FILTERING SOLUTION TO RELATIVE ATTITUDE DETERMINATION PROBLEM USING MULTIPLE CONSTRAINTS

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In this paper a filtering solution for the relative attitude and relative position of a formation of two spacecraft with multiple constraints is shown. The solution for the relative attitude and position between the two spacecraft is obtained only using line-of-sight measurements between them and a common (unknown) object observed by both spacecraft. The constraint used in the solution is a triangle constraint on the vector observations. This approach is extended to multiple objects by apply this constraint for each common object. Simulation runs to study the performance of the approach are shown.

INTRODUCTION

The attitude is determined as the angular departure from some reference. Attitude sensors provide either arc lengths or dihedral angle observations that are known in a reference coordinate system. The angle measurements can be combined to determine entire directions. Oftentimes these directions are LOS observations to an observed object such as a star, the Sun, the Earth's magnetic field vector or landmarks. Since the attitude of an object is described by a 3×3 orthogonal rotation matrix with determinant $+1$, it has three independent parameters; two of which describe an axis and the third the rotation about this axis. Therefore at least two unit vector measurements are needed to determine the attitude. But since each unit vector contains two independent pieces of information, the attitude is over-determined in this case. Therefore it is convenient to divide attitude determination algorithms into two classes: 1) deterministic solutions where the minimal scalar measurements are used, and 2) over-deterministic where more than the minimal scalar measurement set is used to determine the attitude.

Most inertial navigation systems used for vehicles incorporate the Global Positioning System (GPS) along with inertial measurement units providing both inertial position and attitude. If relative information is required then these measurements must be converted to relative coordinates. Although GPS can be used to provide relative information using pseudolites, GPS and GPS-like signals are susceptible to interference and jamming, among other issues. Therefore developing GPS-less navigation systems is currently an active area of research. Recent research concerning vision-based navigation for uninhabited air vehicles indicates that relative navigation can be achieved using camera-based images. Line-of-sight (LOS) vectors between vehicles in formation can be used for relative navigation and in particular relative attitude determination.

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In this paper a constrained relative attitude and position filtering solution of a formation of two vehicles is considered where gyro measurements are used to propagate the attitude and relative motion equations are used to propagate the position. The constraint used in the solution is a triangle constraint on the vector observations. The point-by-point solution (no filtering) for the relative attitude between the two vehicles can be obtained only using line-of-sight measurements between them and a common (unknown) object observed by both vehicles, but its accuracy depends on the measurement geometry. This accuracy issue is overcome in this paper by considering a filtering solution.

The organization of this paper is as follows. First the problem statement is shown followed by a summary of the relative position equations of motion. The point-by-point solution for the case with one common object is then shown, followed by the point-by-point solution for the multiple common object case along with the error covariance expression. Next a Kalman filter for estimating the relative attitude and position is shown using multiple constraints. Finally, simulation results are shown for both the point-by-point solution and the filtering cases.

PROBLEM

Each vehicle measures a LOS from itself to the other vehicle in the formation and as well as a common object and or landmark. In the case of a formation spacecraft the distance between the reference object and the vehicle may be comparable to the distance between the two frames. Therefore the parallax issue needs to be resolved by an origin transformation. This typically requires associated range information which introduces more error into the algorithm.[1] Here it is assumed that parallel beams between the vehicles exists, so that range information is not required. For example, for a laser communication system a feedback device can be employed to ensure parallel beams are given in real-time. The LOS vectors between vehicles are denoted as $\mathbf{v}_1$ and $\mathbf{w}_1$. The measurements can be related to each other through the attitude matrix mapping given by, $\mathbf{w}_1 = A_{\mathcal{B}_1}^{\mathcal{B}_2}\mathbf{v}_1$.

It is well known that using a single pair of LOS vectors between the two vehicles does not provide enough information for a complete three-axis relative attitude solution. In particular to determine the full attitude we must also determine the rotation angle about the LOS direction. We then consider the following property of the attitude matrix: $(A_I^{\mathcal{B}_2\,T}\mathbf{w}_2)^T A_I^{\mathcal{B}_1\,T}\mathbf{v}_2 = \mathbf{w}_2^T A_I^{\mathcal{B}_2} A_I^{\mathcal{B}_1\,T}\mathbf{v}_2 = \mathbf{w}_2^T A_{\mathcal{B}_1}^{\mathcal{B}_2}\mathbf{v}_2$, which means that the attitude matrix preserves the angle between vectors. This allows us to write, $d = \mathbf{w}_2^T A_{\mathcal{B}_1}^{\mathcal{B}_2}\mathbf{v}_2$, where d is the cosine of the angle between the two LOS vectors to the common object and we denote $A_{\mathcal{B}_1}^{\mathcal{B}_2}$ as just A. In terms of the quaternion, the attitude matrix is given by

$$A(\mathbf{q}) = \Xi^T(\mathbf{q})\Psi(\mathbf{q}) \tag{1}$$

where

$$\Xi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{2a}$$

$$\Psi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times 3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{2b}$$

with

$$[\mathbf{a}\times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{3}$$

for any general 3×1 vector $\mathbf{a}$ defined such that $[\mathbf{a}\times]\mathbf{b} = \mathbf{a} \times \mathbf{b}$.[2] In Ref. [1] this angle is determined from two LOS vectors measured to and from a third vehicle in a three-vehicle formation. This requires an extra LOS vector between the two vehicles and the third vehicle, which is not the case here however, as will be seen.

Since for the measurement geometry considered here all measurement vectors lie on a common plane, then a plane constraint can also be applied to solve for this rotation angle. First the LOS vector between the two frames can be aligned through an initial rotation, then a rotation angle about this direction can be found such that when this rotation is applied the angle between the measurements add up to π or the vectors lie on a common plane. The third reference object in the formation doesn't need to communicate its LOS observations to the other two vehicles for the solution of their relative attitude. Therefore a very powerful conclusion can be made from this observation: choice of the common third object in the formation is arbitrary and can be any common reference point, with unknown position, when the geometrical condition is applied.

Now a condition is applied that is present in the observations by means of a constraint, i.e. we know the form of the geometry considered: the observation vectors constitute the legs of a triangle or they lie on a common plane. The planar constraint can be simply written as, $0 = \mathbf{w}_2^T[\mathbf{w}_1\times]A\mathbf{v}_2$. This equation is basically a condition that all the observation are perpendicular to a vector which is perpendicular to any two observations. To determine the full attitude between the $\mathcal{B}_1$ and $\mathcal{B}_2$ frames the attitude matrix must satisfy the following equations:

$$\mathbf{w}_1 = A(\mathbf{q})\mathbf{v}_1 \tag{4a}$$
$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times]A(\mathbf{q})\mathbf{v}_2 \tag{4b}$$

An equivalent approach is to define two vectors, $\mathbf{w}^*$ and $\mathbf{v}^*$, that are both perpendicular to the plane. These two vector can be written as

$$\mathbf{w}^* = \frac{[\mathbf{w}_2\times]\mathbf{w}_1}{\|\mathbf{w}_2 \times \mathbf{w}_1\|} \tag{5a}$$
$$\mathbf{v}^* = \frac{[\mathbf{v}_2\times]\mathbf{v}_1}{\|\mathbf{v}_2 \times \mathbf{v}_1\|} \tag{5b}$$

Then by aligning this vectors one can solve for the attitude matrix. The equation relating $\mathbf{w}^*$ and $\mathbf{v}^*$ is given by

$$\mathbf{w}_1 = A(\mathbf{q})\mathbf{v}_1 \tag{6a}$$
$$\mathbf{w}^* = A(\mathbf{q})\mathbf{v}^* \tag{6b}$$

Solving for the attitude in Eq. (6) is equivalent to solving Eq. (4), but by writing the constraint in the latter form, this solution can now be extended to multiple common objects. The general problem when there exists more the one common object is given by

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{7a}$$
$$\mathbf{w}_1^* = A(\mathbf{q})\mathbf{v}_1^* \tag{7b}$$
$$\vdots \tag{7c}$$
$$\mathbf{w}_n^* = A(\mathbf{q})\mathbf{v}_n^* \tag{7d}$$

Equation (7) can be used to determine the attitude but when gyro measurements are made available the dynamics can be used to filter the body-frame measurements. The rotational dynamics are given by a first-order differential equation:

$$\dot{\mathbf{q}} = \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega}^{d_2}_{d_1/d_2} \tag{8a}$$

The nonlinear truth model is assumed to be driven by a white Gaussian noise process. The truth model is given by

$$\mathbf{x}_{k+1} = \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k, t\right) + G_k \mathbf{w}_k \tag{9a}$$

$$\tilde{\mathbf{y}}_k = \mathbf{h}\left(\mathbf{x}_k\right) + \boldsymbol{\upsilon}_k \tag{9b}$$

