A New Stochastic Control Paradigm Employing Large Deviations Theory

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This paper presents a paradigm shifting approach to the control of dynamic systems under uncertainty governed by stochastic differential equations (SDEs). Large Deviations (LD) techniques are employed to arrive at a control law for a broad class of nonlinear systems minimizing path deviations. Thereby, a shift from point-in-time to sample path statistics is suggested. A suitable formal control framework which leverages embedded Freidlin-Wentzell theory is proposed and described in detail. This includes the precise definition of the control objective and comprises an accurate discussion of the adaptation of the Freidlin-Wentzell theorem. The new control design is enabled by the transformation of an ill-posed control objective into a well-conditioned sequential optimization problem. For the first time, this allows for an LD based stochastic control design applicable to a comprehensive class of nonlinear systems. This work includes a short numerical evaluation using two benchmark problems. The proposed control paradigm allows for addressing the stochastic cost control problem as a special case. The numerical examples furnish proof of the successful design.

I. Introduction

Since the days of the Apollo Guidance Computer, the task of integrating state information from different sources into a controller under uncertainty has evolved into the centerpiece of interest in a tremendous variety of fields. The Kalman-Bucy Filter and its subsequent derivatives have triumphantly conquered the world of engineering since their inception in 1960/61 (see [1] and [2]). Almost every consumer and industrial application associated with model-based estimation for systems governed by stochastic differential equations (SDEs), from cell phones to satellites, relies on some version of an integrated Kalman Filter. For nonlinear as well as large-scale distributed systems, the extended and unscented Kalman filter, particle filter, and Gaussian sum approaches are just a few examples of the rich development of optimal estimation methods.

Historically, the focus with respect to dynamic systems under uncertainty has mainly revolved around the observer problem instead of the stochastic control challenge. The reason for this might root in the certainty equivalence principle: Although in general not valid for nonlinear systems, the application of the certainty equivalence principle to specific nonlinear systems has been proven to be successful for many cases. Nevertheless, a true applicable extension of optimal control methods to stochastic systems is desired. Stochastic optimal control techniques - in a classic sense - minimize the mean or expectation of a performance index. Yet, a simple example illustrates the shortcomings of this approach and the necessity of an extension: Assume a financial setting in which a portfolio strategy is sought which maximizes the average rate of return or profit while the underlying stock or option pricing is determined by SDEs. Of what use would such a strategy be if, for example, the associated probability of bankruptcy were 0.8? The resulting policy would simply be meaningless. Although not of engineering nature, the analogy of this example to tracking problems, is evident.

Several approaches have been suggested to incorporate higher order statistics, for example the k^{th} moment or cumulant, into the cost function. Yet, outcomes have been limited to linear systems and quadratic cost functions. Here, we establish a new control paradigm embedding Large Deviations (LD) techniques. The resulting shift from point in time probability laws to sample function statistics represents a significant change in the prevailing doctrine for stochastic control problems. This shift allows to address a large general class of nonlinear systems while not suffering from the curse of dimensionality and while providing a physically more meaningful interpretation.

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II. Background: Deterministic and Stochastic Optimal Control

The theory of (deterministic) optimal control is a well-refined and highly successful domain in control theory ([3], [4] and [5]). The unifying feature of all optimal control tasks is their objective: minimization of some cost function or performance measure. Besides a few exceptions (e.g. minimum time control), optimal control problems can be cast in a standard form: Find an admissible control $\boldsymbol{u}^*(t)$ that results in an admissible trajectory $\boldsymbol{x}^*(t)$ minimizing the performance measure (index):

$$J(\boldsymbol{u}) = \theta\big(\boldsymbol{x}(T), T\big) + \int_0^T \phi\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big)\ dt \tag{1}$$
$$\text{subject to} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{b}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\ .$$

There are three main approaches to solve these kinds of problems: calculus of variations, the Hamilton-Jacobi-Bellman equation or dynamic programming (for discrete systems).* The solution via calculus of variations is based on Pontryagin's minimum principle (see [6] and [3]) and leads, in general, to an open-loop or feedforward control while the Hamilton-Jacobi-Bellman equation allows for a feedback solution. The performance criterion in Eq. (1) is of the general $\mathcal{H}_2$-form, but several variations exist: For instance, the final time T can be either fixed (finite horizon), infinite, or indefinite.† A large selection of different optimal control tasks and associated cost functions can be found in [8].

