Linearized Analysis of Inertial Navigation Employing Common Frame Error Representations

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This paper expands upon previous work that explores a new paradigm for inertial navigation systems. Errors in filter applications using inertial navigation system equations have been previously defined from an abstract vector point-of-view. For example, the error in velocity has always been expressed using a straight difference of the truth minus the estimate without regard to each of the vector's frame representations. In previous work an alternative vector state-error is defined using common coordinates over all vector error realizations, thereby providing a true-to-life representation of the actual errors. A modified extended Kalman filter was derived that employs the alternative vector state error representation. Here a linearized analysis of the new error equations is conducted to determine effects on INS performance. Schuler and Foucault frequencies are calculated using a stationary analysis.

I. Introduction

The earliest known practical application of an inertial navigation system (INS) is attributed to the German V-2 missile in 1942 [1], which employed a gyroscope, an airspeed sensor and an altimeter. A simple compass heading with a predetermined amount of fuel was used to guide the rocket to a target in a crude but effective manner. Later applications by the United States led to inertial guidance systems for ballistic missiles that could be launched from both land platforms and sea vessels. The space age brought about more accurate INS sensors, including inertial measurement units (IMUs) made up of three gyroscopes and three accelerometers mounted on a beryllium cube. Modern-day applications of INS with IMUs include aircraft navigation [2], underwater vehicles [3], and robotic systems [4].

It is well-known that all IMUs drift. For example the Apollo gyroscopes drifted about one milliradian per hour. This drift was corrected by "realigning" the inertial platform periodically through sighting on stars. This optical sighting measurements were fused with IMU data to 1) determine the drift in the IMU, and 2) propagate the inertial navigation equations using the IMU in "dynamic model replacement" mode [5] when optical sightings where not available. The workhorse for this data fusion was accomplished using the Kalman filter [6], more precisely Potter's square root extended Kalman filter (EKF) [7]. Straightforward application of the EKF for INS applications can be complicated by the choice of the attitude representation though. All minimal representations of the attitude are subject to singularity issues for certain rotations [8]. The quaternion [9] representation is now becoming mainstream because of its lack of singularity and bilinear kinematics relationship. However, handling the norm constraint is problematic. A practical solution to this problem involves using a local (minimal) error representation, such as the small angle approximation, while maintaining the quaternion as the global attitude representation. Rules of quaternion multiplication are employed in the linearization process, which maintain the norm to within the first-order approximation in the EKF. This led to the "multiplicative EKF" (MEKF) [10]. Higher-order approaches using this local/global methodology have been applied with the sigma-point Kalman filter [11], particle filter [12], as well as other filters and observers [13].

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In most INS applications the state vector usually consists of the attitude, position, velocity, and IMU calibration parameters such as drifts, scale factors and misalignments. Because position-type measurements are usually only given, e.g. pseudoranges to GPS satellites, the observability of the attitude and gyroscope calibration parameters is weak, which depends on the degree of motion of the vehicle [14]. Since the early days of employing the EKF for INS applications, and even modern-day applications, the state errors are defined as a simple difference between the truth and the estimate. Reference [15] argues that a new state-error definition is required in which some state-error quantities are defined using elements expressed in a common frame, which provides a realistic framework to describe the actual errors. The errors are put into a common frame using the estimated attitude error, which led to the "geometric EKF" (GEKF). The GEKF provides extra transport terms, due to error-attitude coupling with the states, in the filter dynamics that can provide better convergence characteristics than the standard MEKF. The work in [15] focuses strictly on attitude estimation, which incorporates only "body-frame" errors. Reference [16] extended this work to the full INS formulation for both the ECEF and NED coordinate systems. The main difference between the work in [15] and [16] is how errors that are expressed in some reference-frame coordinate system are handled. The newly derived INS NED filter from [16] is summarized here, then a linearized analysis is conducted for the new error metric. The analysis shows how additional terms creates a stronger coupling between the attitude and velocity states. A stationary analysis is also performed on the linearized state matrix to determine the Schuler and Foucault frequencies.

The organization of this paper proceeds as follows. First a review of the quaternion kinematics is shown, followed by a review of the GEKF approach. Then, the theory behind errors expressed in reference frame coordinates is developed, which leads to a generalized theory that unified errors expressed in either body or reference frame coordinates. Then, the INS formulation based on the new theory is summarized for the NED frame using an EKF. Next, a linearized analysis of the new error metric is shown, highlighting the new additional coupling terms and the Schuler and Foucault frequencies form a stationary analysis. Finally, conclusions are drawn based upon the developed theory.

II. Reference Frames

In this section the reference frames used to derive the INS EKF formulations are summarized, as shown in Figure 1:

- Earth-Centered-Inertial (ECI) Frame: denoted by $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$. The $\hat{\mathbf{i}}_1$ axis points toward the vernal equinox direction (also known as the "First Point of Aries" or the "vernal equinox point"), the $\hat{\mathbf{i}}_3$ axis points in the direction of the North pole and the $\hat{\mathbf{i}}_2$ axis completes the right-handed system (note that the $\hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2$ axes are on the equator, which is the fundamental plane). The ECI frame is non-rotating with respect to the stars (except for precession of equinoxes) and the Earth turns relative to this frame. Vectors described using ECI coordinates will have a superscript I (e.g., $\mathbf{p}^I$).

- Earth-Centered-Earth-Fixed (ECEF) Frame: denoted by $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. This frame is similar to the ECI frame with $\hat{\mathbf{e}}_3 = \hat{\mathbf{i}}_3$; however, the $\hat{\mathbf{e}}_1$ axis points in the direction of the Earth's prime meridian, and the $\hat{\mathbf{e}}_2$ axis completes the right-handed system. Unlike the ECI frame, the ECEF frame rotates with the Earth. The rotation angle is denoted by Θ in Figure 1. Vectors described using ECEF coordinates will have a superscript E (e.g., $\mathbf{p}^E$).

- North-East-Down (NED) Frame: denoted by $\{\hat{\mathbf{n}}, \hat{\mathbf{e}}, \hat{\mathbf{d}}\}$. This frame is used for local navigation purposes. It is formed by fitting a tangent plane to the geodetic reference ellipse at a point of interest [17]. The $\hat{\mathbf{n}}$ axis points true North, the $\hat{\mathbf{e}}$ points East, and the $\hat{\mathbf{d}}$ axis completes the right-handed system, which points in the direction of the interior of the Earth perpendicular to the reference ellipsoid. Vectors described using ECI coordinates will have a superscript N (e.g., $\mathbf{p}^N$).

- Body Frame: denoted by $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$. This frame is fixed onto the vehicle body and rotates with it. Conventions typically depend on the particular vehicle. Vectors described using body-frame coordinates will have a superscript B (e.g., $\mathbf{p}^B$).

The ECEF position vector is useful since this gives a simple approach to determine the longitude and latitude of a user. The Earth's geoid can be approximated by an ellipsoid of revolution about its minor axis. A common ellipsoid model is given by the World Geodetic System 1984 model (WGS-84), with semimajor

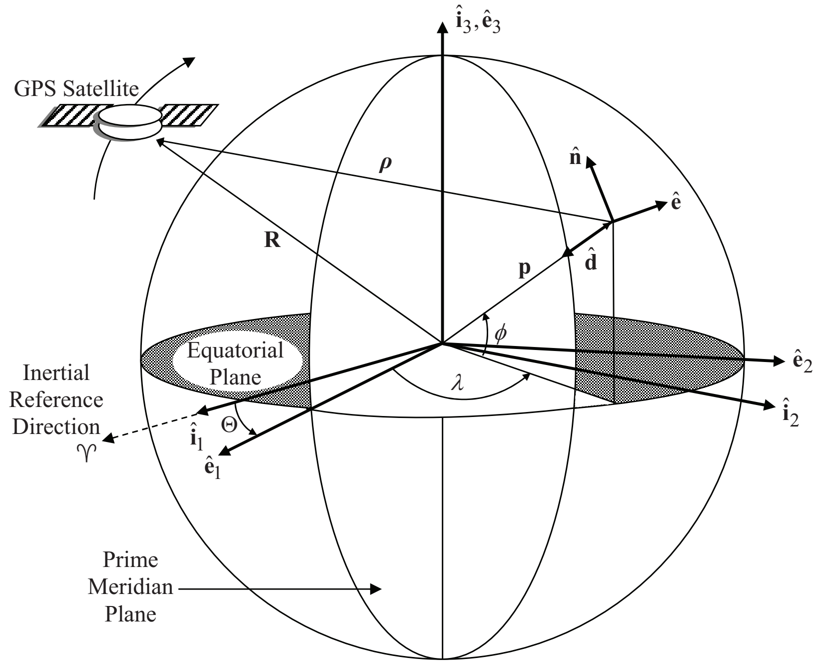

Figure 1. Definitions of Various Reference Frames

axis $a = 6,378,137.0$ m and semiminor axis $b = 6,356,752.3142$ m. The eccentricity of this ellipsoid is given by $e = 0.0818$. The geodetic coordinates are given by the latitude ϕ, longitude λ and height h. To determine the ECEF position vector, the length of the normal to the ellipsoid is first computed, given by

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{1}$$

Then, given the observer geodetic quantities ϕ, λ and h, the observer ECEF position coordinates are computed using

$$x = (N + h) \cos\phi \cos\lambda \tag{2a}$$
$$y = (N + h) \cos\phi \sin\lambda \tag{2b}$$
$$z = [N(1 - e^2) + h] \sin\phi \tag{2c}$$

The conversion from ECEF to geodetic coordinates is not straightforward, but a closed-form solution is given in Ref. [18]. The conversion from ECEF coordinates to NED coordinates involves a rotation matrix from the known latitude and longitude, denoted by A_E^N. By the definition of the NED frame, a vehicle is fixed within this frame. This frame serves to define local directions for the velocity vector determined in a frame in which the vehicle has motion, such as the ECEF frame. The NED frame is generally not used to provide a vehicle's positional coordinates, but rather to provide local directions along which the velocities may be indicated. The positions are determined by relating the velocity $\mathbf{v}^N$ with the derivatives of latitude, longitude and height, and integrating the resulting equations. The attitude matrix which maps the NED frame to the vehicle body frame is denoted by A_N^B. Note that the transformation from the ECEF to the body frame is simply given by $A_E^B = A_N^B A_E^N$.

III. Quaternion Kinematics

This section provides a brief review of quaternion kinematics. For more details see Refs. [5, 9]. The attitude matrix, A, maps from the reference frame to the vehicle body frame according to $A\,\mathbf{r}$, where $\mathbf{r}$ is a component vector given with respect to the reference frame. The quaternion is a four-dimensional vector, defined as

$$\mathbf{q} \triangleq \begin{bmatrix} \boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{3}$$

with

$$\boldsymbol{\varrho} \triangleq [q_1 \ \ q_2 \ \ q_3]^T = \mathbf{e}\sin(\vartheta/2) \tag{4a}$$
$$q_4 = \cos(\vartheta/2) \tag{4b}$$

where $\mathbf{e}$ is the unit Euler axis and ϑ is the rotation angle [9]. A quaternion parameterizing an attitude satisfies a single constraint given by $\|\mathbf{q}\| = 1$. In terms of the quaternion, its associated attitude matrix is given by

$$A(\mathbf{q}) = \Xi^T(\mathbf{q})\Psi(\mathbf{q}) \tag{5}$$

with

$$\Xi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix}, \quad \Psi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{6}$$

where $I_{3\times3}$ is a 3×3 identity matrix, and $[\boldsymbol{\varrho}\times]$ is the cross product matrix, defined by

$$[\boldsymbol{\varrho}\times] \triangleq \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{7}$$

An advantage to using quaternions is that the attitude matrix is quadratic in the parameters and also does not involve transcendental functions. For small angles the vector part of the quaternion is approximately equal to half angles so that $\boldsymbol{\varrho} \approx \boldsymbol{\alpha}/2$ and $q_4 \approx 1$, where $\boldsymbol{\alpha}$ is a vector of the roll, pitch and yaw angles. The attitude matrix can then be approximated by $A \approx I_{3\times3} - [\boldsymbol{\alpha}\times]$ which is valid to within first-order in the angles.