A linear approximation is made to the true system so that the extended Kalman filter (EKF) can be applied. To accomplish this, the nonlinear functions in Eq. (9) are approximated by first order Taylor series. The derivative operators are calculated locally and used to define a linear system that is locally tangent to the true nonlinear system:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k, t\right) \approx \mathbf{f}\left(\bar{\mathbf{x}}_k, \mathbf{u}_k, t\right) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}_k} \left[\mathbf{x}_k - \bar{\mathbf{x}}_k\right] \tag{10a}$$

$$\mathbf{h}\left(\mathbf{x}_k\right) \approx \mathbf{h}\left(\bar{\mathbf{x}}_k\right) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}_k} \left[\mathbf{x}_k - \bar{\mathbf{x}}_k\right] \tag{10b}$$

Linearizing the system about its current state, from Eq. (10), the following derivative operators are defined at the current state estimate:

$$\Phi_k\left(\hat{\mathbf{x}}_k, t\right) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_k} \tag{11a}$$

$$H_k\left(\hat{\mathbf{x}}_k, t\right) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_k} \tag{11b}$$

The system states are chosen to be the relative quaternions mapping $\mathcal{D}_1$ to $\mathcal{D}_2$ and each vehicle gyro bias vector, $\boldsymbol{\beta}_{d_1}$ and $\boldsymbol{\beta}_{d_2}$, and the relative position vector:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q}^T_{d_1/d_2} & \boldsymbol{\beta}^T_{d_1} & \boldsymbol{\beta}^T_{d_2} & \mathbf{X}^T \end{bmatrix}^T \tag{12}$$

Just as the derivative of position is exactly velocity, being that kinematics contain no errors, the quaternion kinematics do not contain process noise. The gyro bias drift, $\boldsymbol{\eta}_u$, as well as the error in the gyro measurements, $\boldsymbol{\eta}_v$, are assumed to be random processes. These noise processes are assumed to have the known properties. Such knowledge is available for a given set of gyros from testing and calibration:

$$\dot{\mathbf{q}}_{d_1/d_2} = \frac{1}{2}\Xi\left(\mathbf{q}_{d_1/d_2}\right)\boldsymbol{\omega}_{d_1/d_2} \tag{13a}$$

$$\boldsymbol{\omega}_{d_2} = \tilde{\mathbf{g}}_{d_2} - \boldsymbol{\beta}_{d_2} - \boldsymbol{\eta}_{d_2 v} \tag{13b}$$

$$\boldsymbol{\omega}_{d_1} = \tilde{\mathbf{g}}_{d_1} - \boldsymbol{\beta}_{d_1} - \boldsymbol{\eta}_{d_1 v} \tag{13c}$$

$$\dot{\boldsymbol{\beta}}_{d_2} = \boldsymbol{\eta}_{d_2 u} \tag{13d}$$

$$\dot{\boldsymbol{\beta}}_{d_1} = \boldsymbol{\eta}_{d_1 u} \tag{13e}$$

$$\boldsymbol{\omega}_{d_1/d_2} = \boldsymbol{\omega}_{d_1} + A(\mathbf{q}_{d_1/d_2})\boldsymbol{\omega}_{d_2} \tag{14}$$

where the properties of $\boldsymbol{\eta}_v$ and $\boldsymbol{\eta}_u$ are given by

$$\boldsymbol{\eta}_v \sim \mathcal{N}\left(\mathbf{0}, \sigma_v^2 \cdot I\right) \tag{15a}$$

$$\boldsymbol{\eta}_u \sim \mathcal{N}\left(\mathbf{0}, \sigma_u^2 \cdot I\right) \tag{15b}$$

The estimate model is given by

$$\dot{\hat{\mathbf{q}}}_{d_i/d_2} = \frac{1}{2}\Xi\left(\hat{\mathbf{q}}_{d_i/d_2}\right)\hat{\boldsymbol{\omega}}_{d_i/d_2} \tag{16a}$$

$$\hat{\boldsymbol{\omega}}_{d_2} = \tilde{\mathbf{g}}_{d_2} - \hat{\boldsymbol{\beta}}_{d_2} \tag{16b}$$

$$\hat{\boldsymbol{\omega}}_{d_i} = \tilde{\mathbf{g}}_{d_i} - \hat{\boldsymbol{\beta}}_{d_i} \tag{16c}$$

$$\dot{\hat{\boldsymbol{\beta}}}_{d_2} = \mathbf{0} \tag{16d}$$

$$\dot{\hat{\boldsymbol{\beta}}}_{d_i} = \mathbf{0} \tag{16e}$$

These equations employ assumed dynamics that dictate a state orbit that the estimate is thought to follow. The state dynamic model is used in the absence of measurements to propagate current state knowledge through time.

With the truth and estimate models fully described, the required parameters for the EKF can be determined. The error-state vector is $\boldsymbol{\Delta}\mathbf{x} = \left[\boldsymbol{\delta\alpha}_{d_2}^{d_1\,T} \quad \boldsymbol{\Delta\beta}_{d_1}^T \quad \boldsymbol{\Delta\beta}_{d_2}^T \quad \boldsymbol{\Delta}\mathbf{X}\right]^T$. Taylor series approximations to first order will be used to locally linearize the system dynamics. This work derives an EKF based on using LOS measurements and the planar constraint to observe the attitude and gyros to filter the estimates. An EKF that estimates the relative attitude and gyro biases for the three formation-flying spacecraft is desired in practice because more accurate estimates can be obtained as compared with those determined from deterministic results.

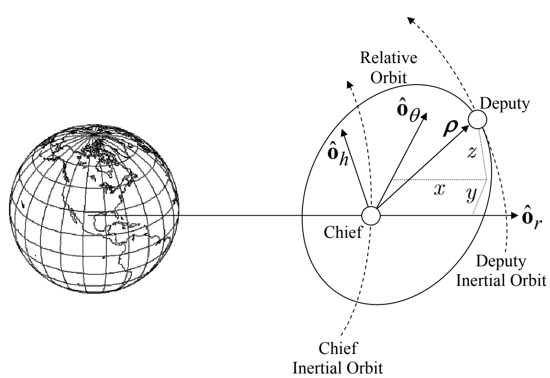


Figure 1. General Type of Spacecraft Formation with Relative Motion

RELATIVE ORBITAL MOTION EQUATIONS

The spacecraft about which all other spacecraft are orbiting is referred to as the chief. The remaining spacecraft are referred to as the deputies. The relative orbit position vector, l, is expressed in components by $\boldsymbol{\rho} = [x\ y\ z]^T$, shown in Figure 1. The vector triad $\{\hat{\mathbf{o}}_r,\ \hat{\mathbf{o}}_\theta,\ \hat{\mathbf{o}}_h\}$ is known as the Hill coordinate frame, where $\hat{\mathbf{o}}_r$ is in the orbit radius direction, $\hat{\mathbf{o}}_h$ is parallel with the orbit momentum vector and $\hat{\mathbf{o}}_\theta$ completes the triad. A complete derivation of the relative equations of motion for eccentric orbits can be found in Ref. [3]. If the relative orbit coordinates are small compared to the chief orbit radius, then the equations of motion are given by[4]

$$\ddot{x} - x\dot{\theta}^2\left(1 + 2\frac{r_c}{p}\right) - 2\dot{\theta}\left(\dot{y} - y\frac{\dot{r}_c}{r_c}\right) = w_x \tag{17a}$$

$$\ddot{y} + 2\dot{\theta}^2\left(\dot{x} + 2x\frac{r_c}{p}\right) - y\dot{\theta}\left(1 - \frac{r_c}{p}\right) = w_x \tag{17b}$$

$$\ddot{z} + x\dot{\theta}^2\frac{r_c}{p} = w_x \tag{17c}$$

where p is semilatus rectum of the chief, r_c is the chief orbit radius and $\dot{\theta}$ is true anomaly rate of the chief. Also, w_x, w_y and w_z are acceleration disturbances which are modeled as zero-mean Gaussian white-noise processes, with variances given by σ_x^2, σ_y^2 and σ_z^2 , respectively. The true anomaly acceleration and chief orbit-radius acceleration are given by

$$\ddot{\theta} = -2\frac{\dot{r}_c}{r_c}\dot{\theta} \tag{18a}$$

$$\ddot{r}_c = r_c\dot{\theta}^2\left(1 - \frac{r_c}{p}\right) \tag{18b}$$

If the chief satellite orbit is assumed to be circular so that $\dot{r}_c = 0$ and $p = r_c$, then the relative equations of motion reduce to the simple form known as the CW equations (with disturbances added here):

$$\ddot{x} - 3n^2x - 2n\dot{y} = w_x \tag{19a}$$

$$\ddot{y} + 2n\dot{x} = w_x \tag{19b}$$

$$\ddot{z} + n^2\dot{z} = w_x \tag{19c}$$

where $n = \dot{\theta}$ is the mean motion.