If the dynamic system is subject to uncertainty and hence becomes an Itō diffusion (see definition 4 in the appendix), the primary performance index of Eq. (1), now being dependent on a large number of random variables,‡ turns into a random variable itself. Equation (1) being a random variable renders the task of 'minimizing' meaningless since further information is required. In general, information on the primary performance measure§ is incomplete such that a secondary formulation becomes necessary. In classic stochastic optimal control, the minimization of the expectation of the random process generated by Eq. (1) replaces the standard objective function and is in many texts regarded as 'the' stochastic optimal control problem:

$$\bar{J}(\boldsymbol{u}) = \mathcal{E}\left\{\theta\big(\boldsymbol{x}_\epsilon(T), T\big)\Big) + \int_{t_0}^T \phi\big(\boldsymbol{x}_\epsilon(t), \boldsymbol{u}(t), t\big)\ dt\right\} \tag{2}$$
$$\text{subject to} \quad d\boldsymbol{x}_\varepsilon(t) = \boldsymbol{b}\left(\boldsymbol{x}_\varepsilon(t), \boldsymbol{u}(t), t\right)\ dt\ +\ \sqrt{\varepsilon}\ \boldsymbol{G}(\boldsymbol{x}_\varepsilon(t))\ d\boldsymbol{\mathcal{B}}_t$$
$$\boldsymbol{x}(t_0) = x_0$$

where $\boldsymbol{\mathcal{B}}_t$ denotes Brownian motion of unit variance. Surprising difficulties and shortcomings arise from this seemingly simple extension. One fundamental difference between deterministic and stochastic control is the fact that open- and closed-loop are equivalent in the deterministic sense but not in the stochastic case.

Although the problem of stochastic optimal control has been addressed since the late 1960s,¶ mainstream control engineering often regards it to be on the fringes. One aspect particularly appears to induce the missing spread of statistical methods: Uncertainty in a dynamic system has repeatedly been approached from a state availability or estimation point of view, i.e. the uncertainty of the system is lumped into the quest of detecting the true state. According to the certainty-equivalence principle or separation principle,‖ a linear dynamic system subject to white noise in both states and measurements will be *optimally* controlled with respect to the criterion of Eq. (2) if an LQR (linear quadratic regulator) minimizing the deterministic performance measure is based on a separately designed Kalman filter's estimates. This combined Linear Quadratic Gaussian control (LQG) represents the most widespread optimal control technique for systems under uncertainty.**

Even though the certainty equivalence principle is not established for nonlinear systems in general, the extension of LQG control to those system (via repeated or continuous linearization) has been proven highly successful in application

*It is assumed without loss of generalization that the initial condition and initial time are specified. It could be argued that there are only two main approaches de facto as the Hamilton-Jacobi-Bellman equation along with the dynamic programming principle both result from Bellman's principle of optimality.

†For details, see [7], chapter 6.

‡In continuous time, these are actually infinitely many.

§The probability density function conditioned on $\boldsymbol{u}(t)$.

¶For example, consider the treatment in Åström's early standard: [9].

‖Although often used interchangeably, there is a subtle difference between the certainty-equivalence principle and the separation principle. For a discussion see [10].