Successive rotations can be accomplished using quaternion multiplication. Here we adopt the convention of Ref. [10] who multiply the quaternions in the same order as the attitude matrix multiplication (in contrast to the usual convention established by Hamilton). A successive rotation using quaternions can be accomplished using

$$A(\mathbf{q}')A(\mathbf{q}) = A(\mathbf{q}' \otimes \mathbf{q}) \tag{8}$$

The composition of the quaternions is bilinear, with

$$\mathbf{q}' \otimes \mathbf{q} = [\Psi(\mathbf{q}') \ \ \mathbf{q}']\,\mathbf{q} = [\Xi(\mathbf{q}) \ \ \mathbf{q}]\,\mathbf{q}' \tag{9}$$

Also, the inverse quaternion is defined by

$$\mathbf{q}^{-1} \triangleq \begin{bmatrix} -\boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{10}$$

Note that $\mathbf{q} \otimes \mathbf{q}^{-1} = [0 \ \ 0 \ \ 0 \ \ 1]^T \triangleq \mathbf{I}_q$, which is the identity quaternion.

With attitude parameterized by the quaternion $\mathbf{q}$, the physical model is then the quaternion kinematics, given by

$$\dot{\mathbf{q}} = \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \mathbf{q} \tag{11}$$

where $\boldsymbol{\omega} \triangleq \boldsymbol{\omega}^B_{B/I}$ is the angular velocity vector of the B frame relative to the I frame expressed in B coordinates, and

$$\Omega(\boldsymbol{\omega}) \triangleq \begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \tag{12}$$

Also, the derivative of $\mathbf{q}^{-1}$ can be shown to be given by [19]

$$\dot{\mathbf{q}}^{-1} = -\frac{1}{2}\mathbf{q}^{-1} \otimes \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \tag{13}$$

The gyro measurement model is given by

$$\tilde{\boldsymbol{\omega}}^B_{B/I} = (I_{3\times3} + \mathcal{K}_g)\boldsymbol{\omega}^B_{B/I} + \boldsymbol{\beta}_g + \boldsymbol{\eta}_{gv} \tag{14a}$$

$$\dot{\boldsymbol{\beta}}_g = \boldsymbol{\eta}_{gu} \tag{14b}$$

where $\boldsymbol{\beta}_g$ is the gyro "bias", $\mathcal{K}_g$ is a diagonal matrix of gyro scale factors, and $\boldsymbol{\eta}_{gv}$ and $\boldsymbol{\eta}_{gu}$ are zero-mean Gaussian white-noise processes with spectral densities given by $\sigma^2_{gv} I_{3\times3}$ and $\sigma^2_{gu} I_{3\times3}$, respectively. The accelerometer measurement model is given by

$$\tilde{\mathbf{a}}^B = (I_{3\times3} + \mathcal{K}_g)\mathbf{a}^B + \boldsymbol{\beta}_a + \boldsymbol{\eta}_{av} \tag{15a}$$

$$\dot{\boldsymbol{\beta}}_a = \boldsymbol{\eta}_{au} \tag{15b}$$

where $\boldsymbol{\beta}_a$ is the accelerometer "bias", $\mathcal{K}_a$ is a diagonal matrix of accelerometer scale factors, and $\boldsymbol{\eta}_{av}$ and $\boldsymbol{\eta}_{au}$ are zero-mean Gaussian white-noise processes with spectral densities given by $\sigma^2_{av} I_{3\times3}$ and $\sigma^2_{au} I_{3\times3}$, respectively. We should note that most manufacturers give values for σ_{gv} and σ_{av}, but not σ_{gu} and σ_{au}. The scale factors are assumed to be small enough so that the approximation $(I + \mathcal{K})^{-1} \approx (I - \mathcal{K})$ is valid for both the gyros and acclerometers. Simulating gyro and accelerometer using computers is not easy since continuous measurements cannot be generated using digital computers. A discrete-time simulation is possible using the spectral densities though [5]. The gyro measurement can be simulated using

$$\tilde{\boldsymbol{\omega}}_{k+1} = \boldsymbol{\omega}_{k+1} + \frac{1}{2}(\boldsymbol{\beta}_{g_{k+1}} + \boldsymbol{\beta}_{g_k}) + \left(\frac{\sigma^2_{gv}}{\Delta t} + \frac{1}{12}\sigma^2_{gu}\,\Delta t\right)^{1/2} \mathbf{N}_{gv_k} \tag{16a}$$

$$\boldsymbol{\beta}_{g_{k+1}} = \boldsymbol{\beta}_{g_k} + \sigma_{gu}\,\Delta t^{1/2}\mathbf{N}_{gu_k} \tag{16b}$$

where the subscript k denotes the k^{th} time-step, Δt denotes the sampling interval, and $\mathbf{N}_{gv_k}$ and $\mathbf{N}_{gu_k}$ are zero-mean Gaussian white-noise processes with covariance each given by the identity matrix. Replacing $\boldsymbol{\omega}_{k+1}$ with $(I_3 + \mathcal{K}_g)\,\boldsymbol{\omega}_{k+1}$ in Eq. (16a) provides the discrete-time model for Eq. (14). A similar model can be employed for the discrete-time accelerometer measurement.

IV. Geometric Filtering

This section provides a review of the GEKF (more details can be found in Ref. [15]). Ideally, the appropriate filter would employ state errors between the true variables ($\mathbf{q}$, $\boldsymbol{\beta}$) and their corresponding estimates ($\hat{\mathbf{q}}$, $\hat{\boldsymbol{\beta}}$) defined by

$$\mathbf{dq} \triangleq \mathbf{q} \otimes \hat{\mathbf{q}}^{-1} \equiv \begin{bmatrix}\mathbf{d}\boldsymbol{\varrho}^T & \mathrm{d}q_4\end{bmatrix}^T \tag{17a}$$

$$\mathbf{d}\boldsymbol{\beta} \triangleq A^T(\mathbf{dq})\boldsymbol{\beta} - \hat{\boldsymbol{\beta}} \tag{17b}$$

where $\boldsymbol{\beta}$ is any state expressed in body-frame coordinates, such a strapdown gyro-bias state, and all realizations of $\mathbf{d}\boldsymbol{\beta}$ are expressed within the mean (estimated) coordinate frame. Also, $A(\mathbf{dq})$ is the attitude-error matrix that maps mean-frame quantities to their respective true frames. The first-order approximation of the attitude error-quaternion $\mathbf{dq}$ is given by $\mathbf{dq} \approx [\frac{1}{2}\mathbf{d}\boldsymbol{\alpha}^T \;\; 1]^T$ [10], where $\mathbf{d}\boldsymbol{\alpha}$ is a vector of small, roll, pitch and yaw angles for any rotation sequence. Then the rotation matrix in Eq. (17b) is approximated by

$$A(\mathbf{dq}) \equiv A(\mathbf{d}\boldsymbol{\alpha}) \approx I_{3\times3} - [\mathbf{d}\boldsymbol{\alpha}\times] \tag{18}$$

with $\mathbf{d}\boldsymbol{\alpha} = 2\,\mathbf{d}\boldsymbol{\varrho}$. Because the series expansions deriving the EKF are truncated to first-order, a linearized approximation is adequate because higher-order terms will ultimately be discarded. Thus, the error definitions are related according to

$$\mathbf{q} - \hat{\mathbf{q}} \approx \frac{1}{2}\Xi(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\alpha} \tag{19a}$$

$$\boldsymbol{\beta} - \hat{\boldsymbol{\beta}} \approx [\hat{\boldsymbol{\beta}}\times]\mathbf{d}\boldsymbol{\alpha} + \mathbf{d}\boldsymbol{\beta} \tag{19b}$$

where the approximation to the attitude matrix in Eq. (18) is used in obtaining Eq. (19b). Equation (19a) is recognized as the mapping employed by the "reduced covariance" approach to deriving the MEKF [10]. Assembling the components of Eq (19) leads to

$$\Delta\mathbf{x} \approx C\mathbf{dx} \tag{20}$$

where $\Delta\mathbf{x} \triangleq [(\mathbf{q}-\hat{\mathbf{q}})^T \;\; (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T]^T$, $\mathbf{dx} \triangleq [\mathbf{d}\boldsymbol{\alpha}^T \;\; \mathbf{d}\boldsymbol{\beta}^T]^T$, and the "error map" C is defined according to

$$C \triangleq \begin{bmatrix} \frac{1}{2}\Xi(\hat{\mathbf{q}}) & 0_{4\times3} \\ [\hat{\boldsymbol{\beta}}\times] & I_{3\times3} \end{bmatrix} \tag{21}$$

where $0_{m\times n}$ is an $m \times n$ matrix of zeros.

A. Propagation

The dynamic model is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\, \mathbf{w}) \tag{22}$$

where $\mathbf{x}$ is the $n\times1$ state vector, and $\mathbf{w}$ is the zero-mean Gaussian process noise vector with spectral density given by Q. Following classical developments of the EKF, the estimated dynamics follow

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}) \tag{23}$$

The standard EKF approximates the local dynamics with the truncated Taylor series expansion, given by

$$\dot{\mathbf{x}} \approx \mathbf{f}(\hat{\mathbf{x}}) + F_a\Delta\mathbf{x} + G_a\mathbf{w} \tag{24}$$

which is expanded about the approximate conditional mean $\hat{\mathbf{x}}$ (and about the mean process noise vector, $\hat{\mathbf{w}} = \mathbf{0}$). Also, F_a and G_a are the standard linearized state and process noise matrices, respectively. In the GEKF the standard EKF error definition $\Delta\mathbf{x}$ is replaced using the relationship of Eq. (20). The state dynamics in the GEKF are given by

$$\dot{\mathbf{x}} = \mathbf{f}(\hat{\mathbf{x}}) + F_aC\mathbf{dx} + G_a\mathbf{w} \tag{25}$$

The error dynamics in the GEKF can be shown to be given by

$$\mathbf{d}\dot{\mathbf{x}} = F_g\mathbf{dx} + G_g\mathbf{w} \tag{26}$$

where

$$F_g = (C^TC)^{-1}C^T(F_aC - \dot{C}) \tag{27a}$$
$$G_g = (C^TC)^{-1}C^TG_a \tag{27b}$$

Reference [15] proves that both F_g and G_g are unique. The GEKF error-covariance, $\mathcal{P}$, is governed by the differential equation

$$\dot{\mathcal{P}} = F_g\mathcal{P} + \mathcal{P}F_g^T + G_gQG_g^T \tag{28}$$

Equations (23) and (28) comprise the propagation stage of the GEKF algorithm.

B. Update

Consider the following discrete-time measurement:

$$\tilde{\mathbf{y}}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k \tag{29}$$

where $\mathbf{v}_k$ is a zero-mean Gaussian noise process with covariance R_k. Expanding $\mathbf{h}_k(\mathbf{x}_k)$ in a Taylor series about the *a priori* state estimate, $\hat{\mathbf{x}}_k^-$, and truncating to first-order leads to

$$\mathbf{h}_k(\mathbf{x}_k) \approx \mathbf{h}_k(\hat{\mathbf{x}}_k^-) + H_kC_k^-\mathbf{dx}_k^- \tag{30}$$

where $C_k^- \triangleq C(t_k^-)$ and $\mathbf{dx}_k^-$ denote the *a priori* error map and state error, respectively, and H_k is the usual EKF sensitivity matrix. The update in the GEKF is given by the usual form:

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k \left[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_k^-) \right] \tag{31}$$

The gain equation in the GEKF is different than the standard EKF though. This is given by

$$K_k = C_k^- \mathcal{P}_k^- [C_k^-]^T H_k^T \left[H_k C_k^- \mathcal{P}_k^- [C_k^-]^T H_k^T + R_k \right]^{-1} \tag{32}$$

Also, the error-covariance update is different, which is given by

$$\mathcal{P}_k^+ = \bar{M}_k \left\{ [I_{(n-1)\times(n-1)} - \bar{K}_k \bar{H}_k] \mathcal{P}_k^- [I_{(n-1)\times(n-1)} - \bar{K}_k \bar{H}_k]^T + \bar{K}_k R_k \bar{K}_k^T \right\} \bar{M}_k^T \tag{33}$$

where $\bar{H}_k \triangleq H_k C_k^-$ and $\bar{K}_k \triangleq \mathcal{P}_k^- \bar{H}_k^T [\bar{H}_k \mathcal{P}_k^- \bar{H}_k + R_k]^{-1}$, and the transformation $\bar{M}_k$ is given by

$$\bar{M}_k \triangleq \left([C_k^+]^T C_k^+ \right)^{-1} [C_k^+]^T C_k^- \tag{34}$$

Note that as $\hat{\mathbf{x}}_k^-$ approaches $\hat{\mathbf{x}}_k^+$, the transformation $\bar{M}_k$ approaches identity. Note that the state is reduced by 1 as evident by the use of the $(n-1) \times (n-1)$ identity matrix in Eq. (33). This is due to the fact that a local (minimal) error representation is used for the attitude error, as discussed in Ref. [10]. Equations (31) and (33) define the GEKF update stage. A summary of the GEKF is shown in Table 1. It should be noted that the quaternion update arises from a multiplicative update [10], even though it can be written as an additive update, i.e. the update maintains quaternion normalization to within first order.

