{\delta\alpha}_{N,i}^T \ \ \Delta\boldsymbol{\beta}_i]$. If the state vector $\mathbf{x}$ is instead written in terms of MRPs, i.e. $\bar{\mathbf{x}}^T \equiv [\mathbf{p}_1^T \ \ \dots \ \ \mathbf{p}_N^T \ \ \boldsymbol{\beta}^T]$, then the small angle attitude errors can be expressed using Eq. (11):

$$\boldsymbol{\delta\alpha}_{j,i} = k_{j,i}B^T(\hat{\mathbf{p}}_{j,i})\left(\mathbf{p}_j - \hat{\mathbf{p}}_{j,i}\right) \tag{12}$$

where $k_{j,i} \equiv 4/\left(1+\hat{\mathbf{p}}_{j,i}^T\hat{\mathbf{p}}_{j,i}\right)^2$. The i^{th} error vector can then be expressed as

$$\Delta\mathbf{x}_i = \begin{bmatrix} k_{1,i}B^T(\mathbf{p}_{1,i}) & 0_{3\times3} & \dots & 0_{3\times3} & 0_{3\times n_\beta} \\ 0_{3\times3} & k_{2,i}B^T(\mathbf{p}_{2,i}) & \dots & 0_{3\times3} & 0_{3\times n_\beta} \\ 0_{3\times3} & 0_{3\times3} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \dots & k_{N,i}B^T(\mathbf{p}_{N,i}) & \vdots \\ 0_{n_\beta\times3} & 0_{n_\beta\times3} & \dots & 0_{n_\beta\times3} & I_{n_\beta} \end{bmatrix} \left(\bar{\mathbf{x}}_{\text{CI}} - \hat{\bar{\mathbf{x}}}_i\right)$$
$$\equiv \mathcal{B}_i\left(\bar{\mathbf{x}}_{\text{CI}} - \hat{\bar{\mathbf{x}}}_i\right) \tag{13}$$

where n_β is the dimension length of $\boldsymbol{\beta}$ and I_z is the $z\times z$ identity matrix. Because MRPs are unconstrained, Eq. (13) can be directly substituted into Eq. (2) resulting in the following quadratic cost function:

$$J(\bar{\mathbf{x}}_{\text{CI}}) = \bar{\mathbf{x}}_{\text{CI}}^T\mathcal{R}\bar{\mathbf{x}}_{\text{CI}} - 2\bar{\mathbf{x}}_{\text{CI}}^T\mathcal{S} + \mathcal{T} \tag{14}$$

given the following definitions:

$$\mathcal{C}_i \equiv \gamma_i\mathcal{B}_i^T P_i^{-1}\mathcal{B}_i \tag{15a}$$
$$\mathcal{R} \equiv \sum_{i=1}^{n}\mathcal{C}_i \tag{15b}$$
$$\mathcal{S} \equiv \sum_{i=1}^{n}\mathcal{C}_i\hat{\bar{\mathbf{x}}}_i \tag{15c}$$
$$\mathcal{T} \equiv \sum_{i=1}^{n}\hat{\bar{\mathbf{x}}}_i^T\mathcal{C}_i\hat{\bar{\mathbf{x}}}_i \tag{15d}$$

Note $P_i \equiv E\{\Delta\mathbf{x}_i\Delta\mathbf{x}_i^T\}$ is the reduced form of the covariance which includes the small roll, pitch and yaw attitude angle errors. This covariance is invariant whether quaternions or MRPs are the attitude parametrization as the errors associated with each are identical to within first order.[22] The optimal solution to Eq. (14) is

$$\bar{\mathbf{x}}_{\text{CI}} = \mathcal{R}^{-1}\mathcal{S} \tag{16}$$

A. Local Error Representation

While Eq. (16) provides a valid solution to the CI fusion, numeric difficulties can surface when the quaternion-to-MRP conversion approaches the MRP singularity. In addition, fusing MRPs near the unit sphere can cause difficulties as discussed before. To overcome these difficulties, a local error representation in terms of MRPs is used. Definition of a local error utilizes the concept of a reference attitude from which each of the other attitudes can be represented as a perturbation about the reference. With proper selection of the reference attitude, each of the perturbations will be sufficiently small such that the small angle approximation in Eq. (12) is valid. This allows the MRP CI approach outlined in the previous section to be applied without having to deal with the singularity and while maintaining a numerically well-conditioned problem. In this section, for clarity, we drop the subscript j on $\hat{\mathbf{q}}_{j,i}$ and associated quantities. All of the results presented are valid for each quaternion within the state vector without coupling between the quaternions.