**Note the complexity of the term *optimality*: The Kalman Filter in LQG is a minimum variance estimator (second central moment) and the LQR an expectation minimizer (first moment). Hence, 'optimality' refers to two different statistical orders. From a stochastic point of view, a certainty equivalence principle in terms of 'optimality' without further elaboration can potentially result in rather confusing settings.

for over 50 years, especially when combined with robust control synthesis. This might be one of the contributing factors why comprehensive research on pure stochastic control and its application has only infrequently surfaced during the last 40 years. Another reason might stem from the fact that stochastic control was historically met with a strong competition by the development of robust control from the 1970s to 1990s. While robust techniques, such as H_∞ control, specifically address disturbances in the system, it is not motivated by statistical reasoning. In particular, its ability to cope much better with model errors[††] than early LQG design caused the focus of the control community to move away from statistical control.

However, there are still serious deficiencies associated with classic stochastic control in general. The most evident inherent flaw lies in the fact that only the expectation of a performance criterion is minimized. The introductory investment example has already illustrated the meaninglessness of certain strategies if only the mean is considered. This insufficiency was realized as early as in the 1960s, and different solution methods were promoted subsequently. Some authors assign the label *statistical control* to those extensions of the minimum mean optimal control. Even though the term stochastic control suggests that it encompasses all control techniques under uncertainty, its historic use in literature almost exclusively addresses the *mean* cost problem.

III. Previous Work

Statistical optimal control can be divided into four major groups which share certain interconnections: stochastic optimal control devoted to the *mean* of the performance measure;[‡‡] minimum cost variance control (MCV) minimizing the *variance* of the performance index with its expectation constrained to a prescribed level; risk-sensitive control (RS) minimizing the *tail* of the state's probability distribution; cost-cumulant control (kCC) minimizing either a specific k^{th} *cumulant* or a linear combination of a finite number of cumulants. Note that classic stochastic control corresponds to first cumulant control while MCV coincides with second cumulant control. A short survey and reference list on statistical control can be found in [11]. As the mean and the variance correspond to the first and second cumulant, respectively, classic stochastic control and MCV simply could be integrated in kCC with respect to nomenclature. However, the development of these methods differs in both its history of advancement and its employed techniques. Unfortunately, several publications use 'cumulant control' in their titles, but solely treat the MCV case, leading to confusion in surveys like [11]. Therefore, two distinct short discussions are provided in this study. For a more detailed account, it is referred to [10].

A. Minimum Variance Control

Some early research on minimal (output) variance control strategies was performed by Åström during the 1960s and is summarized in [9]. However, his transfer function based ideas on minimizing the output variance for discrete linear systems with memory have not been met with widespread reception and hardly any further development appeared. Sain's initial comprehensive research in stochastic control problems in [12] and [13] centered around minimal cost variance control of linear time-invariant dynamic systems with quadratic costs. The mean of the performance index has been constrained to a constant value, $\mathcal{E}\{J(u)\} = D$ which is lower bound by the infimal cost of the associated mean performance problem (optimal stochastic control problem). Thus, a set of admissible controls, $\mathcal{U}$, with an associated space of admissible trajectories, $\mathcal{X}$, is defined. Then, the variance of $J(u)$ is minimized over the sets $\mathcal{U}$ and $\mathcal{X}$. In [14], Souza and Sain consider the same cost variance problem for the estimation problem while Cosenza analyzes MCV for the discrete case in [15]. Sain and Liberty further advance the MCV approach of [13] in [16] to time-varying linear systems providing an open-loop optimal feedback solution. The control strategy can now be expressed via the solution of a two-point boundary value problem of $4n$ coupled equations. Relationships between RS control and MCV are explored by Sain and Won in [17] and by Won in [18]. Won solves the full-state feedback MCV problem for linear time-varying systems with quadratic costs in [17] for a finite horizon. Sain et al. present in [19] a detailed and thorough mathematical theory on cost mean and variance control with connections to risk-sensitive control, summarizing previous research. The closed-loop state feedback solution is extended to an infinite horizon cost function by Won in [20]. Linear MCV theory is completed as a subset of Pham et al.'s solution of the kCC problem including noisy output feedback.