V. Geometric Body and Reference Frame Errors

The derivation in Ref. [15] assumes that errors exist in only the body frame, which will be shown explicitly shortly. For example, gyro biases are associated with a vehicle's body frame, and all realizations of the stochastic errors are given with respect to this frame. However, INS applications also have errors associated with respect to some reference frame. For example, the velocity is defined with respect to a specified reference frame, and all realizations of the stochastic errors are given with respect to this frame. A complete characterization of errors in both the body frame and reference frame requires four frames: 1) True Reference R, Mean Reference $\hat{R}$, True Body B, and Mean Body $\hat{B}$. The body-frame error-quaternion is given by

$$\mathbf{dq}_B = \mathbf{q}_{BR} \otimes \mathbf{q}_{R\hat{R}} \otimes \mathbf{q}_{\hat{R}\hat{B}} \tag{35}$$

where $\mathbf{q}_{BR}$ maps from the true reference to the true body, $\mathbf{q}_{R\hat{R}}$ maps from the mean reference to the true reference, and $\mathbf{q}_{\hat{R}\hat{B}}$ maps from the mean body to the mean reference. The reference-frame error-quaternion is given by

$$\mathbf{dq}_R = \mathbf{q}_{\hat{R}\hat{B}} \otimes \mathbf{q}_{\hat{B}B} \otimes \mathbf{q}_{BR} \tag{36}$$

where $\mathbf{q}_{\hat{R}\hat{B}}$ maps from the mean body to the mean reference, $\mathbf{q}_{\hat{B}B}$ maps from the true body to the mean body, and $\mathbf{q}_{BR}$ maps from the true reference to the true body. These conventions are chosen so that if $\mathbf{q}_{R\hat{R}}$ and $\mathbf{q}_{\hat{B}B}$ are both the identity quaternions, then the error-quaternions would follow the Ref. [10] for the body-frame error, and Ref. [20] for the reference-frame error conventions directly. The true and estimated quaternions are equivalent to

$$\mathbf{q} \equiv \mathbf{q}_{BR} \tag{37a}$$
$$\hat{\mathbf{q}} \equiv \mathbf{q}_{\hat{B}\hat{R}} \tag{37b}$$

Then

$$\mathbf{dq}_B = \mathbf{q} \otimes \mathbf{dq}_R^{-1} \otimes \hat{\mathbf{q}}^{-1} \tag{38a}$$
$$\mathbf{dq}_R = \hat{\mathbf{q}}^{-1} \otimes \mathbf{dq}_B^{-1} \otimes \mathbf{q} \tag{38b}$$

and

$$\mathbf{dq}_B^{-1} = \hat{\mathbf{q}} \otimes \mathbf{dq}_R \otimes \mathbf{q}^{-1} \tag{39a}$$
$$\mathbf{dq}_R^{-1} = \mathbf{q}^{-1} \otimes \mathbf{dq}_B \otimes \hat{\mathbf{q}} \tag{39b}$$

Table 1. Geometric Extended Kalman Filter

Parameter	Value
Model	$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t),\, \mathbf{w}(t)], \quad E\{\mathbf{w}(t)\mathbf{w}^T(\tau)\} = Q(t)\delta(t-\tau)$ $\tilde{\mathbf{y}}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k, \quad E\{\mathbf{v}_k\mathbf{v}_k^T\} = R_k$
Initialize	$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ $\mathcal{P}(t_0) = E\{\mathbf{dx}_0\,\mathbf{dx}_0^T\}$
Gain	$\bar{K}_k = \mathcal{P}_k^- \bar{H}_k^T [\bar{H}_k \mathcal{P}_k^- \bar{H}_k^T + R_k]^{-1}$ $\bar{H}_k = H_k C_k^-$ $H_k = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right\vert_{\hat{\mathbf{x}}_k^-}$ $\Delta\mathbf{x}_k = C_k \mathbf{dx}_k$
Update	$\begin{bmatrix}\hat{\mathbf{q}}_k^+ \\ \hat{\boldsymbol{\beta}}_k^+\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{q}}_k^- \\ \hat{\boldsymbol{\beta}}_k^-\end{bmatrix} + C_k^- \bar{K}_k [\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_k^-)]$ $\hat{\mathbf{q}}_k^+ \leftarrow \hat{\mathbf{q}}_k^+ / \Vert\hat{\mathbf{q}}_k^+\Vert$ $\mathcal{P}_k^+ = \bar{M}_k \left\{[I_{(n-1)\times(n-1)} - \bar{K}_k\bar{H}_k]\mathcal{P}_k^-[I_{(n-1)\times(n-1)} - \bar{K}_k\bar{H}_k]^T + \bar{K}_k R_k \bar{K}_k^T\right\} \bar{M}_k^T$ $\bar{M}_k \triangleq \left([C_k^+]^T C_k^+\right)^{-1} [C_k^+]^T C_k^-$
Propagation	$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}})$ $\dot{\mathcal{P}} = F_g\mathcal{P} + \mathcal{P}F_g^T + G_g Q G_g^T$ $F_g = (C^TC)^{-1}C^T(F_aC - \dot{C}), \quad G_g = (C^TC)^{-1}C^T G_a$ $F_a = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right\vert_{\hat{\mathbf{x}}}, \quad G_a = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right\vert_{\hat{\mathbf{x}}}$

It is now shown that these four frames are not independent of each other. This is first done by deriving the kinematics for $\mathbf{dq}_B$ and $\mathbf{dq}_R$. Taking the time derivative of the expressions in Eq. (38) gives

$$\mathbf{d\dot{q}}_B = \dot{\mathbf{q}} \otimes \mathbf{dq}_R^{-1} \otimes \hat{\mathbf{q}}^{-1} + \mathbf{q} \otimes \mathbf{d\dot{q}}_R^{-1} \otimes \hat{\mathbf{q}}^{-1} + \mathbf{q} \otimes \mathbf{dq}_R^{-1} \otimes \dot{\hat{\mathbf{q}}}^{-1} \tag{40a}$$

$$\mathbf{d\dot{q}}_R = \dot{\hat{\mathbf{q}}}^{-1} \otimes \mathbf{dq}_B^{-1} \otimes \mathbf{q} + \hat{\mathbf{q}}^{-1} \otimes \mathbf{d\dot{q}}_B^{-1} \otimes \mathbf{q} + \hat{\mathbf{q}}^{-1} \otimes \mathbf{dq}_B^{-1} \otimes \dot{\mathbf{q}} \tag{40b}$$

Taking the time derivative of the expressions in Eq. (39) gives

$$\mathbf{d\dot{q}}_B^{-1} = \dot{\hat{\mathbf{q}}} \otimes \mathbf{dq}_R \otimes \mathbf{q}^{-1} + \hat{\mathbf{q}} \otimes \mathbf{d\dot{q}}_R \otimes \mathbf{q}^{-1} + \hat{\mathbf{q}} \otimes \mathbf{dq}_R \otimes \dot{\mathbf{q}}^{-1} \tag{41a}$$

$$\mathbf{d\dot{q}}_R^{-1} = \dot{\mathbf{q}}^{-1} \otimes \mathbf{dq}_B \otimes \hat{\mathbf{q}} + \mathbf{q}^{-1} \otimes \mathbf{d\dot{q}}_B \otimes \hat{\mathbf{q}} + \mathbf{q}^{-1} \otimes \mathbf{dq}_B \otimes \dot{\hat{\mathbf{q}}} \tag{41b}$$

Substituting Eqs. (39b) and (41b) into Eq. (40a) gives

$$\mathbf{d\dot{q}}_B = \dot{\mathbf{q}} \otimes \mathbf{q}^{-1} \otimes \mathbf{dq}_B + \mathbf{q} \otimes \dot{\mathbf{q}}^{-1} \otimes \mathbf{dq}_B + \mathbf{d\dot{q}}_B + \mathbf{dq}_B \otimes \dot{\hat{\mathbf{q}}} \otimes \hat{\mathbf{q}}^{-1} + \mathbf{dq}_B \otimes \hat{\mathbf{q}} \otimes \dot{\hat{\mathbf{q}}}^{-1} \tag{42}$$

Substituting Eqs. (11) and (13), and their respective estimated quantities, given by the following kinematics relations:

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\begin{bmatrix}\hat{\boldsymbol{\omega}} \\ 0\end{bmatrix} \otimes \hat{\mathbf{q}} \tag{43a}$$

$$\dot{\hat{\mathbf{q}}}^{-1} = -\frac{1}{2}\hat{\mathbf{q}}^{-1} \otimes \begin{bmatrix}\hat{\boldsymbol{\omega}} \\ 0\end{bmatrix} \tag{43b}$$

into Eq. (42) gives $\mathbf{0} = \mathbf{0}$. This clearly shows that the four frames are not independent.

Another approach provides the same conclusion. Taking the time derivative of $\mathbf{dq}_R \otimes \mathbf{dq}_R^{-1} = \mathbf{I}_q$ leads to

$$\mathbf{d\dot{q}}_R^{-1} = -\mathbf{dq}_R^{-1} \otimes \mathbf{d\dot{q}}_R \otimes \mathbf{dq}_R^{-1} \tag{44}$$

Using this equation in Eq. (40a) leads to

$$\mathbf{d\dot{q}}_B^{-1} = -\mathbf{dq}_B^{-1} \otimes \mathbf{d\dot{q}}_B \otimes \mathbf{dq}_B^{-1} \tag{45}$$

which shows that $\mathbf{dq}_B \otimes \mathbf{dq}_B^{-1} = \mathbf{I}_q$, since Eq. (45) can be derived from taken the time derivative of $\mathbf{dq}_B \otimes \mathbf{dq}_B^{-1} = \mathbf{I}_q$. This again shows that the four frames are not independent of each other.

Reference [15] states that every realization of the geometrically defined error must be expressed with respect to the same coordinate basis. Therefore, in order to express the body-frame frame errors when errors exist in both the body and reference frames it is argued here that $\mathbf{dq}_B$ is the quaternion that maps the mean body-frame vectors to the true body-frame vectors *given* the reference frame. The same analogy holds for $\mathbf{dq}_R$. The errors are now defined using a conditional probability, given by

$$\mathbf{dq}_B \triangleq E\{\mathbf{q}_{BR} \otimes \mathbf{q}_{R\hat{R}} \otimes \mathbf{q}_{\hat{R}\hat{B}} \,|\, \hat{R}\} = \mathbf{q} \otimes \hat{\mathbf{q}}^{-1} \tag{46a}$$

$$\mathbf{dq}_R \triangleq E\{\mathbf{q}_{\hat{R}\hat{B}} \otimes \mathbf{q}_{\hat{B}B} \otimes \mathbf{q}_{BR} \,|\, \hat{B}\} = \hat{\mathbf{q}}^{-1} \otimes \mathbf{q} \tag{46b}$$

Since the mean reference frame is given in the definition of Eq. (46a) then it is not treated as a random variable from a conditional point-of-view, which resolves the frame independence issue. Note that $\hat{B}$ and $\hat{R}$ are still random variables. The conditional probability definition is required to ensure that the errors in the body and reference frames have physical meaning.