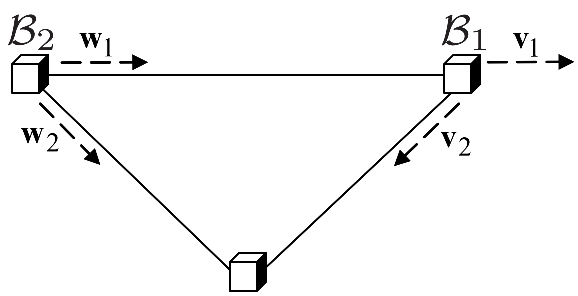


Figure 2. Observation Geometry

DETERMINISTIC CASE: ONE CONSTRAINT

This section summarizes the constrained attitude solution. More details can be found in Ref. [5]. Considering the measurements shown in Figure 2, to determine the full attitude between the $\mathcal{B}_2$ and $\mathcal{B}_1$ frames the attitude matrix must satisfy the following measurement equations:

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{20a}$$

$$d = \mathbf{w}_2^T A\mathbf{v}_2 \tag{20b}$$

It is assumed that $|d| \leq 1$; otherwise a solution will not exist. Here it is assumed that the LOS vectors $\mathbf{v}_1$ and $\mathbf{w}_1$ are parallel. Also note that from Figure 2 no observation information is required from the third object to either $\mathcal{B}_1$ or $\mathcal{B}_2$. Hence, no information such as position is required for this object to determine the relative attitude. A solution for the attitude satisfying Eq. (20) is discussed in Ref. [6] and will be utilized to form a solution for the constrained problem discussed here. The solution for the rotation matrix that satisfies Eq. (20) can be found by first finding a rotation matrix that satisfies that first equation and then finding the angle that one must rotate about the reference direction to align the two remaining vectors such that their dot product is equivalent to that measured in the remaining frame in the formation. The first rotation can be found by rotating about any direction by any angle, where $B = R(\mathbf{n}_1, \theta)$ is a general rotation about some axis rotation that satisfies Eq. (20a). The choice of the initial rotation axis is arbitrary, here the vector between the two reference direction vectors is used and the rotation is as follows:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T \mathbf{w}_1)} - I_{3\times 3} \tag{21}$$

where $\mathbf{n}_1 = (\mathbf{w}_1 + \mathbf{v}_1)$ and $\theta = \pi$. This rotation matrix will align the LOS vectors between frames, but the frames could still have some rotation about this vector, so therefore the angle about this axis must be determined to solve the second equation. To do so the vector $\mathbf{w}^*$ is first defined, which is the vector produced after applying the rotation B on the vector $\mathbf{v}_2$. This will allow us to determine the second rotation needed to map $\mathbf{v}_2$ properly to the $\mathcal{B}_2$ frame with $\mathbf{w}^* = B\,\mathbf{v}_2$. Since the rotation axis is the $\mathbf{w}_1$ vector, this vector will be invariant under this transformation and the solution to the full attitude can be written as $A = R(\mathbf{n}_2, \theta)\, B$.

Consider solving for the rotation angle using the planar constraint, the constraint can be written as the following:

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times] R(\mathbf{n}_2, \theta)\, \mathbf{w}^* \tag{22}$$

Substituting the second rotation matrix into Eq. (22), and with $\mathbf{n}_2 = \mathbf{w}_1$, leads to

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times][\mathbf{w}_1 \mathbf{w}_1^T - \cos(\theta)[\mathbf{w}_1 \times]^2 \mathbf{w}^* - \sin(\theta)[\mathbf{w}_1 \times]\mathbf{w}^*] \tag{23}$$

Expanding out this expression gives

$$\left(\mathbf{w}_2^T [\mathbf{w}_1 \times]\mathbf{w}^*\right) \cos(\theta) = \left(\mathbf{w}_2^T [\mathbf{w}_1 \times]^2 \mathbf{w}^*\right) \sin(\theta) \tag{24}$$

Notice that if Eq. (24) is divided by -1 then the resulting equation would be unchanged but the solution for the angle θ would differ by π. Therefore, using the planar constraint the solution for the angle θ can be written as $\theta = \beta + \phi$, where

$$\beta = \text{atan2}(\mathbf{w}_2^T [\mathbf{w}_1 \times]\mathbf{w}^*, \mathbf{w}_2^T [\mathbf{w}_1 \times]^2 \mathbf{w}^*) \tag{25}$$

and $\phi = 0$ or π. An ambiguity exists when using this approach but it is important to note that one of the possible solutions for this approach is equivalent to the triangle constraint case.

Finally the solution for the attitude is given by $A = R(\mathbf{w}_1, \theta)\, B$. The solution is now summarized:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T \mathbf{w}_1)} - I_{3\times 3} \tag{26a}$$

$$R(\mathbf{w}_1, \theta) = I_{3\times 3}\cos(\theta) + (1 - \cos(\theta))\mathbf{w}_1\mathbf{w}_1^T - \sin(\theta)[\mathbf{w}_1 \times] \tag{26b}$$

$$\theta = \text{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \pi \tag{26c}$$

$$A = R(\mathbf{w}_1, \theta)\, B \tag{26d}$$

This result shows that for any formation of two vehicles a deterministic solution will exist using one direction and one angle. Due to the fact that this case is truly deterministic there is no need to minimize a cost function and the solution will always be the maximum likelihood one. It is very important to note that without the resolution of the attitude ambiguity any covariance development might not have any meaning since although the covariance might take a small value if the wrong possible attitude is used then the error might be fairly large and not bounded by the attitude covariance.

The solution in Eq. (26) can be rewritten without the use of any transcendental functions. The following relationships can be derived:

$$\cos(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{27a}$$

$$\sin(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{27b}$$

This leads to $\cos(\theta) = -b/c$ and $\sin(\theta) = -a/c$ with

$$a = \mathbf{w}_2^T[\mathbf{w}_1\times]\left([\mathbf{w}_1\times] + [\mathbf{v}_1\times]\right)[\mathbf{v}_1\times]\mathbf{v}_2 \tag{28a}$$

$$b = \mathbf{w}_2^T[\mathbf{w}_1\times]\left([\mathbf{w}_1\times][\mathbf{v}_1\times] - I_{3\times 3}\right)[\mathbf{v}_1\times]\mathbf{v}_2 \tag{28b}$$

$$c = (1 + \mathbf{v}_1^T\mathbf{w}_1)\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\| \tag{28c}$$

Note that $c = \sqrt{a^2 + b^2}$. Then the matrix R is given by

$$R = -\frac{b}{c}\, I_{3\times 3} + \left(1 + \frac{b}{c}\right)\mathbf{w}_1\mathbf{w}_1^T + \frac{a}{c}\,[\mathbf{w}_1\times] \tag{29}$$

Noting that $\mathbf{w}_1\mathbf{w}_1^T B = \mathbf{w}_1\mathbf{v}_1^T$ then the solution in Eq. (26d) can be rewritten as

$$\begin{aligned} A = {} & \frac{b}{c}\left(I_{3\times 3} - \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} + \mathbf{w}_1\mathbf{v}_1^T\right) \\ & + \frac{a}{c}\,[\mathbf{w}_1\times]\left(\frac{\mathbf{v}_1\mathbf{w}_1^T + \mathbf{v}_1\mathbf{v}_1^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times 3}\right) + \mathbf{w}_1\mathbf{v}_1^T \end{aligned} \tag{30}$$

Note in practice the measured quantities from the previous section are used in place of the observed quantities shown in Eq. (26), and Eqs. (28) and (30).

The covariance matrix for an attitude estimate is defined as the covariance of a small angle rotation taking the true attitude to the estimated attitude. Typically small Euler angles are used to parameterize the attitude error-matrix. Reference [5] derives the attitude error-covariance for the constrained solution by using the attitude matrix with respect to the small angle errors. The attitude error-covariance is given by

$$P = \left(\begin{bmatrix} -[A_{\text{true}}\mathbf{v}_2\times] \\ \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix} \begin{bmatrix} R_{\Delta_1} & R_{\Delta_1\Delta_2} \\ \\ R_{\Delta_1\Delta_2}^T & R_{\Delta_2} \end{bmatrix}^{-1} \begin{bmatrix} -[A_{\text{true}}\mathbf{v}_2\times] \\ \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix}^T \right)^{-1} \tag{31}$$

where

$$R_{\Delta_1} = R_{w_1} + A_{\text{true}} R_{v_1} A_{\text{true}}^T \tag{32a}$$

$$\begin{aligned} R_{\Delta_2} = {} & \mathbf{w}_2^T[A_{\text{true}}\mathbf{v}_2\times]R_{w_1}[A_{\text{true}}\mathbf{v}_2\times]\mathbf{w}_2 + (A_{\text{true}}\mathbf{v}_2)^T[\mathbf{w}_1\times]R_{w_2}[\mathbf{w}_1\times](A_{\text{true}}\mathbf{v}_2) \\ & + \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}R_{v_2}A_{\text{true}}^T[\mathbf{w}_1\times]\mathbf{w}_2 \end{aligned} \tag{32b}$$

$$R_{\Delta_1\Delta_2} = -R_{w_1}[A_{\text{true}}\mathbf{v}_2\times]\mathbf{w}_2 \tag{32c}$$

This expression is a function of the true attitude, A_{true}, but the true attitude can effectively be replaced with the estimated attitude to within first order.