††This includes deterministic disturbances.

‡‡Linear quadratic Gaussian control (LQG) forms the most most basic consideration of stochastic optimal control.

B. k-th Cumulant Control

The advancement of k^{th} cost cumulant control is closely related in both time period and authorship. In kCC, the MCV technique is stretched beyond the variance by minimizing a weighted sum of higher order moments: Liberty and Hartwig extend the analysis of performance index statistics for quadratic cost functions and linear systems by developing a characteristic function generating cost cumulants in the time domain in [21] and [22] and provide evolution equations as well as probability densities for the k^{th} cost cumulant. This is the starting point of k^{th} cumulant control. Pham et al. solve the kCC problem for a closed-loop state feedback on a finite horizon in [23]. Subsequently, Pham et al. are able to drop the state availability requirement in [23] as Pham solves the k^{th} cumulant control problem for linear systems in finite and infinite time-horizon settings for both state-feedback and output feedback control. Results are further published in [24] and [25] marking the completion of the linear kCC. The control performance is tested via a structural (base-isolated building) benchmark problem in [25]. Despite more than 40 years of effort, research outcomes for higher-order statistical control are still limited to linear dynamic systems with quadratic costs. Pham supplies by far the largest body of work on cumulant cost control beyond the variance. His dissertation ([26]) is the most rigorous and comprehensive account of the entities of MCV and kCC. Furthermore, he has successfully applied kCC to a variety of structural benchmark problems in [23], [27], [24] and [25].

C. Risk-Sensitive Stochastic Optimal Control

Risk-sensitive (RS) stochastic optimal control, sometimes also called linear exponential quadratic control, is an extension of the classic stochastic optimal control problem, and was developed mostly in between the 1970s and the 1990s. The fundamental idea consists of penalizing large total cost variations (of the quadratic criterion) disproportionately high. Therefore, instead of minimizing the average of the quadratic cost, the average of the exponential of the quadratic cost is minimized:

$$J_\mu(\boldsymbol{u}) = \mu\,\mathcal{E}\left\{\exp\left(\mu\left(\frac{1}{2}\int_0^T \|\boldsymbol{x}(\tau)\|_Q^2 + \|\boldsymbol{u}(\tau)\|_R^2\,d\tau\right)\right)\right\}\ . \tag{3}$$

Here, the parameter μ allows adjustment for the desired risk-sensitivity: While $\mu > 0$ corresponds to risk-sensitive behavior, $\mu < 0$ yields a risk-seeking strategy. The limit for $\mu \to 0$ recovers the standard (risk-neutral) LQG control. The risk-sensitive criterion is introduced by Jacobson in [28]. In his original work, he addresses both a negative and positive design parameter μ for the completely observed case and for discrete as well as continuous systems. The risk-sensitive method is further extended by Speyer et al. in [29] to the partially observed case for discrete time systems including the continuous time terminal cost situation. Speyer considers risk-sensitive problems under measurement noise in [30]. Kumar and van Schuppen propose the general solution of the partially observed linear exponential quadratic control problem for continuous time systems in [31] without considering any plant or process noise.

Whittle is able to arrive at the general solution of the partially observable risk-sensitive optimal control problem for discrete systems in [32] by replacing the exponential-of-integral (EOI) statement with the logarithm-exponential-of-integral problem (LEOI):

$$J_\mu(\boldsymbol{u}) = -\frac{2}{\mu}\ \ln\left(\mathcal{E}\left\{\exp\left(-\frac{\mu}{2}J(\boldsymbol{u})\right)\right\}\right)$$
$$\text{with}\quad J(\boldsymbol{u}) = \boldsymbol{x}_N^T H \boldsymbol{x}_N + \sum_{i=1}^{N-1}(\boldsymbol{x}_i^T Q \boldsymbol{x}_i + \boldsymbol{u}_i^T R \boldsymbol{u}_i)\ .$$