Now that the body and reference error frame definitions have been established, the relationships between them are derived. Suppose that representations of some true vector are given in the body frame, denoted by $\boldsymbol{\beta}_B$, and in the reference frame, denoted by $\boldsymbol{\beta}_R$. The mapping between these two vectors is given by the attitude matrix $A(\mathbf{q})$ with

$$\boldsymbol{\beta}_B = A(\mathbf{q})\boldsymbol{\beta}_R \tag{47}$$

The attitude matrix equivalent of Eq. (46a) for the body-frame error definition is given by

$$A_B(\mathbf{dq}) = A(\mathbf{q})A^T(\hat{\mathbf{q}}) \tag{48}$$

Solving Eq. (48) for $A(\mathbf{q})$, and substituting the resultant into Eq. (47) gives

$$\boldsymbol{\beta}_B = A_B(\mathbf{dq})A(\hat{\mathbf{q}})\boldsymbol{\beta}_R \tag{49}$$

The conditional body-frame estimate, i.e. given the mean reference frame, for $\boldsymbol{\beta}_B$ is given by

$$\hat{\boldsymbol{\beta}}_B \triangleq E\{\boldsymbol{\beta}_B|\hat{R}\} = A(\hat{\mathbf{q}})\boldsymbol{\beta}_R \tag{50}$$

Using Eq. (50) in Eq. (49) gives

$$\boldsymbol{\beta}_B = A_B(\mathbf{dq})\hat{\boldsymbol{\beta}}_B \tag{51}$$

These equations are consistent with previously defined body-frame error:

$$\mathbf{d}\boldsymbol{\beta}_B \triangleq A_B^T(\mathbf{dq})\boldsymbol{\beta}_B - \hat{\boldsymbol{\beta}}_B \tag{52}$$

Substituting Eq. (49) and using Eq. (50), or substituting Eq. (50) and using Eq. (49), in Eq. (52) both give $\mathbf{d}\boldsymbol{\beta}_B = \mathbf{0}$. This analysis shows why Eq. (46a) is called the "body-referenced" error in Ref. [10], although it is not explicitly shown there. It is derived here explicitly to show the consistency of the geometrically defined body-error representation. The "unframed" error definition in Ref. [10] is not consistent with the geometrically defined body-error representation though. This unframed body-error definition is given by

$$\boldsymbol{\delta\beta}_B \triangleq \boldsymbol{\beta}_B - \hat{\boldsymbol{\beta}}_B \tag{53}$$

which is the same error definition used in all filter implementations, dating back to the earliest days of attitude estimation and inertial navigation, before the geometrically consistent filter in Ref. [15] was derived. Substituting Eq. (51) into Eq. (53) shows that $\boldsymbol{\delta\beta}_B$ clearly does not achieve the goal of having errors represented in common frames.

The attitude matrix equivalent of Eq. (46b) for the reference-frame error definition is given by

$$A_R(\mathbf{dq}) = A^T(\hat{\mathbf{q}})A(\mathbf{q}) \tag{54}$$

which is consistent with the definition given in Ref. [20]. Solving this equation for $A^T(\mathbf{q})$, and substituting it into $\boldsymbol{\beta}_R = A^T(\mathbf{q})\boldsymbol{\beta}_B$ gives

$$\boldsymbol{\beta}_R = A_R^T(\mathbf{dq})A^T(\hat{\mathbf{q}})\boldsymbol{\beta}_B \tag{55}$$

The conditional reference-frame estimate, i.e. given the mean body frame, for $\boldsymbol{\beta}_R$ is given by

$$\hat{\boldsymbol{\beta}}_R \triangleq E\{\boldsymbol{\beta}_R|\hat{B}\} = A^T(\hat{\mathbf{q}})\boldsymbol{\beta}_B \tag{56}$$

Solving Eq. (55) for $\boldsymbol{\beta}_B$, and substituting the resultant into Eq. (56) gives

$$\hat{\boldsymbol{\beta}}_R = A_R(\mathbf{dq})\boldsymbol{\beta}_R \tag{57}$$

Equations (51) and (57) have a similar form but use two different attitude-error representations. Also, note that $A_R(\mathbf{dq}) \neq A_B^T(\mathbf{dq})$. Whereas $A_B(\mathbf{dq})$ maps mean-frame body quantities to their respective true body frames, $A_R(\mathbf{dq})$ maps true-frame reference quantities to their respective mean reference frames. This is consistent with the following reference-frame error definition:

$$\mathbf{d}\boldsymbol{\beta}_R \triangleq A_R(\mathbf{dq})\boldsymbol{\beta}_R - \hat{\boldsymbol{\beta}}_R \tag{58}$$

Substituting Eq. (57) into Eq. (58) gives $\mathbf{d}\boldsymbol{\beta}_R = \mathbf{0}$, which is the desired result. As with the unframed body-error representation, the unframed reference-error representation, defined by

$$\boldsymbol{\delta\beta}_R \triangleq \boldsymbol{\beta}_R - \hat{\boldsymbol{\beta}}_R \tag{59}$$

does not yield a frame consistent representation, which is commonly employed in INS filter applications.

The body-frame error-kinematics follow [15]

$$\dot{A}_B(\mathbf{dq}) = -A_B(\mathbf{dq})[\mathbf{d}\boldsymbol{\omega}_B\times] \tag{60}$$

where

$$\mathbf{d}\boldsymbol{\omega}_B \triangleq A_B^T(\mathbf{dq})\boldsymbol{\omega}_B - \hat{\boldsymbol{\omega}}_B \tag{61}$$

It is explicitly stated here that the true and estimated angular velocities are expressed in body-frame coordinates by the subscript B. Note that Eq. (61) shows a mapping of the body-frame angular velocities into a common frame, which is the mean frame in this case. This is akin to what is commonly seen in attitude control designs, where the desired angular velocity and actual angular velocity are also mapped into a common frame [21]. The reference-frame error-kinematics can be shown to be given by

$$\dot{A}_R(\mathbf{dq}) = -[\mathbf{d}\boldsymbol{\omega}_R\times]A_R(\mathbf{dq}) \tag{62}$$

where

$$\mathbf{d}\boldsymbol{\omega}_R \triangleq A^T(\hat{\mathbf{q}})(\boldsymbol{\omega}_B - \hat{\boldsymbol{\omega}}_B) \tag{63}$$

Note that a straight difference of $\boldsymbol{\omega}_B$ and $\hat{\boldsymbol{\omega}}_B$ now appears. This seems unnatural in the context of common frame error representations discussed in this paper. But this "unframed" difference actually comes about from the reference-frame error definition given by Eq. (46b), in which the mean body-frame is assumed to be given. By this conditional expectation any reference-frame error definition involving body-frame vectors does not require that the body-frame vectors be mapped into a common frame. Thus the unframed angular velocity in Eq. (63) is perfectly reasonable under this definition.

It is important to note that any reference-frame vector would still be mapped into a common frame under Eq. (46b) though. The respective true and estimated reference-frame angular velocities are given by

$$\boldsymbol{\omega}_R = A^T(\mathbf{q})\boldsymbol{\omega}_B \tag{64a}$$

$$\hat{\boldsymbol{\omega}}_R = A^T(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}_B \tag{64b}$$

They are also referred to as the "space-referenced angular velocity" in Ref. [5]. Solving Eq. (64) for $\boldsymbol{\omega}_B$ and $\hat{\boldsymbol{\omega}}_B$, and substituting their resultants into Eq. (63) gives

$$\mathbf{d}\boldsymbol{\omega}_R = A_R(\mathbf{dq})\boldsymbol{\omega}_R - \hat{\boldsymbol{\omega}}_R \tag{65}$$

where Eq. (54) has been used. It is now seen that the reference-frame angular velocities are mapped into a common frame.

The same analogy can be shown when reference-frame vectors are used in the body-frame error-kinematics. Solving Eq. (64) for $\boldsymbol{\omega}_B$ and $\hat{\boldsymbol{\omega}}_B$, and substituting their resultants into Eq. (61) gives

$$\mathbf{d}\boldsymbol{\omega}_B = A^T(\hat{\mathbf{q}})(\boldsymbol{\omega}_R - \hat{\boldsymbol{\omega}}_R) \tag{66}$$

where Eq. (48) has also been used. It is seen here that a straight difference of the reference-frame angular velocities is now given, which is a result of the body-frame error definition given by conditional expectation in Eq. (46b).

In INS applications errors will be defined both in the body and reference frames. For example, body-frame gyro biases and reference-frame velocity vectors are employed in an INS filter. At first, it would seem that the filter design would require both Eqs. (60) and (62) to fully describe the error-kinematics for both frames. But this is not required because $A_B(\mathbf{dq})$ is related to $A_R(\mathbf{dq})$ through

$$A_R(\mathbf{dq}) = A^T(\hat{\mathbf{q}})A_B(\mathbf{dq})A(\hat{\mathbf{q}}) \tag{67}$$

Thus, error-kinematics for either $A_B(\mathbf{dq})$ or $A_R(\mathbf{dq})$ can be used, and then Eq. (67) can be employed to map between frames. Here, the body-frame error-kinematics in Eq. (60) will be employed from this point forward, and any reference-frame errors will be mapped by Eq. (67) so that only $A_B(\mathbf{dq})$ needs to be employed for both error definitions. Hence, the GEKF in Table 1 can still be directly employed even when reference-frame errors are present. From this point forward it will be evident when body-frame vectors and reference-frame vectors are used, so the subscripts B and R will be dropped to simplify the notation.

VI. Inertial Navigation Filter Applications

In this section the implementation equations for the EKF and GEKF using the NED formulation are shown. Here the quaternion maps quantities from the N frame to the B frame. The truth equations are given by

$$\dot{\mathbf{q}} = \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega}_{B/N}^B \tag{68a}$$

$$\dot{\phi} = \frac{v_N}{R_\phi + h} \tag{68b}$$

$$\dot{\lambda} = \frac{v_E}{(R_\lambda + h)\cos\phi} \tag{68c}$$

$$\dot{h} = -v_D \tag{68d}$$

$$\dot{v}_N = -\left[\frac{v_E}{(R_\lambda + h)\cos\phi} + 2\omega_e\right] v_E \sin\phi + \frac{v_N v_D}{R_\phi + h} + a_N \tag{68e}$$

$$\dot{v}_E = \left[\frac{v_E}{(R_\lambda + h)\cos\phi} + 2\omega_e\right] v_N \sin\phi + \frac{v_E v_D}{R_\lambda + h} + 2\omega_e v_D \cos\phi + a_E \tag{68f}$$

$$\dot{v}_D = -\frac{v_E^2}{R_\lambda + h} - \frac{v_N^2}{R_\phi + h} - 2\omega_e v_E \cos\phi + g + a_D \tag{68g}$$

where $\boldsymbol{\omega}_{B/N}^B$ is the angular velocity of the B frame relative to the N frame expressed in B coordinates, and

$$R_\phi = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} \tag{69a}$$

$$R_\lambda = \frac{a}{(1-e^2\sin^2\phi)^{1/2}} \tag{69b}$$

The local gravity, g, using WGS-84 parameters, is given by

$$\begin{aligned} g = 9.780327(1 + 5.3024\times10^{-3}\sin^2\phi - 5.8\times10^{-6}\sin^2 2\phi) \\ -(3.0877\times10^{-6} - 4.4\times10^{-9}\sin^2\phi)h + 7.2\times10^{-14}h^2 \ \ \text{m/sec}^2 \end{aligned} \tag{70}$$

where h is measured in meters. Note that Eq. (68a) cannot be used directly with the gyro measurement. However, this problem can be overcome by using the following identity:

$$\boldsymbol{\omega}_{B/I}^B = \boldsymbol{\omega}_{B/N}^B + \boldsymbol{\omega}_{N/I}^B \tag{71}$$

Solving Eq. (71) for $\boldsymbol{\omega}_{B/N}^B$ and substituting $\boldsymbol{\omega}_{N/I}^B = A_N^B(\mathbf{q})\boldsymbol{\omega}_{N/I}^N$ yields

$$\boldsymbol{\omega}_{B/N}^B = \boldsymbol{\omega}_{B/I}^B - A_N^B(\mathbf{q})\boldsymbol{\omega}_{N/I}^N \tag{72}$$

where

$$\boldsymbol{\omega}_{N/I}^N = \omega_e \begin{bmatrix} \cos\phi \\ 0 \\ -\sin\phi \end{bmatrix} + \begin{bmatrix} \dfrac{v_E}{R_\lambda + h} \\ -\dfrac{v_N}{R_\phi + h} \\ -\dfrac{v_E\tan\phi}{R_\lambda + h} \end{bmatrix} \tag{73}$$

Now, Eq. (68a) can be related to the gyro measurements. Also, the acceleration variables are related to the accelerometer measurements through

$$\mathbf{a}^N \triangleq \begin{bmatrix} a_N \\ a_E \\ a_D \end{bmatrix} = A_B^N(\mathbf{q})\mathbf{a}^B = A_B^N(\mathbf{q})(I_{3\times3} - \mathcal{K}_a)(\tilde{\mathbf{a}}^B - \boldsymbol{\beta}_a - \boldsymbol{\eta}_{av}) \tag{74}$$

where $\mathbf{a}^B$ is the acceleration vector in body coordinates, and $A_B^N(\mathbf{q})$ is the matrix transpose of $A_N^B(\mathbf{q})$.