OVER-DETERMINISTIC CASE: MULTIPLE CONSTRAINTS

When multiple common objects are acquired a deterministic solution is no longer achievable and therefore the solution given previously is no longer valid. The observation is written using the following notation:

$$\tilde{\mathbf{s}}_1 = \tilde{\mathbf{w}}_1, \quad \tilde{\mathbf{r}}_1 = \tilde{\mathbf{v}}_1 \tag{33a}$$

$$\tilde{\mathbf{s}}_i = \frac{[\tilde{\mathbf{w}}_i\times]\tilde{\mathbf{w}}_1}{\|\tilde{\mathbf{w}}_i \times \tilde{\mathbf{w}}_1\|}, \quad \tilde{\mathbf{r}}_i = \frac{[\tilde{\mathbf{v}}_i\times]\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_i \times \tilde{\mathbf{v}}_1\|} \tag{33b}$$

where $\tilde{\mathbf{s}}_1$ and $\tilde{\mathbf{r}}_1$ denote the vectors that are derived from the LOS observations and therefore are corrupted with measurement noise, denoted by the $\tilde{\cdot}$ notation. The vectors $\tilde{\mathbf{s}}_i$ and $\tilde{\mathbf{r}}_i$, for $i > 1$, are the vectors associated with the multiple planar constraints. Notice that all vectors in Eq. (33) use the observations $\tilde{\mathbf{w}}_1$ or $\tilde{\mathbf{v}}_1$, resulting in cross-correlation between the vectors in Eq. (33). Then the attitude estimation problem can be written as follows:

$$\tilde{\mathbf{s}}_1 = A\tilde{\mathbf{r}}_1 \tag{34a}$$

$$\tilde{\mathbf{s}}_i = A\tilde{\mathbf{r}}_i \tag{34b}$$

The measurement models for $\tilde{\mathbf{s}}_i$ and $\tilde{\mathbf{r}}_i$ are given by

$$\tilde{\mathbf{s}}_1 = \mathbf{s}_1 + \boldsymbol{\Delta}\mathbf{s}_1, \quad \tilde{\mathbf{r}}_1 = \mathbf{r}_1 + \boldsymbol{\Delta}\mathbf{r}_1 \tag{35a}$$

$$\tilde{\mathbf{s}}_i = \mathbf{s}_i + \boldsymbol{\Delta}\mathbf{s}_i, \quad \tilde{\mathbf{r}}_i = \mathbf{r}_i + \boldsymbol{\Delta}\mathbf{r}_i \tag{35b}$$

Here both the measurement LOS vectors and the reference vectors contain uncertainty and therefore two noise terms are needed, denoted by $\boldsymbol{\Delta}_{\mathbf{s}_i}$ and $\boldsymbol{\Delta}_{\mathbf{r}_i}$. Equations (34) and (33) are used to write the effective noise of the LOS equations as

$$\boldsymbol{\Delta}_1 = \boldsymbol{\Delta}\mathbf{s}_1 - A\boldsymbol{\Delta}\mathbf{r}_1 \tag{36a}$$

$$\boldsymbol{\Delta}_i = \boldsymbol{\Delta}\mathbf{s}_i - A\boldsymbol{\Delta}\mathbf{r}_i \tag{36b}$$

where $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_i$ denote the noise in the first and i^{th} LOS equation, respectively, and the noise terms are functions of the measurements $\{\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i\}$. To describe the optimal attitude estimation problem the noise statistics of $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_i$ need to be described with respect to the known noise statistics of the measurements $\{\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i\}$. For now it is assumed that $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_i$ are both zero-mean Gaussian random variables.

Under this assumption the covariance expression can be written as

$$R_{\boldsymbol{\Delta}_1} = R_{\boldsymbol{\Delta}\mathbf{s}_1} + AR_{\boldsymbol{\Delta}\mathbf{r}_1}A^T \tag{37a}$$

$$R_{\boldsymbol{\Delta}_i} = R_{\boldsymbol{\Delta}\mathbf{s}_i} + AR_{\boldsymbol{\Delta}\mathbf{r}_i}A^T \tag{37b}$$

Then the LOS equations can be cast in matrix vector form, given by

$$\mathbf{y}(\tilde{\mathbf{s}}_1, \ldots, \tilde{\mathbf{s}}_n) = \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_n) + \boldsymbol{\Delta} \tag{38}$$

where $\mathbf{h}(A, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_n) = \begin{bmatrix}(\hat{A}\tilde{\mathbf{r}}_1)^T & \ldots & (\hat{A}\tilde{\mathbf{r}}_n)^T\end{bmatrix}^T$, $\mathbf{y}(\tilde{\mathbf{s}}_1, \ldots, \tilde{\mathbf{s}}_n) = \begin{bmatrix}\tilde{\mathbf{s}}_1^T & \ldots & \tilde{\mathbf{s}}_n^T\end{bmatrix}^T$, and $\boldsymbol{\Delta} = \begin{bmatrix}\boldsymbol{\Delta}_1^T & \ldots & \boldsymbol{\Delta}_i^T\end{bmatrix}^T$. The covariance for $\boldsymbol{\Delta}$ can be written in the following form:

$$\mathcal{R} = \begin{bmatrix} R_{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_1} & \ldots & R_{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_i} & \ldots & R_{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_n} \\ \vdots & \ddots & \cdot & \cdot & \vdots \\ R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_1} & \cdot & R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_i} & \cdot & R_{\boldsymbol{\Delta}_i\boldsymbol{\Delta}_n} \\ \vdots & \cdot & \cdot & \ddots & \vdots \\ R_{\boldsymbol{\Delta}_n\boldsymbol{\Delta}_1} & \ldots & R_{\boldsymbol{\Delta}_n\boldsymbol{\Delta}_i} & \ldots & R_{\boldsymbol{\Delta}_n\boldsymbol{\Delta}_n} \end{bmatrix} \tag{39}$$

where the terms of this covariance matrix can be found by taking $E\{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_1^T\}$. This covariance matrix will be derived in the next section. Then given this representation the optimal attitude estimation problem can be stated as

$$\begin{aligned} &\min J = \Big(\mathbf{y}(\tilde{\mathbf{s}}_1, \ldots, \tilde{\mathbf{s}}_n) - \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_n)\Big)^T \mathcal{R}^{-1} \Big(\mathbf{y}(\tilde{\mathbf{s}}_1, \ldots, \tilde{\mathbf{s}}_n) - \mathbf{h}(\hat{A}, \tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_n)\Big) \\ &\quad \text{s.t.} \;\; I_{3\times 3} - \hat{A}\,\hat{A}^T = 0_{3\times 3} \end{aligned} \tag{40}$$

In the filtering case we can use Eq. 38 as our observation function but we need to derive an expression for $\mathcal{R}$. The expression for $\mathcal{R}$ is derived in the next section.

Covariance Expressions

For simplicity consider the following notation:

$$\tilde{\mathbf{s}}_1 = \boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i) \tag{41}$$

This simplifies the derivation of the covariance matrix. The terms needed for the covariance derivation are given by

$$\begin{aligned} \boldsymbol{\alpha}(\mathbf{w}_1, \mathbf{w}_i) = \|\mathbf{w}_i \times \mathbf{w}_1\|^{-1}, \quad \boldsymbol{\alpha}_{\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_i) &= \frac{1}{2}\|\mathbf{w}_i \times \mathbf{w}_1\|^{-3}([\mathbf{w}_i\times]\mathbf{w}_1)^T[\mathbf{w}_i\times], \\ \boldsymbol{\alpha}_{\mathbf{w}_i}(\mathbf{w}_1, \mathbf{w}_i) &= -\frac{1}{2}\|\mathbf{w}_i \times \mathbf{w}_1\|^{-3}([\mathbf{w}_i\times]\mathbf{w}_1)^T[\mathbf{w}_1\times] \end{aligned} \tag{42a}$$

$$\boldsymbol{\beta}(\mathbf{w}_1, \mathbf{w}_i) = [\mathbf{w}_i\times]\mathbf{w}_1, \quad \boldsymbol{\beta}_{\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_i) = [\mathbf{w}_i\times], \quad \boldsymbol{\beta}_{\mathbf{w}_i}(\mathbf{w}_1, \mathbf{w}_i) = -[\mathbf{w}_1\times] \tag{42b}$$