He also establishes that the LEOI problem corresponds to the minimization of a linear combinations of all the cumulants associated with the stochastic quadratic cost,[§§] and introduces a new LEQG certainty-equivalence principle. The same solution is obtained by Bensoussan and van Schuppen a few years later for continuous systems using the EOI approach in [33]. The first culmination of RS optimal control is reached in the early 1990s: Whittle publishes his book [34] presenting the RS maximum principle; Bensoussan's first edition of [35] contains all solutions to EOI control up to this point, including partial observations. In [36], Whittle connects LD techniques with risk-sensitivity by employing asymptotic probabilities to establish an RS maximum principle for the case of partially observed states. An application analysis for RS control is found in [37] when this technique is applied by Won et al. to a structural problem under seismic disturbances. A summary of the development of RS control theory together with a discussion about relations between RS, MCV, and k^{th} cumulant control is provided by Won et al. in [11].

[§§] Indeed, expanding the logarithm and exponential function into a power series each results after some tedious algebraic manipulations in the following series: $J_\mu = \kappa_1 + \frac{\mu}{4}\kappa_2 + \frac{\mu^2}{24}\kappa_3 + \frac{\mu^3}{192}\kappa_4 + \ldots$.

IV. The New Idea

Many control designs have been developed over the last 50 years, but applicable techniques based on statistical control still seem to be of limited availability in comparison to other control techniques. As previously discussed, the most significant weakness of mean cost designs originates from the fact that there are many situations in which the expectation of the performance matter does not admit a meaningful control problem. In terms of engineering applications, there exists an ample variety of potential concerns or constraints like operational limits, bifurcative characteristics, fatigue, tracking limits, disproportionately high costs for large deviations, to name a few. MCV control and its successor kCC are mostly limited to linear systems and quadratic costs. Unfortunately, the theoretical advancement of statistical control has followed a significantly different course than the development of estimation theory: While advanced estimation methods kept their low order (moment) character, i.e. they remained based on the first two (central) moments, significant efforts went into the improvement of the propagation through nonlinear systems. Statistical optimal control, on the other hand, has almost exclusively focused on linear systems (and quadratic costs) and has sought improvement through incorporation of either higher order moments or cumulants, for that matter. Yet, it remains unclear how the design parameters, i.e. the assigned weights to the different cumulants in the minimization process, should be determined.

A. New Paradigm

This work claims that both current statistical control and optimal estimation techniques are caught in the 'point-in-time trap': Stochastic dynamic systems are consistently interpreted from an ensemble statistics aspect. The mean or the variance at a certain point in time t is considered and − where appropriate − minimized. This is highly intuitive for the first two moments and for linear systems, but quickly loses its significance. It is really not of interest to consider the probability of deviation over all ensemble functions at a certain point of time, the central question should rather address the probability that a particular sample path, i.e. a particular realization, deviates from its mean during a certain time interval. The 'point-in-time trap' becomes even more evident from a computational point of view: all nonlinear current statistical optimal control problems (similar to estimation techniques) suffer from the curse of dimensionality as the probabilistic expressions require the evaluation of multidimensional integrals over the entire (high-dimensional) state-space.[¶¶] The required numerical approximation technique always has to rely on a suitably selected grid whose discretization complexity will grow exponentially with the dimension of the state space.

The origin of this trap, i.e. the the standard angle of stochastic analysis, might intrinsically be connected to the definition of a stochastic process: In its basic form a stochastic process is defined as a family of indexed random variables with index set T, i.e. $\{X(\omega)_t\}_{t\in T}$. This gives rise to the intuitive interpretation that a real-valued stochastic process is created by executing a random experiment on the sample space Ω at every time t. Thereby, the sample space Ω can simply be a finite set of numbers or the continuum.[***] The outcome of this repeated random experiment then determines the value of the stochastic process at time t.