The estimated quantities, assuming ω_e is exact, are given by

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\Xi(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}_{B/N}^B \tag{75a}$$

$$\hat{\boldsymbol{\omega}}_{B/N}^B = (I_{3\times3} - \hat{\mathcal{K}}_g)(\tilde{\boldsymbol{\omega}}_{B/I}^B - \hat{\boldsymbol{\beta}}_g) - A_N^B(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}_{N/I}^N \tag{75b}$$

$$\dot{\hat{\phi}} = \frac{\hat{v}_N}{\hat{R}_\phi + \hat{h}} \tag{75c}$$

$$\dot{\hat{\lambda}} = \frac{\hat{v}_E}{(\hat{R}_\lambda + \hat{h})\cos\hat{\phi}} \tag{75d}$$

$$\dot{\hat{h}} = -\hat{v}_D \tag{75e}$$

$$\dot{\hat{v}}_N = -\left[\frac{\hat{v}_E}{(\hat{R}_\lambda + \hat{h})\cos\hat{\phi}} + 2\omega_e\right]\hat{v}_E\sin\hat{\phi} + \frac{\hat{v}_N\hat{v}_D}{\hat{R}_\phi + \hat{h}} + \hat{a}_N \tag{75f}$$

$$\dot{\hat{v}}_E = \left[\frac{\hat{v}_E}{(\hat{R}_\lambda + \hat{h})\cos\hat{\phi}} + 2\omega_e\right]\hat{v}_N\sin\hat{\phi} + \frac{\hat{v}_E\hat{v}_D}{\hat{R}_\lambda + \hat{h}} + 2\omega_e\hat{v}_D\cos\hat{\phi} + \hat{a}_E \tag{75g}$$

$$\dot{\hat{v}}_D = -\frac{\hat{v}_E^2}{\hat{R}_\lambda + \hat{h}} - \frac{\hat{v}_N^2}{\hat{R}_\phi + \hat{h}} - 2\omega_e\hat{v}_E\cos\hat{\phi} + \hat{g} + \hat{a}_D \tag{75h}$$

$$\hat{\mathbf{a}}^N \equiv \begin{bmatrix} \hat{a}_N \\ \hat{a}_E \\ \hat{a}_D \end{bmatrix} = A_B^N(\hat{\mathbf{q}})\hat{\mathbf{a}}^B \tag{75i}$$

$$\hat{\mathbf{a}}^B = (I_{3\times3} - \hat{\mathcal{K}}_a)(\tilde{\mathbf{a}}^B - \hat{\boldsymbol{\beta}}_a) \tag{75j}$$

$$\dot{\hat{\boldsymbol{\beta}}}_g = \mathbf{0} \tag{75k}$$

$$\dot{\hat{\boldsymbol{\beta}}}_a = \mathbf{0} \tag{75l}$$

$$\dot{\hat{\mathbf{k}}}_g = \mathbf{0} \tag{75m}$$

$$\dot{\hat{\mathbf{k}}}_a = \mathbf{0} \tag{75n}$$

Also, $\hat{\boldsymbol{\omega}}_{N/I}^N$, $\hat{R}_\phi$, $\hat{R}_\lambda$, and $\hat{g}$ are evaluated at the current estimates.

The EKF formulation is shown in Ref. [19], which is not repeated here for brevity. The global state and local state-error vectors in the GEKF are defined as

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q} \\ \mathbf{p}^N \\ \mathbf{v}^N \\ \boldsymbol{\beta}_g \\ \boldsymbol{\beta}_a \\ \mathbf{k}_g \\ \mathbf{k}_a \end{bmatrix}, \quad \mathbf{dx} \triangleq \begin{bmatrix} \mathbf{d}\boldsymbol{\alpha} \\ \mathbf{dp}^N \\ \mathbf{dv}^N \\ \mathbf{d}\boldsymbol{\beta}_g \\ \mathbf{d}\boldsymbol{\beta}_a \\ \mathbf{dk}_g \\ \mathbf{dk}_a \end{bmatrix} \tag{76}$$

where $\mathbf{p}^N = [\phi \;\; \lambda \;\; h]^T$ and $\mathbf{v}^N = [v_N \;\; v_E \;\; v_D]^T$. The matrix F_a is given by

$$F_a = \begin{bmatrix} F_{a_{11}} & F_{a_{12}} & F_{a_{13}} & F_{a_{14}} & 0_{4\times3} & F_{a_{16}} & 0_{4\times3} \\ 0_{3\times4} & F_{a_{22}} & F_{a_{23}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ F_{a_{31}} & F_{a_{32}} & F_{a_{33}} & 0_{3\times3} & F_{a_{35}} & 0_{3\times3} & F_{a_{37}} \\ 0_{3\times4} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times4} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times4} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times4} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix} \tag{77}$$

with

$$F_{a_{11}} = \frac{1}{2}\Omega(\hat{\boldsymbol{\omega}}_{B/N}^B) - \Xi(\hat{\mathbf{q}})\left[A_N^B(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}_{N/I}^N\times\right]\Xi^T(\hat{\mathbf{q}}) \tag{78a}$$

$$F_{a_{12}} = -\frac{1}{2}\Xi(\hat{\mathbf{q}})A_N^B(\hat{\mathbf{q}}) \left.\frac{\partial\boldsymbol{\omega}_{N/I}^N}{\partial\mathbf{p}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N}, \quad F_{a_{13}} = -\frac{1}{2}\Xi(\hat{\mathbf{q}})A_N^B(\hat{\mathbf{q}}) \left.\frac{\partial\boldsymbol{\omega}_{N/I}^N}{\partial\mathbf{v}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N} \tag{78b}$$

$$F_{a_{14}} = -\frac{1}{2}\Xi(\hat{\mathbf{q}})(I_{3\times3} - \hat{\mathcal{K}}_g), \quad F_{a_{16}} = -\frac{1}{2}\Xi(\hat{\mathbf{q}})\mathbb{D}(\tilde{\boldsymbol{\omega}}_{B/I}^B - \hat{\boldsymbol{\beta}}_g) \tag{78c}$$

$$F_{a_{22}} = \left.\frac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{p}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N}, \quad F_{a_{23}} = \left.\frac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{v}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N} \tag{78d}$$

$$F_{a_{31}} = -2[A_B^N(\hat{\mathbf{q}})\hat{\mathbf{a}}^B\times]\Psi^T(\hat{\mathbf{q}}) \tag{78e}$$

$$F_{a_{32}} = \left.\frac{\partial\dot{\mathbf{v}}^N}{\partial\mathbf{p}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N}, \quad F_{a_{33}} = \left.\frac{\partial\dot{\mathbf{v}}^N}{\partial\mathbf{v}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N} \tag{78f}$$

$$F_{a_{35}} = -A_B^N(\hat{\mathbf{q}})(I_{3\times3} - \hat{\mathcal{K}}_a), \quad F_{a_{37}} = -A_B^N(\hat{\mathbf{q}})\mathbb{D}(\tilde{\mathbf{a}}^B - \hat{\boldsymbol{\beta}}_a) \tag{78g}$$

The angular velocity partials are given by

$$\frac{\partial \boldsymbol{\omega}_{N/I}^N}{\partial \mathbf{p}^N} = \begin{bmatrix} -\omega_e \sin\phi - \dfrac{v_E}{(R_\lambda+h)^2}\dfrac{\partial R_\lambda}{\partial \phi} & 0 & -\dfrac{v_E}{(R_\lambda+h)^2} \\ \dfrac{v_N}{(R_\phi+h)^2}\dfrac{\partial R_\phi}{\partial \phi} & 0 & \dfrac{v_N}{(R_\phi+h)^2} \\ -\omega_e \cos\phi - \dfrac{v_E \sec^2\phi}{R_\lambda+h} + \dfrac{v_E \tan\phi}{(R_\lambda+h)^2}\dfrac{\partial R_\lambda}{\partial \phi} & 0 & \dfrac{v_E \tan\phi}{(R_\lambda+h)^2} \end{bmatrix} \tag{79a}$$

$$\frac{\partial \boldsymbol{\omega}_{N/I}^N}{\partial \mathbf{v}^N} = \begin{bmatrix} 0 & \dfrac{1}{R_\lambda+h} & 0 \\ -\dfrac{1}{R_\phi+h} & 0 & 0 \\ 0 & -\dfrac{\tan\phi}{R_\lambda+h} & 0 \end{bmatrix} \tag{79b}$$

with

$$\frac{\partial R_\lambda}{\partial \phi} = \frac{a\, e^2 \sin\phi\cos\phi}{(1-e^2\sin^2\phi)^{3/2}} \tag{80a}$$

$$\frac{\partial R_\phi}{\partial \phi} = \frac{3a(1-e^2)e^2 \sin\phi\cos\phi}{(1-e^2\sin^2\phi)^{5/2}} \tag{80b}$$

The position partials are given by

$$\frac{\partial \dot{\mathbf{p}}^N}{\partial \mathbf{p}^N} = \begin{bmatrix} -\dfrac{v_N}{(R_\phi+h)^2}\dfrac{\partial R_\phi}{\partial \phi} & 0 & -\dfrac{v_N}{(R_\phi+h)^2} \\ -\dfrac{v_E \sec\phi}{(R_\lambda+h)^2}\dfrac{\partial R_\lambda}{\partial \phi} + \dfrac{v_E \sec\phi\tan\phi}{R_\lambda+h} & 0 & -\dfrac{v_E\sec\phi}{(R_\lambda+h)^2} \\ 0 & 0 & 0 \end{bmatrix} \tag{81a}$$

$$\frac{\partial \dot{\mathbf{p}}^N}{\partial \mathbf{v}^N} = \begin{bmatrix} \dfrac{1}{R_\phi+h} & 0 & 0 \\ 0 & \dfrac{\sec\phi}{R_\lambda+h} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{81b}$$