First consider that a CI fused quaternion can be expressed as a small rotation about one of the individual estimates:

$$\mathbf{q}_{\text{CI}} = \boldsymbol{\delta}\mathbf{q}_i \otimes \hat{\mathbf{q}}_i \tag{17}$$

where $\otimes$ is the quaternion composition operator.[5] If there exists some reference quaternion $\bar{\mathbf{q}}$ such that the CI fused quaternion estimate satisfies $\mathbf{q}_{\text{CI}} = \boldsymbol{\delta}\bar{\mathbf{q}} \otimes \bar{\mathbf{q}}$, then it follows that

$$\boldsymbol{\delta}\mathbf{q}_i \otimes \hat{\mathbf{q}}_i = \boldsymbol{\delta}\bar{\mathbf{q}} \otimes \bar{\mathbf{q}} \tag{18}$$

Relating the reference quaternion to the individual estimate as

$$\bar{\mathbf{q}} = \boldsymbol{\delta}\mathbf{q}_{\hat{\mathbf{q}}_i \to \bar{\mathbf{q}}} \otimes \hat{\mathbf{q}}_i \tag{19}$$

and substituting into Eq. (18) leads to

$$\boldsymbol{\delta}\mathbf{q}_i \otimes \hat{\mathbf{q}}_i = \boldsymbol{\delta}\bar{\mathbf{q}} \otimes \boldsymbol{\delta}\mathbf{q}_{\hat{\mathbf{q}}_i \to \bar{\mathbf{q}}} \otimes \hat{\mathbf{q}}_i \tag{20}$$

Thus the individual errors can be expressed as

$$\boldsymbol{\delta}\mathbf{q}_i = \boldsymbol{\delta}\bar{\mathbf{q}} \otimes \Delta\mathbf{q}_i \tag{21}$$

where $\Delta\mathbf{q}_i$ is compact notation for $\boldsymbol{\delta}\mathbf{q}_{\hat{\mathbf{q}}_i \to \bar{\mathbf{q}}}$. Rearranging Eq. (21) as

$$\boldsymbol{\delta}\bar{\mathbf{q}} = \boldsymbol{\delta}\mathbf{q}_i \otimes \Delta\mathbf{q}_i^{-1} \tag{22}$$

and converting directly to MRPs results in

$$\mathbf{p}_{\boldsymbol{\delta}\bar{\mathbf{q}}} = \boldsymbol{\delta}\mathbf{p}_i \odot (-\mathbf{p}_{\Delta\mathbf{q}_i}) \tag{23}$$

where the relationship $\mathbf{p}_{\Delta\mathbf{q}_i^{-1}} = -\mathbf{p}_{\Delta\mathbf{q}_i}$ has been used. Using Eq. (11) allows the errors in the individual MRP estimates to be written as

$$\boldsymbol{\delta}\mathbf{p}_i = \left(\frac{1}{1+\mathbf{p}_{\Delta\mathbf{q}_i}^T\mathbf{p}_{\Delta\mathbf{q}_i}}\right)^2 B^T(-\mathbf{p}_{\Delta\mathbf{q}_i})\left[\boldsymbol{\delta}\mathbf{p}_{\boldsymbol{\delta}\bar{\mathbf{q}}} - (-\mathbf{p}_{\Delta\mathbf{q}_i})\right] \tag{24}$$

which can be used directly in Eq. (12) without any further modification. Note that now, rather than estimating the fused MRP corresponding to the fused attitude, a correction to the reference attitude is being estimated. Once the correction $\boldsymbol{\delta}\mathbf{p}_{\boldsymbol{\delta}\bar{\mathbf{q}}}$ is estimated with Eq. (16), it can be converted to the corresponding quaternion correction via Eq. (5). The CI fused quaternion is then determined from $\mathbf{q}_{\text{CI}} = \boldsymbol{\delta}\bar{\mathbf{q}} \otimes \bar{\mathbf{q}}$.