However, another possible interpretation executes a single random experiment on the sample space Ω, now containing all possible path realizations, at time $t = 0$. Hence, the outcome of the random experiment picks a certain path, and the value of the stochastic process at time t corresponds to the value of the particular sample path. It is a pure academic question to decide which interpretation represents reality better. Yet, all observed real measurements allow only for the computation of time averages or statistics, respectively, rather than ensemble ones. The property of ergodicity then allows for interchanging the time average with the ensemble average. It is the purpose of this work to approach SDEs from a different angle in comparison to standard approaches, i.e. ergodicity is not utilized as sample path statistics are employed. It is claimed that this is a more consistent analysis when the probability of a sample path's deviation from its nominal (unperturbed) solution is concerned as ergodicity addresses averages and not probabilities directly. It is the goal to present a sophisticated controller which in addition exhibits a clear structure and simplicity in design such that it is appropriate for engineering applications. The new control paradigm is free from the inherent weaknesses associated with current methods for nonlinear and non-Gaussian systems which arise from the fact that the number of parameters necessary to fully describe the system is infinite.[†††] In addition, any method based on a finite number of parameters requires not only additional knowledge of the system, but still suffers from the curse of dimensionality.

The foundation of the presented new approach emerges from LD theory and specifically is built upon the concept of sample path large deviations: Schilder's theorem and the Freidlin-Wentzell theory provide the necessary building blocks. As this study constitutes the first treatment of a new control paradigm, the main attention is on the consistent

[¶¶]This might be via the Chapman-Kolmogorov, Fokker-Planck, Kushner, or Zakai equation depending on the quantity of interest.

[***]For engineering applications, it is assumed that real-valued random variables or random processes are of concern.

[†††]This prevails in any form, be it either the (continuous) characteristic equation, the probability distribution, the probability density function (if existent) or the moments (if existent).

advancement of results for different dynamic systems under uncertainty. Therefore, full state knowledge is assumed while the associated partial observations or estimation problem is left for later research.

B. Motivating Example

The path dependence of the exit probability for a simple nonlinear system under open-loop control is demonstrated via a Monte-Carlo simulation.‡‡‡ Assume that an Itō diffusion is generated by the simple nonlinear stochastic differential equation

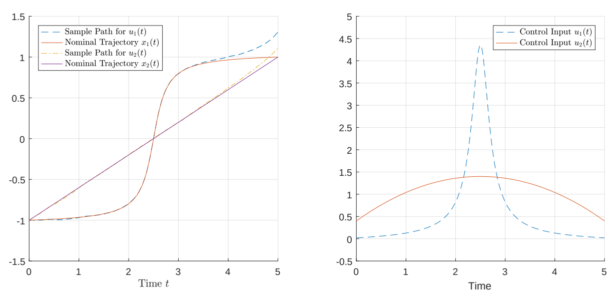
$$dx_\varepsilon(t) = \left[x^2(t) - 1 + u(t)\right] dt + \sqrt{\varepsilon}\, d\mathcal{B}_t \tag{4}$$

with $\mathcal{B}_t$ being Brownian motion of unit variance. Four different control inputs are considered, i.e.

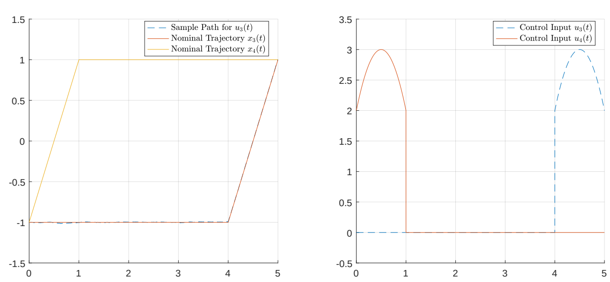
$$u_1(t) = \frac{5}{\alpha}\frac{1}{1+(5t-12.5)^2} - \left(\frac{1}{\alpha}\tan^{-1}(5t-12.5)\right)^2 + 1 \qquad u_3(t) = \begin{cases} 0 & \text{for } 0 \le t < 4 \\ 2-(2t-9)^2+1 & \text{for } 4 \le t \le 5 \end{cases}$$