The velocity partials are given by

$$\frac{\partial \dot{\mathbf{v}}^N}{\partial \mathbf{p}^N} = \begin{bmatrix} Y_{11} & 0 & Y_{13} \\ Y_{21} & 0 & Y_{23} \\ Y_{31} & 0 & Y_{33} \end{bmatrix}, \quad \frac{\partial \dot{\mathbf{v}}^N}{\partial \mathbf{v}^N} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & 0 \end{bmatrix} \tag{82}$$

where

$$Y_{11} = -\frac{v_E^2 \sec^2\phi}{R_\lambda+h} + \frac{v_E^2\tan\phi}{(R_\lambda+h)^2}\frac{\partial R_\lambda}{\partial \phi} - 2\omega_e v_E\cos\phi - \frac{v_N v_D}{(R_\phi+h)^2}\frac{\partial R_\phi}{\partial \phi} \tag{83a}$$

$$Y_{13} = \frac{v_E^2\tan\phi}{(R_\lambda+h)^2} - \frac{v_N v_D}{(R_\phi+h)^2} \tag{83b}$$

$$
\begin{aligned}
Y_{21} &= \frac{v_E\, v_N \sec^2\phi}{R_\lambda + h} - \frac{v_E\, v_N \tan\phi}{(R_\lambda + h)^2}\frac{\partial R_\lambda}{\partial \phi} + 2\omega_e v_N \cos\phi \\
&\quad - \frac{v_E\, v_D}{(R_\lambda + h)^2}\frac{\partial R_\lambda}{\partial \phi} - 2\omega_e v_D \sin\phi
\end{aligned} \tag{83c}
$$

$$
Y_{23} = -v_E\left[\frac{v_N \tan\phi + v_D}{(R_\lambda + h)^2}\right] \tag{83d}
$$

$$
Y_{31} = \frac{v_E^2}{(R_\lambda + h)^2}\frac{\partial R_\lambda}{\partial \phi} + \frac{v_N^2}{(R_\phi + h)^2}\frac{\partial R_\phi}{\partial \phi} + 2\omega_e v_E \sin\phi + \frac{\partial g}{\partial \phi} \tag{83e}
$$

$$
Y_{33} = \frac{v_E^2}{(R_\lambda + h)^2} + \frac{v_N^2}{(R_\phi + h)^2} + \frac{\partial g}{\partial h} \tag{83f}
$$

and

$$
Z_{11} = \frac{v_D}{R_\phi + h}, \quad Z_{12} = -\frac{2 v_E \tan\phi}{R_\lambda + h} - 2\omega_e \sin\phi, \quad Z_{13} = \frac{v_N}{R_\phi + h} \tag{84a}
$$

$$
Z_{21} = \frac{v_E \tan\phi}{R_\lambda + h} + 2\omega_e \sin\phi, \quad Z_{22} = \frac{v_D + v_N \tan\phi}{R_\lambda + h}, \quad Z_{23} = \frac{v_E}{R_\lambda + h} + 2\omega_e \cos\phi \tag{84b}
$$

$$
Z_{31} = -\frac{2 v_N}{R_\phi + h}, \quad Z_{32} = -\frac{2 v_E}{R_\lambda + h} - 2\omega_e \cos\phi \tag{84c}
$$

with

$$
\begin{aligned}
\frac{\partial g}{\partial \phi} &= 9.780327[1.06048 \times 10^{-2} \sin\phi\cos\phi \\
&\quad - 4.64 \times 10^{-5}(\sin\phi\cos^3\phi - \sin^3\phi\cos\phi)] + 8.8 \times 10^{-9} h\, \sin\phi\cos\phi
\end{aligned} \tag{85a}
$$

$$
\frac{\partial g}{\partial h} = -3.0877 \times 10^{-6} + 4.4 \times 10^{-9}\sin^2\phi + 1.44 \times 10^{-13} h \tag{85b}
$$

The matrix G_a is given by

$$
G_a = \begin{bmatrix}
-\frac{1}{2}\Xi(\hat{\mathbf{q}})(I_{3\times3} - \hat{\mathcal{K}}_g) & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & -A_B^N(\hat{\mathbf{q}})(I_{3\times3} - \hat{\mathcal{K}}_a) & 0_{3\times3} \\
0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3}
\end{bmatrix} \tag{86}
$$

The matrices C and $\dot{C}$ are given by

$$
C = \begin{bmatrix}
\frac{1}{2}\Xi(\hat{\mathbf{q}}) & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} \\
0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
-[\hat{\mathbf{v}}^N\times]A_B^N(\hat{\mathbf{q}}) & 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
[\hat{\boldsymbol{\beta}}_g\times] & 0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
[\hat{\boldsymbol{\beta}}_a\times] & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
[\hat{\mathbf{k}}_g\times] & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\
[\hat{\mathbf{k}}_a\times] & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3}
\end{bmatrix} \tag{87a}
$$

$$
\dot{C} = \begin{bmatrix}
\frac{1}{4}\Omega(\hat{\boldsymbol{\omega}}_{B/N}^B)\Xi(\hat{\mathbf{q}}) + \frac{1}{2}\Xi(\hat{\mathbf{q}})[\hat{\boldsymbol{\omega}}_{B/N}^B\times] & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} & 0_{4\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
-[\dot{\hat{\mathbf{v}}}^N\times]A_B^N(\hat{\mathbf{q}}) - [\hat{\mathbf{v}}^N\times]A_B^N(\hat{\mathbf{q}})[\hat{\boldsymbol{\omega}}_{B/N}^B\times] & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3}
\end{bmatrix} \tag{87b}
$$

VII. Linearized Error Dynamics

A rigorous first-order analysis can be used to determine how the system will perform over short duration flights. The first navigation component to be investigated will be the orientation. Errors in the orientation of the body frame with respect to the navigation frame are parameterized by the small angle vector $\mathbf{d\alpha}$. The kinematic expression for $\mathbf{d\dot{\alpha}}$ is given by [15]

$$\mathbf{d\dot{\alpha}} = -[\mathbf{d\omega}^B_{B/N}\times]\mathbf{d\alpha} - \mathbf{d\omega}^B_{B/N} \tag{88}$$

While for the filter formulation the attitude errors are written in the body frame coordinates, for the standard linearized error analysis these errors are written in the navigation frame. This is simply done by including an additional attitude mapping as

$$\mathbf{d\dot{\alpha}} = -A(\mathbf{q})[\mathbf{d\omega}^B_{B/N}\times]\mathbf{d\alpha} - A(\mathbf{q})\mathbf{d\omega}^B_{B/N} \tag{89}$$

where the error in the rotation rate of the navigation frame with respect to the body frame, $\mathbf{d\omega}^B_{B/N}$, is based on the newly defined frame consistent error metric given as

$$\mathbf{d\omega}^B_{B/N} \triangleq A^T_B(\mathbf{dq})\boldsymbol{\omega}^B_{B/N} - \hat{\boldsymbol{\omega}}^{\hat{B}}_{B/N} \tag{90}$$

Here $\boldsymbol{\omega}^B_{B/N}$ is the angular rate given in the true frame, $\hat{\boldsymbol{\omega}}^{\hat{B}}_{B/N}$ is the estimated angular rate given in the mean body frame, and $A^T_B(\mathbf{dq})$ is the frame consistent mapping. These angular rates can be broken down into the body angular rate that is sensed by the gyro and the rotation rate of the navigation frame with respect to the inertial frame as

$$\boldsymbol{\omega}^B_{B/N} = \boldsymbol{\omega}^B_{B/I} - A^B_N\boldsymbol{\omega}^N_{N/I} \tag{91a}$$

$$\hat{\boldsymbol{\omega}}^{\hat{B}}_{B/N} = \hat{\boldsymbol{\omega}}^{\hat{B}}_{B/I} - A^{\hat{B}}_N\hat{\boldsymbol{\omega}}^N_{N/I} \tag{91b}$$

The error mapping, $A^T_B(\mathbf{dq})$, in Eq. (90) is the transpose of the error rotation matrix in Eq. (18), and can be written as

$$A^T_B(\mathbf{dq}) = I_{3\times3} + [\mathbf{d\alpha}\times] \tag{92}$$

Substituting Eq. (92) and $A^T(\mathbf{q}) = A(\mathbf{dq})A^T(\hat{\mathbf{q}})$ into Eq. (90) and multiplying yields

$$\mathbf{d\omega}^B_{B/N} = [\mathbf{d\alpha}\times]\,\boldsymbol{\omega}^B_{B/I} - A^T(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^N_{N/I} + \Delta\boldsymbol{\omega}^B_{B/I} \tag{93}$$

This can now be used in Eq. (89) along with the cross product matrix identity $M\,[\mathbf{x}\times]\,M^T = [(M\mathbf{x})\times]$ for a proper orthogonal 3×3 matrix, M, to give

$$\mathbf{d\dot{\alpha}} = -\left[A(\mathbf{q})\boldsymbol{\omega}^B_{B/I}\times\right]\mathbf{d\alpha} - \Delta\boldsymbol{\omega}^N_{N/I} + A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I} \tag{94}$$

where $\mathbf{d\dot{\alpha}}$ is now the rate of change for the small orientation error angle with its components written in the navigation frame. Using Eq. (71) for the true inertial rate $\boldsymbol{\omega}^B_{B/I}$ in Eq. (94) gives

$$\mathbf{d\dot{\alpha}} = -\left[\left(A(\mathbf{q})\boldsymbol{\omega}^B_{B/N} + \boldsymbol{\omega}^N_{N/I}\right)\times\right]\mathbf{d\alpha} - \Delta\boldsymbol{\omega}^N_{N/I} + A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I} \tag{95}$$

Here there is an additional body angular rate term, $A(\mathbf{q})\boldsymbol{\omega}^B_{B/N}$, seen in the attitude error. Since the inertial rate is potentially a much smaller number compared to the body rate, given there is motion of the vehicle, it can be seen how the body rate will dominate this attitude error term. This creates an additional coupling effect, supporting why there is a greater convergence rate seen in simulations studies in Ref. [16]. A first-order Taylor series expansion can be used to approximate $\Delta\boldsymbol{\omega}^N_{N/I}$ as

$$\mathbf{d\dot{\alpha}} = -\left[\left(A(\mathbf{q})\boldsymbol{\omega}^B_{B/N} + \boldsymbol{\omega}^N_{N/I}\right)\times\right]\mathbf{d\alpha} - C\mathbf{dp} - D\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I} \tag{96}$$

where $C = \left.\frac{\partial\boldsymbol{\omega}^N_{N/I}}{\partial\mathbf{p}}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N}$ and $D = \left.\frac{\partial\boldsymbol{\omega}^N_{N/I}}{\partial\mathbf{v}^N}\right|_{\hat{\mathbf{p}},\hat{\mathbf{v}}^N}$.

Next the error dynamics for the vehicular accelerations needs to be derived. This is done by first considering the new frame consistent error metric for the velocity vector:

$$\mathbf{dv} = A(\hat{\mathbf{q}})A(\mathbf{dq})A^T(\hat{\mathbf{q}})\mathbf{v} - \hat{\mathbf{v}} \tag{97}$$

Taking the first time derivative gives

$$\begin{aligned}\mathbf{d\dot{v}} &= \dot{A}(\hat{\mathbf{q}})A(\mathbf{dq})A^T(\hat{\mathbf{q}})\mathbf{v} + A(\hat{\mathbf{q}})\dot{A}(\mathbf{dq})A^T(\hat{\mathbf{q}})\mathbf{v} \\ &+ A(\hat{\mathbf{q}})A(\mathbf{dq})\dot{A}^T(\hat{\mathbf{q}})\mathbf{v} + A(\hat{\mathbf{q}})A(\mathbf{dq})A^T(\hat{\mathbf{q}})\dot{\mathbf{v}} - \dot{\hat{\mathbf{v}}}\end{aligned} \tag{98}$$

Substituting $\dot{A}(\mathbf{q}) = A(\mathbf{q})\left[\boldsymbol{\omega}^B_{B/N}\times\right]$ and its transpose, along with $\dot{A}(\mathbf{dq}) = -A(\mathbf{dq})\left[\mathbf{d}\boldsymbol{\omega}^B_{B/N}\times\right]$ in Eq. (98) gives

$$\begin{aligned}\mathbf{d\dot{v}} = A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A(\mathbf{dq})A^T(\hat{\mathbf{q}})\mathbf{v} - A(\hat{\mathbf{q}})A(\mathbf{dq})\left[\mathbf{d}\boldsymbol{\omega}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\mathbf{v} \\ -A(\hat{\mathbf{q}})A(\mathbf{dq})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\mathbf{v} + A(\hat{\mathbf{q}})A(\mathbf{dq})A^T(\hat{\mathbf{q}})\dot{\mathbf{v}} - \dot{\hat{\mathbf{v}}}\end{aligned} \tag{99}$$