A first-order Taylor series expansion about the error statistics of the terms $\boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ and $\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ yields:

$$\boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i) = \boldsymbol{\alpha}(\mathbf{w}_1, \mathbf{w}_i) + \boldsymbol{\alpha}_{\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_i)\boldsymbol{\upsilon}_{w1} + \boldsymbol{\alpha}_{\mathbf{w}_i}(\mathbf{w}_1, \mathbf{w}_i)\boldsymbol{\upsilon}_{wi} \tag{43a}$$
$$\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i) = \boldsymbol{\beta}(\mathbf{w}_1, \mathbf{w}_i) + \boldsymbol{\beta}_{\mathbf{w}_1}(\mathbf{w}_1, \mathbf{w}_i)\boldsymbol{\upsilon}_{w1} + \boldsymbol{\beta}_{\mathbf{w}_i}(\mathbf{w}_1, \mathbf{w}_i)\boldsymbol{\upsilon}_{wi} \tag{43b}$$

Note that the expressions for $\boldsymbol{\alpha}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ and $\boldsymbol{\beta}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_i)$ can b written in terms of the true quantities $\mathbf{w}_1$, $\mathbf{w}_i$, and error terms $\boldsymbol{\upsilon}_{wi}$ and $\boldsymbol{\upsilon}_{w1}$. The notation is simplified by removing the first function dependence in $\mathbf{w}_1$ since all terms share this and display the functional dependence of the second term as a superscript, i.e. $\boldsymbol{\beta}^i = \boldsymbol{\beta}(\mathbf{w}_1, \mathbf{w}_i)$.

To determine the error term in $\tilde{\mathbf{s}}_1 = \mathbf{s}_1 + \boldsymbol{\Delta}\mathbf{s}_i$ a Taylor series expansion in Eq. (43) is used and the result is substituted into Eq. (41), giving

$$\mathbf{s}_i + \boldsymbol{\Delta}\mathbf{s}_i = \left(\boldsymbol{\alpha}^i + \boldsymbol{\alpha}^i_{\mathbf{w}_1}\boldsymbol{\upsilon}_{w1} + \boldsymbol{\alpha}^i_{\mathbf{w}_i}\boldsymbol{\upsilon}_{wi}\right) \cdot \left(\boldsymbol{\beta}^i + \boldsymbol{\beta}^i_{\mathbf{w}_1}\boldsymbol{\upsilon}_{w1} + \boldsymbol{\beta}^i_{\mathbf{w}_i}\boldsymbol{\upsilon}_{wi}\right) \tag{44}$$

Rearranging terms and using the fact that $\mathbf{s}_i = \boldsymbol{\alpha}^i\boldsymbol{\beta}^i$, and neglecting higher order terms $\boldsymbol{\upsilon}_{w1}\boldsymbol{\upsilon}_{wi}$, $\boldsymbol{\upsilon}_{w1}\boldsymbol{\upsilon}_{w1}$, and $\boldsymbol{\upsilon}_{wi}\boldsymbol{\upsilon}_{wi}$, then the following expression is given

$$\boldsymbol{\Delta}\mathbf{s}_i = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_1})\boldsymbol{\upsilon}_{w1} + (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})\boldsymbol{\upsilon}_{wi} \tag{45}$$

Then the covariance for $\boldsymbol{\Delta}\mathbf{s}_i$ can be calculated as

$$\begin{aligned} R_{\boldsymbol{\Delta}\mathbf{s}_i\boldsymbol{\Delta}\mathbf{s}_i} = E\{\boldsymbol{\Delta}\mathbf{s}_i\boldsymbol{\Delta}\mathbf{s}_i^T\} &= (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})R_{\mathbf{w}_1}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})^T \\ &+ (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})R_{\mathbf{w}_i}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_i})^T \end{aligned} \tag{46}$$

The cross-correlation between the $\boldsymbol{\Delta}\mathbf{s}_i$ and $\boldsymbol{\Delta}\mathbf{s}_j$ terms can also be written as

$$R_{\boldsymbol{\Delta}\mathbf{s}_i\boldsymbol{\Delta}\mathbf{s}_j} = E\{\boldsymbol{\Delta}\mathbf{s}_i\boldsymbol{\Delta}\mathbf{s}_j^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_1})R_{\mathbf{w}_1}(\boldsymbol{\alpha}^j\boldsymbol{\beta}^j_{\mathbf{w}_1} + \boldsymbol{\beta}^j\boldsymbol{\alpha}^j_{\mathbf{w}_1})^T \tag{47}$$

Finally the cross-correlation between the $\boldsymbol{\Delta}\mathbf{s}_1$ and $\boldsymbol{\Delta}\mathbf{s}_i$ terms can also be written as

$$R_{\boldsymbol{\Delta}\mathbf{s}_i\boldsymbol{\Delta}\mathbf{s}_1} = E\{\boldsymbol{\Delta}\mathbf{s}_i\boldsymbol{\Delta}\mathbf{s}_1^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{w}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{w}_1})R_{\mathbf{w}_1} \tag{48}$$

A similar procedure can be used to compute $\boldsymbol{\Delta}\mathbf{r}_i$ terms, which is summarized here. Similarly, for simplicity the following notation is considered:

$$\tilde{\mathbf{r}}_1 = \alpha(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i)\boldsymbol{\alpha}(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_i) \tag{49}$$

The terms need for the covariance derivation for the terms $\boldsymbol{\Delta}\mathbf{r}_i$ are given by

$$\begin{aligned} \boldsymbol{\alpha}(\mathbf{v}_1, \mathbf{v}_i) = \|\mathbf{v}_i \times \mathbf{v}_1\|^{-1}, \quad \boldsymbol{\alpha}_{\mathbf{v}_1}(\mathbf{v}_1, \mathbf{v}_i) &= \frac{1}{2}\|\tilde{\mathbf{v}}_i \times \mathbf{v}_1\|^{-3}([\mathbf{v}_i\times]\mathbf{v}_1)^T[\mathbf{v}_i\times], \\ \boldsymbol{\alpha}_{\mathbf{v}_i}(\mathbf{v}_1, \mathbf{v}_i) &= -\frac{1}{2}\|\mathbf{v}_i \times \mathbf{v}_1\|^{-3}([\mathbf{v}_i\times]\mathbf{v}_1)^T[\mathbf{v}_1\times] \end{aligned} \tag{50a}$$
$$\boldsymbol{\beta}(\mathbf{v}_1, \mathbf{v}_i) = [\mathbf{v}_i\times]\mathbf{v}_1, \quad \boldsymbol{\beta}_{\mathbf{v}_1}(\mathbf{v}_1, \mathbf{v}_i) = [\mathbf{v}_i\times], \quad \boldsymbol{\beta}_{\mathbf{v}_i}(\mathbf{v}_1, \mathbf{v}_i) = -[\mathbf{v}_1\times] \tag{50b}$$

Then the error term $\boldsymbol{\Delta}\mathbf{r}_i$ can be written in terms of the noise terms $\boldsymbol{\upsilon}_{v1}$ and $\boldsymbol{\upsilon}_{vi}$, resulting in the following expression:

$$\boldsymbol{\Delta}\mathbf{r}_i = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})\boldsymbol{\upsilon}_{w1} + (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_i})\boldsymbol{\upsilon}_{vi} \tag{51}$$

Then the covariance term for $\boldsymbol{\Delta}\mathbf{r}_i$: $R_{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_i}$, $R_{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_j}$, and $R_{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_1}$ are given as follows

$$\begin{aligned} R_{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_i} &= E\{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_i^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})R_{\mathbf{v}_1}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})^T \\ &\quad + (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_i})R_{\mathbf{v}_i}(\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_i} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_i})^T \end{aligned} \tag{52a}$$

$$R_{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_j} = E\{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_j^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})R_{\mathbf{v}_1}(\boldsymbol{\alpha}^j\boldsymbol{\beta}^j_{\mathbf{v}_1} + \boldsymbol{\beta}^j\boldsymbol{\alpha}^j_{\mathbf{v}_1})^T \tag{52b}$$

$$R_{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_1} = E\{\boldsymbol{\Delta}\mathbf{r}_i\boldsymbol{\Delta}\mathbf{r}_1^T\} = (\boldsymbol{\alpha}^i\boldsymbol{\beta}^i_{\mathbf{v}_1} + \boldsymbol{\beta}^i\boldsymbol{\alpha}^i_{\mathbf{v}_1})R_{\mathbf{v}_1} \tag{52c}$$

Then the covariance in Eq. (39) can be formed by using Eq. (42) to form the terms in Eqs. (46), (47) and (48), and using Eq. (49) to form the terms in Eqs. (52a), (52b) and (52c), and combining it all together to solve for the terms in Eq. (37). These terms can be collected into one matrix given by Eq. (39). The covariance for the approximate solution can be formed using the weight to quantify the error in the solution.