$$u_2(t) = +\frac{2}{5} - \frac{4}{25}(t-2.5)^2 + 1 \qquad u_4(t) = \begin{cases} 2-(2t-1)^2+1 & \text{for } 0 \le t < 4 \\ 0 & \text{for } 4 \le t \le 5 \end{cases} \tag{5}$$

where the abbreviation $\alpha = \tan^{-1}(12.5)$ is used for convenience. These control inputs result in four different nominal trajectories on the time interval $[0, 5]$ transitioning from -1 to 1 as depicted in Figure 1. Here, *nominal trajectory* refers to the solution of the deterministic ODE emerging as the limit of Eq. (4) for $\varepsilon \to 0$. The only purpose of these four arbitrarily chosen control inputs is to demonstrate the extent to which the deviation probability depends on a particular nominal path. Certainly, they are not optimal in any stochastic or deterministic sense. In addition, a single sample trajectory for the perturbed Eq. (4) subject to control inputs $u_1(t)$, $u_2(t)$ and $u_3(t)$ is shown in Figure 1. Most sample paths for $u_4(t)$, however, are unstable and omitted. Now, a Monte-Carlo simulation of Eq. (4) is performed for $\sqrt{\varepsilon} = 0.01$



(a) Trajectories for Control Strategies $u_1(t)$ and $u_2(t)$



(b) Trajectories for Control Strategies $u_3(t)$ and $u_4(t)$

Fig. 1 Exemplary Control Strategies for Eq. (4) **for** $x(0) = -1$ **and** $x(5) = 1$

based on an Euler-Maruyama method. Here, the time interval $[0, 5]$ is divided by 10^4 sample points for the numerical

‡‡‡A simple control problem for a time-invariant linear scalar discrete system, subject to a random Gaussian excitation, is discussed as another motivating example in [10]. There, it is shown that the deviation probability remains independent from the control input for a linear open-loop system. However, a different situation emerges if a nonlinear system is affected.

integration and 10^5 sample paths are thus created. The quantity of interest is the empirical probability that a sample path $x_\varepsilon(t)$ deviates more than a distance of K from its nominal solution $x(t)$, i.e.

$$\mathbb{P}_\varepsilon(S) = \mathbb{P}_\varepsilon\Big(|x_\varepsilon(t) - x(t)| \geq K \text{ for some } t \in [0,5]\Big) .$$

For the Monte-Carlo simulation, $K = 0.1$ is chosen and the resulting empirical probabilities are shown in Table 1. Although all nominal trajectories travel from -1 to 1 during the time interval $[0, 5]$, significantly different deviation probabilities arise depending on the specific path taken. In particular, Table 1 reveals an immense difference between the deviation probability of nominal trajectories 3 and 4. Thereby, trajectories 3 and 4 are merely point symmetric to each other with respect to the half point of the time interval.

Table 1 Empirical Deviation Probability for Eq. (4)**,** $\varepsilon = 10^{-4}$**,** $K = 0.1$

	Control Input			
	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u_4(t)$
$\mathbb{P}_\varepsilon(S)$	0.8873	0.5582	0	0.9981

It is the goal of the control paradigm developed in this chapter to choose the particular control $u(t)$ such that the resulting trajectory of the underlying nominal (unperturbed) ODE exhibits the lowest deviation probability. In addition, the new controller will allow to incorporate a large variety of additional constraints on the state as well as the control.

V. Large Deviations Theory

A. LD Overview

LD techniques are of central importance for the newly suggested control paradigm as they provide the theoretical gateway to address the probabilities of interest. For a more solidifying rigorous introduction of the needed concepts in order to enable an in-depth understanding, it is referred to the underlying original work in [10].