Solving Eq. (97) for $\mathbf{v}$, and substituting the resultant into Eq. (99) yields

$$\mathbf{d\dot{v}} = \boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} + \boldsymbol{\delta} - \dot{\hat{\mathbf{v}}} \tag{100}$$

where

$$\boldsymbol{\alpha} \triangleq A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\left(\mathbf{dv} + \hat{\mathbf{v}}\right) \tag{101a}$$

$$\boldsymbol{\beta} \triangleq -A(\hat{\mathbf{q}})A(\mathbf{dq})\left[\mathbf{d}\boldsymbol{\omega}^B_{B/N}\times\right]A^T(\mathbf{dq})A^T(\hat{\mathbf{q}})\left(\mathbf{dv} + \hat{\mathbf{v}}\right) \tag{101b}$$

$$\boldsymbol{\gamma} \triangleq -A(\hat{\mathbf{q}})A(\mathbf{dq})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\mathbf{dq})A^T(\hat{\mathbf{q}})\left(\mathbf{dv} + \hat{\mathbf{v}}\right) \tag{101c}$$

$$\boldsymbol{\delta} \triangleq A(\hat{\mathbf{q}})A(\mathbf{dq})A^T(\hat{\mathbf{q}})\dot{\mathbf{v}} \tag{101d}$$

First examining $\boldsymbol{\alpha}$, it can be expanded to

$$\boldsymbol{\alpha} = A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\mathbf{dv} + A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\hat{\mathbf{v}} \tag{102}$$

Using the cross product identity $M^T\left[\boldsymbol{v}\times\right]M = \left[M\boldsymbol{v}\times\right]$ for orthogonal M gives

$$\boldsymbol{\alpha} = \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{dv} + \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\hat{\mathbf{v}} \tag{103}$$

Next examining $\boldsymbol{\beta}$, using $A(\mathbf{dq}) = I_{3\times3} - [\mathbf{d}\boldsymbol{\alpha}\times]$ and $A^T(\mathbf{dq}) = I_{3\times3} + [\mathbf{d}\boldsymbol{\alpha}\times]$ gives

$$\boldsymbol{\beta} = -A(\hat{\mathbf{q}})\left(I_{3\times3} - [\mathbf{d}\boldsymbol{\alpha}\times]\right)\left[\mathbf{d}\boldsymbol{\omega}^B_{B/N}\times\right]\left(I_{3\times3} + [\mathbf{d}\boldsymbol{\alpha}\times]\right)A^T(\hat{\mathbf{q}})\left(\mathbf{dv} + \hat{\mathbf{v}}\right) \tag{104}$$

Neglecting higher order terms and further simplification leads to

$$\boldsymbol{\beta} \approx [\hat{\mathbf{v}}\times]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\omega}^B_{B/N} \tag{105}$$

Next examining $\boldsymbol{\gamma}$, substituting $A(\mathbf{dq})$ and $A^T(\mathbf{dq})$ gives

$$\boldsymbol{\gamma} = -A(\hat{\mathbf{q}})\left(I_{3\times3} - [\mathbf{d}\boldsymbol{\alpha}\times]\right)\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\left(I_{3\times3} + [\mathbf{d}\boldsymbol{\alpha}\times]\right)A^T(\hat{\mathbf{q}})\left(\mathbf{dv} + \hat{\mathbf{v}}\right) \tag{106}$$

Expanding the parenthesis and neglecting higher order terms gives

$$\begin{aligned}\boldsymbol{\gamma} \approx &- A(\hat{\mathbf{q}})\left[\boldsymbol{\omega}^{\hat{B}}_{B/N}\times\right]A^T(\hat{\mathbf{q}})\mathbf{dv} - A(\hat{\mathbf{q}})\left[\boldsymbol{\omega}^{\hat{B}}_{B/N}\times\right]A^T(\hat{\mathbf{q}})\hat{\mathbf{v}} \\ &- A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right][\mathbf{d}\boldsymbol{\alpha}\times]A^T(\hat{\mathbf{q}})\hat{\mathbf{v}} + A(\hat{\mathbf{q}})[\mathbf{d}\boldsymbol{\alpha}\times]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\end{aligned} \tag{107}$$

Using the cross product identities, $M[\boldsymbol{v}\times]M^T = [M\boldsymbol{v}\times]$ and $[\mathbf{u}\times]\mathbf{v} = -[\mathbf{v}\times]\mathbf{u}$, this can be rewritten as

$$\begin{aligned}\boldsymbol{\gamma} = -\left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{dv} - \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\hat{\mathbf{v}} + A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\mathbf{d}\boldsymbol{\alpha} \\ -A(\hat{\mathbf{q}})\left[\left(\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\right)\times\right]\mathbf{d}\boldsymbol{\alpha}\end{aligned} \tag{108}$$

The last term can be rewritten using the identity $[(\boldsymbol{u}\times\boldsymbol{v})\times] = [\boldsymbol{u}\times][\boldsymbol{v}\times] - [\boldsymbol{v}\times][\boldsymbol{u}\times]$, giving

$$\begin{aligned}\boldsymbol{\gamma} = -\left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{dv} - \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\hat{\mathbf{v}} + A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\mathbf{d}\boldsymbol{\alpha} \\ -A(\hat{\mathbf{q}})\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\mathbf{d}\boldsymbol{\alpha} + A(\hat{\mathbf{q}})\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{d}\boldsymbol{\alpha}\end{aligned} \tag{109}$$

Simplifying gives

$$\boldsymbol{\gamma} = -\left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{dv} - \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\hat{\mathbf{v}} + A(\hat{\mathbf{q}})\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{d}\boldsymbol{\alpha} \tag{110}$$

Finally $\boldsymbol{\delta}$ in Eq. (101d) is simplified, written here for convenience:

$$\boldsymbol{\delta} = A(\hat{\mathbf{q}})A(\mathbf{dq})A^T(\hat{\mathbf{q}})\dot{\mathbf{v}} \tag{111}$$

Here, $\dot{\mathbf{v}}$ can be written as

$$\dot{\mathbf{v}} = f(\mathbf{p},\mathbf{v}) + g(\mathbf{p}) + A(\mathbf{q})\mathbf{a} \tag{112}$$

The functions $f(\mathbf{p},\mathbf{v})$ and $g(\mathbf{p})$ can be approximated using a first-order Taylor series expansion as

$$f(\mathbf{p},\mathbf{v}) + g(\mathbf{p}) \approx f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + \left.\frac{\partial f(\mathbf{p},\mathbf{v})}{\partial \mathbf{p}}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N}\mathbf{dp} + \left.\frac{\partial g(\mathbf{p})}{\partial \mathbf{p}}\right|_{\hat{\mathbf{p}}^N}\mathbf{dp} + \left.\frac{\partial f(\mathbf{p},\mathbf{v})}{\partial \mathbf{v}^N}\right|_{\hat{\mathbf{p}},\hat{\mathbf{v}}^N}\Delta\mathbf{v} \tag{113}$$

where $\Delta\mathbf{v}$ is defined by

$$\Delta\mathbf{v} = -\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\alpha} + \mathbf{dv} \tag{114}$$

Using Eq. (114) in Eq. (113) gives

$$f(\mathbf{p},\mathbf{v}) + g(\mathbf{p}) \approx f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} \tag{115}$$

where

$$M = \left.\frac{\partial f(\mathbf{p},\mathbf{v})}{\partial \mathbf{p}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N} \tag{116a}$$

$$N = \left.\frac{\partial f(\mathbf{p},\mathbf{v})}{\partial \mathbf{v}^N}\right|_{\hat{\mathbf{p}}^N,\hat{\mathbf{v}}^N} \tag{116b}$$

$$O = \left.\frac{\partial g(\mathbf{p})}{\partial \mathbf{p}^N}\right|_{\hat{\mathbf{p}}^N} \tag{116c}$$

Using $A(\mathbf{q}) = A(\hat{\mathbf{q}})A^T(\mathbf{dq})$ and Eq. (115) in Eq. (112) gives

$$\dot{\mathbf{v}} \approx f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})A^T(\mathbf{dq})\mathbf{a} \tag{117}$$

Replacing $A^T(\mathbf{dq})$ with $I_{3\times3} + [\mathbf{d}\boldsymbol{\alpha}\times]$ in the last term, expanding, and neglecting higher order terms gives

$$\dot{\mathbf{v}} \approx f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})\mathbf{a} + A(\hat{\mathbf{q}})\left[\mathbf{d}\boldsymbol{\alpha}\times\right]\mathbf{a} \tag{118}$$

Gathering like terms gives

$$\dot{\mathbf{v}} = f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} + \left(-N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}}) - A(\hat{\mathbf{q}})\left[\mathbf{a}\times\right]\right)\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})\mathbf{a} \tag{119}$$

This can now be used in Eq. (111) along with $A(\mathbf{dq}) = I_{3\times3} - [\mathbf{d}\boldsymbol{\alpha}\times]$ to give

$$\begin{aligned}&\boldsymbol{\delta} \approx A(\hat{\mathbf{q}})\left(I_{3\times3} - [\mathbf{d}\boldsymbol{\alpha}\times]\right)A^T(\hat{\mathbf{q}}) \\ &\quad \left(f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} + \left(-N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}}) - A(\hat{\mathbf{q}})\left[\mathbf{a}\times\right]\right)\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})\mathbf{a}\right)\end{aligned} \tag{120}$$

Expanding and neglecting higher order terms gives

$$\begin{aligned}\boldsymbol{\delta} = f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} + \left(-N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}}) - A(\hat{\mathbf{q}})\left[\mathbf{a}\times\right]\right)\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})\mathbf{a} \\ - A(\hat{\mathbf{q}})\left[\mathbf{d}\boldsymbol{\alpha}\times\right]A^T(\hat{\mathbf{q}})f(\hat{\mathbf{p}},\hat{\mathbf{v}}) - A(\hat{\mathbf{q}})\left[\mathbf{d}\boldsymbol{\alpha}\times\right]A^T(\hat{\mathbf{q}})g(\hat{\mathbf{p}}) - A(\hat{\mathbf{q}})\left[\mathbf{d}\boldsymbol{\alpha}\times\right]\mathbf{a}\end{aligned} \tag{121}$$

Using matrix identities this can be simplified to

$$\begin{aligned}\boldsymbol{\delta} = f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + (M+O)\,\mathbf{dp} \\ + \left(\left[f(\hat{\mathbf{p}},\hat{\mathbf{v}})\times\right]A(\hat{\mathbf{q}}) + \left[g(\hat{\mathbf{p}})\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\right)\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})\mathbf{a}\end{aligned} \tag{122}$$

Combining Eqs. (103), (105), (110), and (122) in Eq.(100) gives

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} \approx \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{dv} + \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\hat{\mathbf{v}} + \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\omega}^B_{B/N} \\ - \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{dv} - \left[A(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\hat{\mathbf{v}} + A(\hat{\mathbf{q}})\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\mathbf{d}\boldsymbol{\alpha} + f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) \\ + (M+O)\,\mathbf{dp} + \left(\left[f(\hat{\mathbf{p}},\hat{\mathbf{v}})\times\right]A(\hat{\mathbf{q}}) + \left[g(\hat{\mathbf{p}})\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\right)\mathbf{d}\boldsymbol{\alpha} + N\mathbf{dv} + A(\hat{\mathbf{q}})\mathbf{a} - \dot{\hat{\mathbf{v}}}\end{aligned} \tag{123}$$

Here $\dot{\hat{\mathbf{v}}}$ is defined by

$$\dot{\hat{\mathbf{v}}} = f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}}) + A(\hat{\mathbf{q}})\hat{\mathbf{a}} \tag{124}$$