RELATIVE ATTITUDE AND POSITION KALMAN FILTER

In this section the necessary equations for both relative attitude and position estimation between two spacecraft are derived. The state vector in the attitude-only estimation formulations shown in the previous section is now appended to include relative position and velocity of the deputy, radius and radial rate of the chief, and the true anomaly and its rate. This appended vector is given by

$$\mathbf{X} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & r_c & \dot{r}_c & \theta & \dot{\theta} \end{bmatrix}^T \tag{53a}$$

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{bmatrix}^T \tag{53b}$$

The nonlinear state-space model follows from Eqs. (17) and (18) as

$$\dot{\mathbf{X}} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 x_{10}^2(1 + 2x_7/p) + 2x_{10}(x_5 - x_2 x_8/x_7) \\ -2x_{10}(x_4 - x_1 x_8/x_7) + x_2 x_{10}^2(1 - x_7/p) \\ -x_7 x_{10}^2 x_3/p \\ x_8 \\ x_7 x_{10}^2(1 - x_7/p) \\ x_{10} \\ -2x_8 x_{10}/x_7 \end{bmatrix} \tag{54}$$

Here it is assumed that p is known perfectly. Any error in p can be incorporated into the process noise vector if needed. The error-state vector for the chief and deputy gyro bias case is now given by

$$\boldsymbol{\Delta}\mathbf{x} = \begin{bmatrix} \boldsymbol{\delta\alpha}^T_{d_2/d_1} & \boldsymbol{\Delta\beta}^T_{d_1} & \boldsymbol{\Delta\beta}^T_{d_2} & \boldsymbol{\Delta}\mathbf{X}^T \end{bmatrix}^T$$

The matrices F and G that are used in the EKF covariance propagation are given by[7]

$$F = \begin{bmatrix} -\left[\hat{\boldsymbol{\omega}}_{d_i/d_2}\times\right] & A(\hat{\mathbf{q}}_{d_i/d_2}) & -I_{3\times 3} & 0_{3\times 10} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 10} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 10} \\ 0_{10\times 3} & 0_{10\times 3} & 0_{10\times 3} & \frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}} \end{bmatrix} \tag{55}$$

$$G = \begin{bmatrix} A(\hat{\mathbf{q}}_{d_i/d_2}) & -I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} \\ 0_{10\times 3} & 0_{10\times 3} & 0_{10\times 3} & \frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}} \end{bmatrix} \tag{56}$$

where $\hat{\mathbf{X}}$ denotes the estimate of $\mathbf{X}$. The partial matrix $\frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}}$ is straightforward to derive and is not shown here for brevity. Defining the new process noise vector as $w \equiv [\boldsymbol{\eta}_{d_2 v} \;\; \boldsymbol{\eta}_{d_1 v} \;\; \boldsymbol{\eta}_{d_2 u} \;\; \boldsymbol{\eta}_{d_1 u}]^T$ where no process noise is add to the relative motion equations, then the new matrix Q is given by

$$Q = \begin{bmatrix} \sigma_{d_1 v}^2 I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \sigma_{d_2 v}^2 I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & \sigma_{d_1 u}^2 I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & \sigma_{d_2 u}^2 I_{3\times 3} \end{bmatrix} \tag{57}$$

The sensitivity matrix is modified to be

$$H_k(\mathbf{q}_k^-, \hat{\boldsymbol{\rho}}^-) = \begin{bmatrix} \left[A(\mathbf{q}_k^-)\hat{\mathbf{r}}_1^- \times\right] & 0_{3\times 3} & 0_{3\times 3} & \frac{\partial \hat{\mathbf{r}}_1^-}{\partial \hat{\boldsymbol{\rho}}^-} & 0_{3\times 7} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left[A(\mathbf{q}_k^-)\hat{\mathbf{r}}_N^- \times\right] & 0_{3\times 3} & 0_{3\times 3} & \frac{\partial \hat{\mathbf{r}}_N^-}{\partial \hat{\boldsymbol{\rho}}^-} & 0_{3\times 7} \end{bmatrix} \tag{58}$$

where $\hat{\mathbf{r}}_i^-$ is given by Eq. 33 evaluated at $\hat{\boldsymbol{\rho}}^- = [\hat{x}^- \;\; \hat{y}^- \;\; \hat{z}^-]$ and the partial matrix $\frac{\partial \hat{\mathbf{r}}_i^-}{\partial \hat{\boldsymbol{\rho}}^-}$. is given by

$$\frac{\partial \hat{\mathbf{r}}_i^-}{\partial \hat{\boldsymbol{\rho}}^-} = \frac{1}{\|\hat{\boldsymbol{\rho}}^-\|} \begin{bmatrix} -\left(y^2+z^2\right) & xy & xz \\ yx & -\left(x^2+z^2\right) & yz \\ zx & yz & -\left(y^2+x^2\right) \end{bmatrix} \tag{59}$$

The EKF can now be executed with these new quantities to estimate both relative attitude and position. In the formulation of this section the chief radius and true anomaly, as well as their respective derivatives, are estimated. If this information is assumed known a priori, then these states can removed and their respective measured values can be added as process noise in the state model.

SIMULATIONS

Three simulation scenarios are studied: a static formation configuration, a dynamic formation configuration, and dynamic formation filtering solution of two vehicles and multiple common targets are considered. Each vehicle has light source devices and focal plane detectors (FPD)s, which produce a set of parallel LOS measurements. In the static formation each vehicle is observing two common objects. As mentioned previously the location of these objects is not required for the attitude solution, only the LOS vectors from each vehicle to the objects are needed. In the dynamic configuration it is assumed that the two vehicles are formation flying spacecraft in a close relative orbit. The two common targets in the dynamic case are two other spacecraft that are also in the formation. They do not measure or communicate LOS observations to the two vehicles for which the relative attitudes are determined.

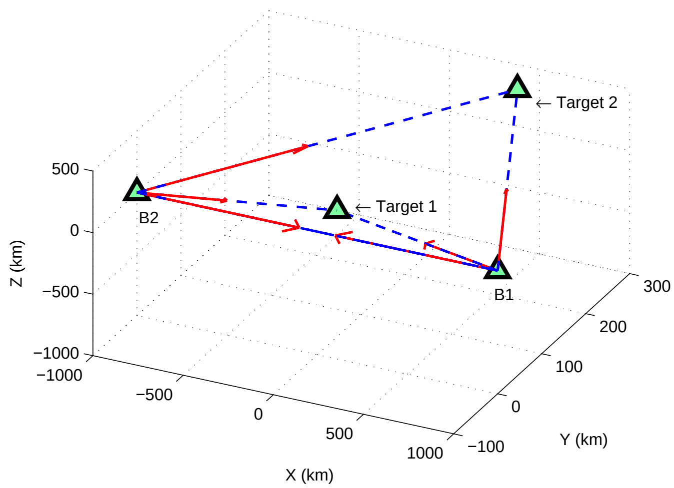


Figure 3. Dynamic Trajectories

Static Formation Simulation

The formation configuration uses the following true location of vehicle and targets:

$$\mathbf{x}_1 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \text{m}, \quad \mathbf{x}_2 = \begin{bmatrix} -1000 \\ 0 \\ 0 \end{bmatrix} \text{m}, \quad \mathbf{x}_3 = \begin{bmatrix} 500 \\ 250 \\ 500 \end{bmatrix} \text{m}, \quad \mathbf{x}_4 = \begin{bmatrix} -500 \\ 250 \\ -800 \end{bmatrix} \text{m} \tag{60}$$

Here vehicle one is at $\mathbf{x}_1$ and vehicle two is at $\mathbf{x}_2$. Targets one and two are at $\mathbf{x}_3$ and $\mathbf{x}_4$, respectively. The LOS truth vectors are determined from locations listed in Eq. (60), others can be found by using the appropriate attitude transformation without noise added. For this configuration the true relative attitude is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{61}$$

For the simulation the LOS vectors are converted into focal-plane coordinates and random noise is added to the true values having covariances described previously, with $\sigma = 17 \times 10^{-6}$ rad. Since each FPD has its own boresight axis, the measurement error-covariance for each FPD is determined with respect to the corresponding sensor frames and must be rotated to the vehicle's body frame as well. The letter S is used to denote sensor frame. The orthogonal transformations for their respective sensor frames, denoted by the subscript, used to orientate the FPD to the specific vehicle, denoted by the superscript, are given by

$$A_{s_{\mathcal{B}_1}}^{\mathbf{v}_1} = \begin{bmatrix} -0.8373 & -0.2962 & 0.4596 \\ -0.2962 & -0.4609 & 0.8366 \\ 0.4596 & -0.8366 & 0.2981 \end{bmatrix}, \quad A_{s_{\mathcal{B}_1}}^{\mathbf{v}_2} = \begin{bmatrix} -0.8069 & 0.4487 & 0.3843 \\ 0.4487 & -0.0423 & 0.8927 \\ 0.3843 & 0.8927 & -0.2355 \end{bmatrix} \tag{62a}$$