The primary advantage of the local error approach is that singularities in the MRP can be strictly avoided through proper selection of the reference attitude, $\bar{\mathbf{q}}$. If the reference attitude is close to the estimates, each $\boldsymbol{\delta}\mathbf{q}_i$ will have a fourth component very close to $+1$. Therefore, the conversion to MRPs will always be defined and numerically well behaved. The choice of reference quaternion is arbitrary so long as the quaternion is sufficiently close to the individual (and by default fused) quaternions. One option that can be considered is selecting the reference quaternion to be equal to one of the individual estimates. This is the approach taken in Ref. [22] for attitude filtering. While computationally efficient, this approach could suffer from the existence of an erroneous quaternion estimate which is relatively far from the other estimates and the optimal fusion. To avoid this, the reference quaternion can be selected as the average quaternion from the set of individual estimates.[23] For the case when state vectors contain more than a single quaternion, the averaging quaternions algorithm reduces to a form consistent with the CI algorithm in order to account for correlations between quaternions within the state vector. Because of this close relation between the CI and the averaging quaternion algorithm, the averaging quaternions algorithm is well suited for determination of an appropriate reference quaternion. For practical applications within the scope of defining a reference attitude, the averaging quaternion can be determined assuming independence among the quaternions within the state vector.

V. Star Tracker Simulation Results

The usefulness of the proposed algorithm is demonstrated by examining the attitude estimation of a spacecraft in low-Earth orbit. The spacecraft is equipped with two star trackers positioned 90° apart. The star trackers have an 8 degree field-of-view and can observe stars down to magnitude 6 with a maximum of 10 stars at any time. The +45 degree tracker observations are corrupted with zero-mean Gaussian white noise using a standard deviation of 3.0 arc-sec, while the −45 degree tracker observations have noise with a standard deviation of 30 arc-sec. A sampling interval of 1 second is assumed for the star observations and gyro measurements. Each tracker is running its own extended Kalman filter using a common gyro (see Fig. 2). Details of the Kalman filter employed can be found in Ref. [18]. The estimated quantities are the spacecraft's attitude and gyro biases, i.e. $\mathbf{x}^T = [\mathbf{q}^T \;\; \boldsymbol{\beta}^T]$.

It was found that the local error MRP approach yielded nearly identical results to the quaternion based algorithms presented in [3]. Errors between the quaternion and local-error MRP algorithm were observed to have an average approximately 5% of the corresponding $\pm 3\sigma$ covariance interval at each time. The attitude estimate errors and respective 3σ bounds can be seen in Fig. 3 which shows only the results of the MRP CI fusion for clarity.