Equation (123) can be simplified to

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} = \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\mathbf{d}\boldsymbol{\omega}^B_{B/N} + (M+O)\,\mathbf{dp} + N\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} \\ + \left(\left[f(\hat{\mathbf{p}},\hat{\mathbf{v}})\times\right]A(\hat{\mathbf{q}}) + \left[g(\hat{\mathbf{p}})\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}}) + A(\hat{\mathbf{q}})\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\right)\mathbf{d}\boldsymbol{\alpha}\end{aligned} \tag{125}$$

where $\Delta\mathbf{a} = \mathbf{a} - \hat{\mathbf{a}}$. Using Eq. (93) gives

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} = \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\left(\left[\mathbf{d}\boldsymbol{\alpha}\times\right]\boldsymbol{\omega}^B_{B/I} - A^T(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^N_{N/I} + \Delta\boldsymbol{\omega}^B_{B/I}\right) + (M+O)\,\mathbf{dp} + N\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} \\ + \left(\left[f(\hat{\mathbf{p}},\hat{\mathbf{v}})\times\right]A(\hat{\mathbf{q}}) + \left[g(\hat{\mathbf{p}})\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}}) + A(\hat{\mathbf{q}})\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right]\right)\mathbf{d}\boldsymbol{\alpha}\end{aligned} \tag{126}$$

Multiplying and gathering like terms gives

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} \approx \left(\left[\left(f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}})\right)\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}}) + A(\hat{\mathbf{q}})\left[A^T(\hat{\mathbf{q}})\hat{\mathbf{v}}\times\right]\left[\hat{\boldsymbol{\omega}}^B_{B/N}\times\right] - \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\left[\boldsymbol{\omega}^B_{B/I}\times\right]\right)\mathbf{d}\boldsymbol{\alpha} \\ + (M+O)\,\mathbf{dp} + N\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} - \left[\hat{\mathbf{v}}\times\right]\Delta\boldsymbol{\omega}^N_{N/I} + \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I}\end{aligned} \tag{127}$$

Rearranging the cross product identity $M\left[\boldsymbol{v}\times\right]M^T = \left[M\boldsymbol{v}\times\right]$ into the form $\left[\boldsymbol{v}\times\right]M^T = M^T\left[M\boldsymbol{v}\times\right]$ allows the last two terms of $\mathbf{d}\boldsymbol{\alpha}$ to be cancelled leaving

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} = \left(\left[\left(f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}})\right)\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\right)\mathbf{d}\boldsymbol{\alpha} \\ + (M+O)\,\mathbf{dp} + N\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} - \left[\hat{\mathbf{v}}\times\right]\Delta\boldsymbol{\omega}^N_{N/I} + \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I}\end{aligned} \tag{128}$$

Solving Eq. (124) for $f(\hat{\mathbf{p}},\hat{\mathbf{v}}) + g(\hat{\mathbf{p}})$ and using in Eq. (128) yields

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} = \left(\left[\left(\dot{\hat{\mathbf{v}}} - A(\hat{\mathbf{q}})\hat{\mathbf{a}}\right)\times\right]A(\hat{\mathbf{q}}) - N\left[\hat{\mathbf{v}}^N\times\right]A(\hat{\mathbf{q}})\right)\mathbf{d}\boldsymbol{\alpha} \\ + (M+O)\,\mathbf{dp} + N\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} - \left[\hat{\mathbf{v}}\times\right]\Delta\boldsymbol{\omega}^N_{N/I} + \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I}\end{aligned} \tag{129}$$

The attitude matrices in $\mathbf{d}\boldsymbol{\alpha}$ can be moved outside of the parenthesis so that the attitude error metric's components are written in the navigation frame, similar to those in Eq. (95). Rearranging the values in the parenthesis gives

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} = \left(\left[\dot{\hat{\mathbf{v}}}\times\right] - N\left[\hat{\mathbf{v}}^N\times\right] - \left[A(\hat{\mathbf{q}})\hat{\mathbf{a}}\times\right]\right)\mathbf{d}\boldsymbol{\alpha} \\ + (M+O)\,\mathbf{dp} + N\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} - \left[\hat{\mathbf{v}}\times\right]\Delta\boldsymbol{\omega}^N_{N/I} + \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}^B_{B/I}\end{aligned} \tag{130}$$

Expanding $\Delta\boldsymbol{\omega}_{N/I}^N$ with a first-order Taylor series expansion and gathering like terms yields

$$\begin{aligned}\mathbf{d}\dot{\mathbf{v}} = &\left(\left[\dot{\hat{\mathbf{v}}}\times\right] - N\left[\hat{\mathbf{v}}^N\times\right] - \left[A(\hat{\mathbf{q}})\hat{\mathbf{a}}\times\right]\right)\mathbf{d}\boldsymbol{\alpha} \\ &+ \left(M + O - \left[\hat{\mathbf{v}}\times\right]C\right)\mathbf{dp} + \left(N - \left[\hat{\mathbf{v}}\times\right]D\right)\mathbf{dv} + A(\hat{\mathbf{q}})\Delta\mathbf{a} + \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}})\Delta\boldsymbol{\omega}_{B/I}^B\end{aligned} \tag{131}$$

where C and D were previously defined in the derivation of $\mathbf{d}\dot{\boldsymbol{\alpha}}$.

As previously stated the positioning error is not effected by the new error metric since this is still a kinematic relationship with the velocity. However, for this error analysis it is important to note that the derivative of the position here is not the velocity $\mathbf{v}^N$, since the position is in geodetic coordinates. The derivative of the position is given by Eqs. (68b)–(68d); considering a perturbation of these equations yields

$$\mathbf{d}\dot{\mathbf{p}} = \frac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{p}^N}\mathbf{dp} + \frac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{v}^N}\mathbf{dv} \tag{132}$$

where $\mathbf{dp} = [d\phi\ \ d\lambda\ \ dh]^T$, $\mathbf{dv}^N = [dv_N\ \ dv_E\ \ dv_D]^T$ and $\frac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{p}^N}$ and $\frac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{v}^N}$ are given in Eq. (81).

Combining the error equations into one set of linear first-order differential equations gives

$$\mathbf{d}\dot{\mathbf{x}} = F\mathbf{dx} + G\mathbf{u} \tag{133}$$

where the state error vector is given by $\mathbf{dx} = [\mathbf{d}\boldsymbol{\alpha}^T\ \ \mathbf{dp}^T\ \ \mathbf{dv}^T]^T$, the forcing input vector is given by $\mathbf{u} = [\Delta\boldsymbol{\omega}_{B/I}^{B\ T}\ \ \Delta\mathbf{a}^T\ \ \Delta\mathbf{g}^T]^T$, with

$$F = \begin{bmatrix} -\left[\left(A(\mathbf{q})\boldsymbol{\omega}_{B/N}^B + \boldsymbol{\omega}_{N/I}^N\right)\times\right] & -C & -D \\ 0 & \dfrac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{p}^N} & \dfrac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{v}^N} \\ \left(\left[\dot{\hat{\mathbf{v}}}\times\right] - N\left[\hat{\mathbf{v}}^N\times\right] - \left[A(\hat{\mathbf{q}})\hat{\mathbf{a}}\times\right]\right) & \left(M + O - \left[\hat{\mathbf{v}}\times\right]C\right) & \left(N - \left[\hat{\mathbf{v}}\times\right]D\right) \end{bmatrix} \tag{134}$$

and

$$G = \begin{bmatrix} A(\hat{\mathbf{q}}) & 0 & 0 \\ 0 & 0 & 0 \\ \left[\hat{\mathbf{v}}\times\right]A(\hat{\mathbf{q}}) & A(\hat{\mathbf{q}}) & 0 \end{bmatrix} \tag{135}$$

For a stationary error analysis it will be assumed that the INS is at rest and therefore the F matrix reduces down to

$$F = \begin{bmatrix} -\left[\boldsymbol{\omega}_{N/I}^N\times\right] & -C & -D \\ 0 & \dfrac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{p}^N} & \dfrac{\partial\dot{\mathbf{p}}^N}{\partial\mathbf{v}^N} \\ \left(-\left[A(\hat{\mathbf{q}})\hat{\mathbf{a}}\times\right]\right) & (M+O) & N \end{bmatrix} \tag{136}$$

Using Eqs. (79), (81), (82), $\boldsymbol{\omega}_{N/I}^N = [\omega_N\ \ \omega_E\ \ \omega_D]^T$ and $A(\hat{\mathbf{q}})\hat{\mathbf{a}} = [a_N\ \ a_E\ \ a_D]^T$the F matrix can be expanded so that the individual components can be examined.

$$F = \begin{bmatrix} 0 & \omega_D & -\omega_E & \omega_e \sin\phi & 0 & 0 & 0 & -\dfrac{1}{R_\lambda + h} & 0 \\ -\omega_D & 0 & \omega_N & 0 & 0 & 0 & \dfrac{1}{R_\phi + h} & 0 & 0 \\ \omega_E & -\omega_N & 0 & \omega_e \cos\phi & 0 & 0 & 0 & \dfrac{\tan\phi}{R_\lambda + h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{R_\phi + h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\sec\phi}{R_\lambda + h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & a_D & -a_E & 0 & 0 & 0 & 0 & -2\omega_e \sin\phi & 0 \\ -a_D & 0 & a_N & 0 & 0 & 0 & 2\omega_e \sin\phi & 0 & 2\omega_e \cos\phi \\ a_E & -a_N & 0 & 0 & 0 & -\dfrac{2g}{R} & 0 & -2\omega_e \cos\phi & 0 \end{bmatrix} \quad (137)$$

where $-\frac{2g}{R}$ is an approximation of $\frac{\partial g}{\partial h}$. Examining Eq. (85b) it can be seen that -3.0877×10^{-6} is the dominating term which can be approximated by$-\frac{2\gamma_a}{a}$, the equatorial gravity term and the semimajor axis. This gravity compensation is unstable, therefore, the vertical dynamics will be removed by removing the sixth and ninth rows and columns. Using $a^N = [0 \ 0 \ g]^T$, $\omega_N = \omega_e \cos\phi$, $\omega_D = \omega_e \sin\phi$ and $\omega_E = 0$ and assuming $R_\lambda \approx R_\phi \approx R_e$ and that the vehicle is located at a known altitude near the sea level yields

$$F = \begin{bmatrix} 0 & \omega_e \sin\phi & 0 & \omega_e \sin\phi & 0 & 0 & -\dfrac{1}{R_e} \\ -\omega_e \sin\phi & 0 & \omega_e \cos\phi & 0 & 0 & \dfrac{1}{R_e} & 0 \\ 0 & -\omega_e \cos\phi & 0 & \omega_e \cos\phi & 0 & 0 & \dfrac{\tan\phi}{R_e} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{R_e} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\sec\phi}{R_e} \\ 0 & g & 0 & 0 & 0 & 0 & -2\omega_e \sin\phi \\ -g & 0 & 0 & 0 & 0 & 2\omega_e \sin\phi & 0 \end{bmatrix} \quad (138)$$

The eigenvalues corresponding to this system are either zero or pure imaginary. For the imaginary eigenvalues, the oscillations fluctuate at frequencies ω_e and $\omega_S \pm \omega_e \sin\phi$, where ω_e is the Earth's rotation rate and ω_S is the Schuler frequency given by

$$\omega_S = \sqrt{\frac{g}{R}} \quad (139)$$

The Foucault frequency modulates the amplitude of these eigenvalues, this is given by

$$\omega_f = \omega_e \sin\phi \tag{140}$$

For a vehicle at 45° latitude the Schuler frequency will 84.4 minutes and the Foucault frequency is 33.9 hours. This corresponds to the same Schuler frequency and Foucault frequency for the standard INS error metric. This is because the additional coupling terms that were seen in the equations for $\mathbf{d}\dot{\boldsymbol{\alpha}}$ and $\mathbf{d}\dot{\mathbf{v}}$ are canceled out by the zero velocity for the stationary analysis. It is important to note that the stationary analysis yields the same fundamental frequencies as the standard INS filter frequencies. But, as previously mentioned the time-varying linearized matrices are different.

VIII. Conclusion

A new error-representation proposed for Inertial Navigation Systems that utilizes a frame consistent error metric for both body-frame and reference-frame errors was derived. It has been presented how the body-frame and reference-frame errors have a direct relationship and, therefore, only one needs to be utilized. A geometric-based Kalman Filter was presented whose previous simulation results showed stronger convergence potentially from more coupling terms. This was confirmed by a linear analysis, where additional terms coupling the attitude and velocity for no stationary analysis. The linear analysis also demonstrated how the new error metric simplifies to the standard INS solution for a stationary analysis, which the same Schuler and Foucault frequencies.

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