$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_1} = \begin{bmatrix} -0.8889 & 0.0644 & 0.4535 \\ 0.0644 & -0.9626 & 0.2630 \\ 0.4535 & 0.2630 & 0.8515 \end{bmatrix}, \quad A_{s_{\mathcal{B}_2}}^{\mathbf{w}_2} = \begin{bmatrix} 0.4579 & -0.0169 & 0.8888 \\ -0.0169 & -0.9998 & -0.0103 \\ 0.8888 & 0.0103 & -0.4581 \end{bmatrix} \tag{62b}$$

$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_3} = \begin{bmatrix} -0.8121 & -0.1287 & -0.5692 \\ -0.1287 & -0.9119 & 0.3898 \\ -0.5692 & 0.3898 & 0.7240 \end{bmatrix}, \quad A_{s_{\mathcal{B}_1}}^{\mathbf{v}_3} = \begin{bmatrix} -0.1171 & -0.1582 & 0.9804 \\ -0.1582 & -0.9716 & -0.1757 \\ 0.9804 & -0.1757 & 0.0887 \end{bmatrix} \tag{62c}$$

The configuration is considered for 1,000 Monte Carlo trials. Measurements are generated in the sensor frame and rotated to the body frame to be combined with the other measurements to determine the full relative attitudes. The wide-FOV measurement model for the FPD LOS covariance is used. Relative attitude angle errors are displayed in Figure 4. Three case are shown: two cases involve only one target and the LOS vectors between the vehicle, and one case uses both targets and the LOS vectors between the vehicles. The first case uses only target one, $\mathbf{x}_3$, which is shown in Figure 4(a). Case 2 uses target 2, $\mathbf{x}_4$, which is shown in Figure 4(b), and Case 3 uses both targets which is shown in Figure 4(c).

Better performance characteristics are given when both targets are used in the constrained solution. Figure 4 shows that the derived attitude-error covariance does indeed bound these errors in a 3σ sense. It is seen that Case 1 and Case 2 have larger roll errors than Case 3. In this scenario since the LOS between the vehicle is in the x-direction, rotation about this direction is not resolved by the LOS observation. The common target observations provide the missing information. Therefore Case 3, which uses both targets, has smaller errors in the roll direction than Case 1 or Case 2 which only use one target. The covariance values for the three cases are given below

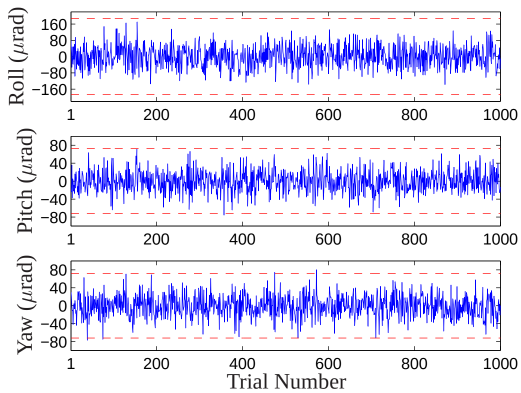
$$P_{\text{case1}} = 1 \times 10^{-8} \begin{bmatrix} 0.2599 & -0.0124 & 0.0335 \\ -0.0124 & 0.0583 & -0.0008 \\ 0.0335 & -0.0008 & 0.0579 \end{bmatrix} \text{rad}^2 \tag{63a}$$

$$P_{\text{case2}} = 1 \times 10^{-8} \begin{bmatrix} 0.5011 & 0.0243 & 0.0470 \\ 0.0243 & 0.0583 & -0.0008 \\ 0.0470 & -0.0008 & 0.0579 \end{bmatrix} \text{rad}^2 \tag{63b}$$

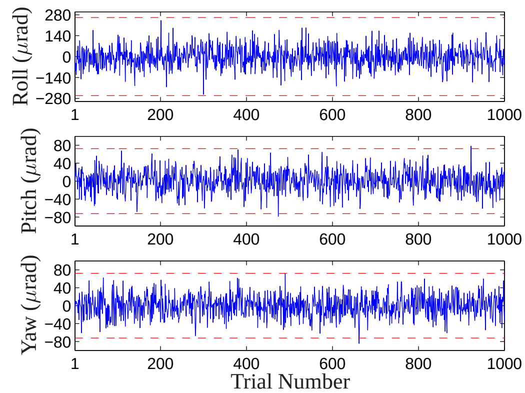
$$P_{\text{case3}} = 1 \times 10^{-8} \begin{bmatrix} 0.1925 & 0.0243 & 0.0470 \\ 0.0243 & 0.0583 & -0.0008 \\ 0.0470 & -0.0008 & 0.0579 \end{bmatrix} \text{rad}^2 \tag{63c}$$

Dynamic Formation Simulation

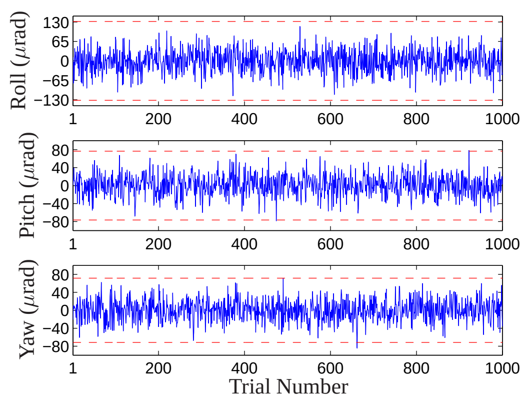
In the dynamic configuration four spacecraft are flying in a closed formation. Their relative positions are unknown as they measure LOS vectors to each other. In this configuration it is assumed that the FPD devices are gimballed allowing constant visibility and, for simplicity, it is assumed that the rotation matrix between the body frames to the sensor frames are those listed for the static simulations.



(a) **Relative Attitude Estimate Errors Case 1: Using Target 1**



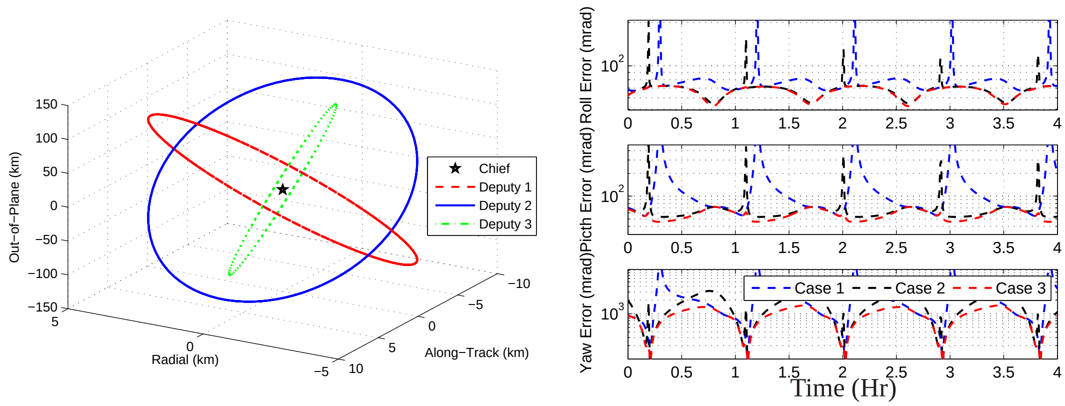
(b) **Relative Attitude Estimate Errors Case 2: Using Target 2**



(c) **Relative Attitude Estimate Errors Case 3: Using both Targets**

Figure 4. Relative Attitude Estimate Errors for the Three Cases

The relative motion trajectories are displayed in Figure 3. The chief spacecraft is in a low-Earth orbit and the orbital element differences are selected such that the deputies' relative motion around the chief have both in-plane and out-plane relative motion. The orbital elements of the chief and the orbital element difference for the deputies are listed in Table 1. These values can be used to produce the orbits using the linearized solution.[8] The sizes of the orbits are chosen such that deputy 1 has a smaller orbit than deputies 2 and 3 who both have similar size orbits in different orientations. Deputy 1 has 5 km of in-plane motion and 100 km of out-of-plane motion, as opposed to the size for deputies 2 and 3 which have 10 km of in-plane motion and 150 km of out-of-plane motion. LOS vectors are converted into focal-plane coordinates and random noise is added to the