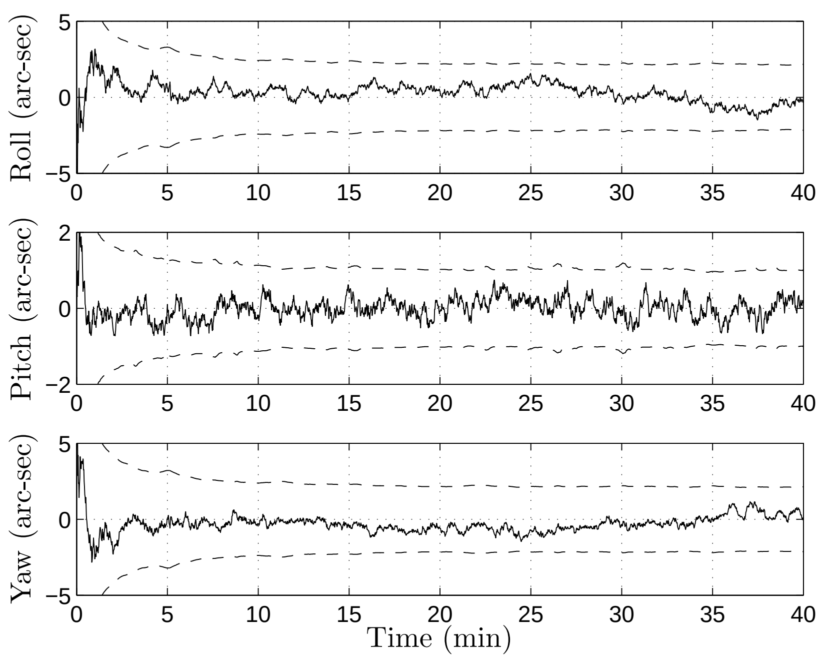


Figure 3. Attitude Estimate Errors and 3σ Bounds

In addition to circumventing the quaternion norm constraint, the proposed algorithm leads to better numeric conditioning of the CI algorithm. This is demonstrated by means of a numerically difficult case where both star trackers have are corrupted by zero-mean Gaussian white noise with a 3.0 arc-second standard deviation. For this case each of the local quaternion estimates are close to one another, leading to singularity issues with the quaternion algorithm.[3] In contrast, with the local error MRP algorithm, as all the individual estimates tend towards the same value, the corresponding local error MRPs tend towards $\mathbf{0}_{3\times 1}$. Correspondingly, the matrix $\mathcal{B}_i$ then tends towards the identity matrix leading to $\mathcal{R} = \sum_{i=1}^{n} \gamma_i P_i^{-1} = P_{\text{CI}}^{-1}$ which we have implicitly assumed to exist.

Figure 4 shows the inverse of the condition number for the required matrix inversion within the quaternion solution,[3] that for $\mathcal{R}$ from Eq. (16) for the local error MRP solution and the inverse condition number for a square root algorithm developed specifically for better handling of numerically difficult situations within the quaternion solution.[3] As seen, with an inverse condition number closer to unity, the local error MRP approach is better numerically conditioned than either of the quaternion algorithms. This means that the local error MRP algorithm will be more robust against round-off errors.

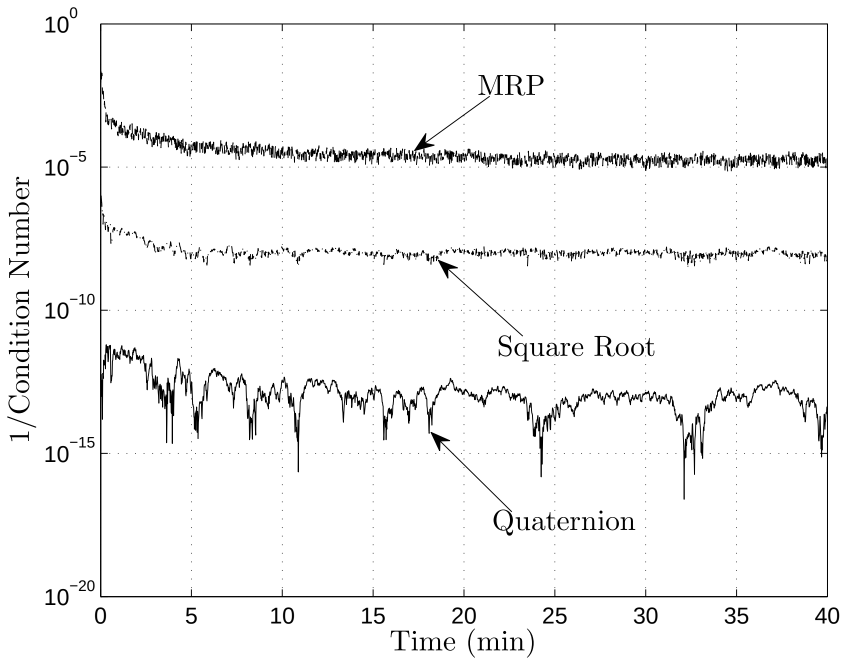


Figure 4. Inverse Condition Number for the Quaternion and Local Error MRP Algorithms

VI. Conclusions

Efficiency and robustness are two important factors in determining the applicability of onboard estimation algorithms. The use of modified Rodrigues parameters to define a local error along with some favorable mathematics has led to the development of an efficient and robust algorithm which overcomes the difficulties reported in existing literature. In addition, the increased numerical conditioning of the local error modified Rodrigues parameter algorithm means that the results are less susceptible to roundoff errors than existing algorithms. The mathematics of the problem were shown to be such that a minimal number of computationally expensive matrix inverses are required meaning that little processing power is required for implementation of the algorithm. Both of the aforementioned observations offer distinct advantages for onboard, real-time implementation given current spacecraft technologies.

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