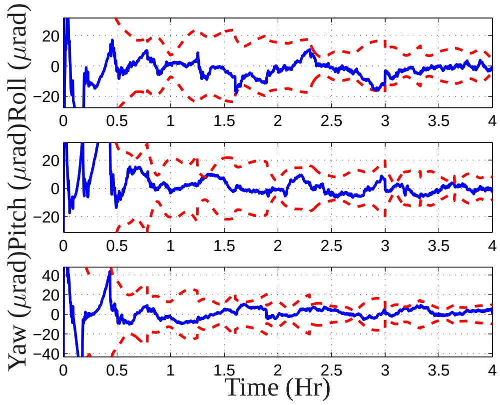


(a) **Dynamic Trajectories** (b) **Comparison of 3σ Bounds for three Cases**

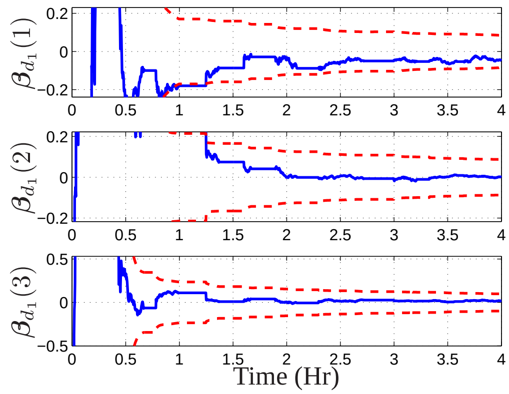
Figure 5. Dynamic Scenario

true values having covariances described previously, with $\sigma = 17 \times 10^{-6}$rad. The algorithms for the deterministic case with one constraint and for the suboptimal solution case are used to provide a point-by-point solution for the relative attitude. The relative attitude that is determined solves the transformation from the chief spacecraft frame, $\mathcal{C}$, to the deputy 1 frame, $\mathcal{D}_1$. These frames are defined as the fixed body frames of each spacecraft and it is assumed both the chief and the deputy 1 spacecraft have inertial fixed attitudes (no attitude dynamics). The algorithm from the deterministic case with one constraint is used to solve for the relative attitude in the deterministic case where only one common target is considered. Here the algorithm from the deterministic case with one constraint is used to solve for the relative attitude between the chief and deputy 1 spacecraft for two cases: one case where deputy 2 is considered as the common object and the other where deputy 3 is considered as a common object. The algorithm from the suboptimal solution case is used to provide a solution for the over-deterministic case where both deputy 2 and deputy 3 are used as common objects to solve for the relative attitude between the chief spacecraft and deputy 1 spacecraft.

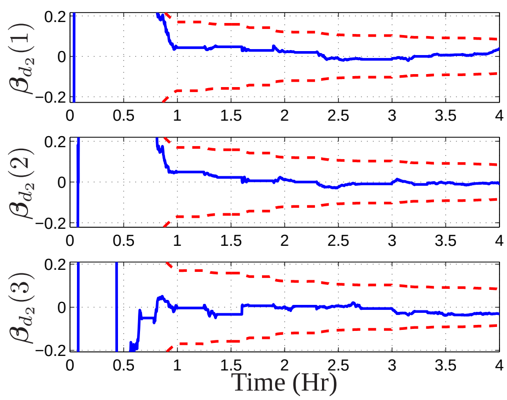
Figure 5 displays the relative attitude errors for the dynamic configuration for all three cases. The magnitude of the relative attitude errors dependence on geometry can clearly be seen. As the LOS geometry changes throughout the trajectory, the 3σ bounds of the errors also change and accurately bound the estimated attitude errors. A large increase in the relative error can be seen as the LOS configuration approaches an extreme condition where $\mathbf{w}_1$, $\mathbf{w}_2$ and $A_{\text{true}}\mathbf{v}_2$ are nearly parallel for Case 1 and Case 2. This results in a near unobservable situation, which is correctly depicted in the covariance. Since Case 3 determines a solution using both deputy 2 and deputy 3 as common objects it avoids unobservable situations as can be seen from Figure 5. In fact Case 3 doesn't have



(a) **Relative Attitude Estimate Errors**



(b) **Gyro Bias β_{d_1} (Deg/Hr)**



(c) **Gyro Bias β_{d_2} (Deg/Hr)**

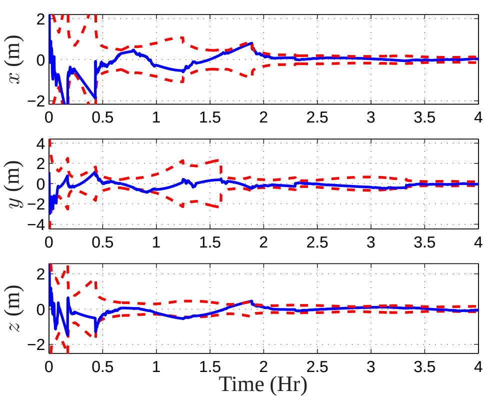
Figure 6. Relative Attitude Estimate Errors for the Filtering Cases

large increases in the attitude error, as in Case 1 and Case 2, since it considers both common objects. When one common object has large error due it approaching a unobservable situation the second common object provides this information and avoids any of the errors getting large. Figure 5(b) shows the 3σ bounds for the three cases plotted in log scale. From this figure it is seen that Case 3 gives the optimal error for two targets and consistently gives the best estimate as expected, since this case uses both common objects.

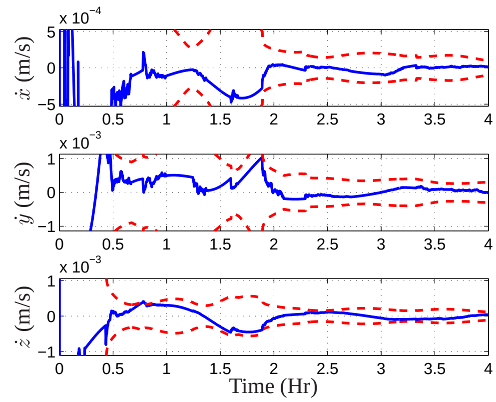
Attitude and Position Filtering

In this example attitude dynamics is considered where the relative attitude that is determined solves the transformation from the chief spacecraft frame, $\mathcal{C}$, to the deputy 1 frame, $\mathcal{D}_1$. The Kalman filter described in the previous section is used to estimate the relative position and attitude between the deputy 1 and chief spacecraft using LOS, gyros, and common targets measurements. The gyro biases are also estimated for in the filter along with chief orbital parameters.

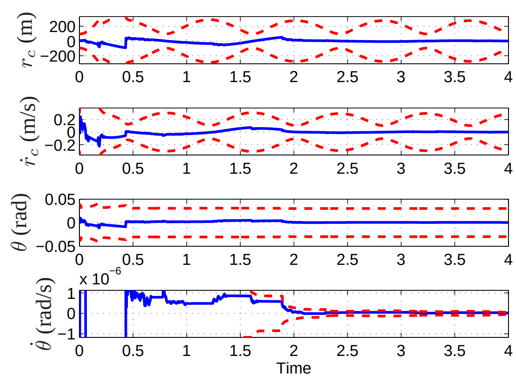
Figure 6 displays the relative attitude errors for the filtering case. From figure 6(a) it is seen that the magnitude of the relative attitude errors dependence on geometry is reduced by the filtering approach. As the LOS geometry changes throughout the trajectory, the 3σ bounds of the errors



(a) **Relative Position Error**



(b) **Relative Velocity Error**



(c) **Chief Orbit Element Errors**

Figure 7. Relative Position State Estimates

do not change as much as the previous examples. The gyro states show good performance and estimation errors are within the 3 σ bounds for the simulation period. Figure 7 displays the relative position estimates along with the chief orbital parameters estimated. It can be seen that relative position is estimated to within 1 m and the relative velocity is estimated to within 1 mm/s. The chief orbital parameters show good performance.

Table 1. Orbital Elements of Chief and Orbital Element Differences for Deputy Spacecraft

	Chief		Deputy1	Deputy2	Deputy3
a (km)	7555.0	δa (km)	0.0	0.0	0.0
e	0.03	δe	-0.477×10^{-3}	0.477×10^{-3}	-0.191×10^{-3}
i (deg)	48.0	δi (deg)	-0.60	-0.60	0.60
Ω (deg)	20.0	$\delta\Omega$ (deg)	-0.60	0.60	1
ω (Deg)	10.0	$\delta\omega$ (deg)	0	0	0
M (Deg)	0.00	δM (deg)	-3.60	3.60	3.60

CONCLUSIONS

In this paper a relative attitude filtering solution for a formation of two vehicles with multiple constraints and gyro measurements was shown. The solution for the relative attitude between the two vehicles was obtained only using line-of-sight measurements between them and a common (unknown) object observed by both vehicles, as well as gyro measurements. The constraint used in the solution is a triangle constraint on the vector observations and gyro measurements. This approach was extended to multiple objects by applying this constraint for each object. Simulation runs showed that good performance is possible with the developed filter shown in this paper.

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