LD theory has its origin in the Scandinavian actuary industry where early results were used for the evaluation of risks. Its birthplace lies within the work of Harold Cramér ([38]), but it is still a very active area of research with applications especially in applied probability for mathematical finance (with emphasis on option pricing, risk estimation for large portfolio losses and stochastic volatility models). In general, probabilities are split in an LD rate and a sequence converging at a subexponential rate.

In contrast to other branches of mathematics, LD theory is far from being a general 'theory' as its name might suggest. It is a branch of probability theory which is rooted in many different areas, and it has grown into a vast, yet loose collection of differently motivated results during its more than 70-year history. In a simplified way, it is a generalization of convergence theorems such as the law of large numbers, ergodic theorems, or the Glivenko-Cantelli theorem: These theorems essentially state that random variables over large populations are approximately equal to their expectation, and deviations become increasingly unlikely, LD theory acknowledges the existence of untypical fluctuations and attempts to characterize them. Its name might be misleading in a second sense as LD theory contains both the law of large numbers and the central limit theorem. Although its origin in risk analysis was purely motivated by application, general LD theory has evolved into a highly sophisticated abstract mathematical construct based on measure theory where sets on the real line have been replaced by abstract topologies. In LD theory, it is not unusual if the same answers are reached by using different paths that seem completely unrelated.

Surprisingly, LD results for stochastic processes have not yet found their way into mainstream control engineering applications although their contributions in other branches of both science and engineering have proven to be indispensable. These are, in particular, problems related to signal processing, information entropy and simulation techniques of rare events. As many of the major developments of LD theory have been already foreseen by Boltzmann,[§§§] it is no wonder that physical applications compromise the largest body of LD application examples, such as equilibrium statistical mechanics, chaotic systems and multifractals, nonequilibrium systems and fluctuation relations, to name a few. By nature, rare event prediction and therefore asymptotic analysis has been applied extensively in context of natural

[§§§]Ellis emphasizes in an account of Boltzmann's contributions to statistical physics in [39] how deeply connected his early results are to modern LD theory.

disasters (hurricanes, avalanches, tsunamies, etc.). LD techniques are inherently tied with simulation methods such as importance sampling.

The adaptation in engineering applications might often be hindered by technical notation and the focus on an abstract formulation. In addition, there is a certain lack of textbook literature on LD theory in traditional engineering notation, preventing its appeal to a broader engineering community. The classic and comprehensive references for LD theory are the well-established books by Dembo and Zeitouni in [40] and by den Hollander in [41]. Both treat the subject in a mathematically substantial and rigorous way in topological settings. Thereby, they fulfill the role of a gold standard. For an engineering audience, however, an account of LD results utilizing some mathematical simplifications, by solely addressing random variables taking values on $\mathbb{R}$ or $\mathbb{R}^d$, might result in a more engaging statement. Such an approach has been presented in the lesser known book by Bucklew in [42]. Although Ellis criticized Bucklew's presentation for a lack of mathematical accuracy in parts,[¶¶¶] he still praised it for its unique attempt in demonstrating LD theory. Other classic references include the older survey by Varadhan in [44] and more recent short discussion with emphasis on application examples in [45]. In particular, Varadhan's notes in [44] provide a consolidated account for the mathematically minded, but readability as a first time access to LD theory is heavily limited due to their highly condensed character. Kallenberg's standard reference on probability in [46] supplies a concise chapter on LD theory, exhibiting important results. In addition, the notes of Ellis in [47] and [39] contribute a discussion centered on statistical mechanics. Touchette supplies an additional excellent series of notes with focus on statistical mechanics in [48] and in the form of a subject primer in [49] including simulation. For years, the introduction by Lewis and Russell in [50] has been a popular subject primer of LD theory. The LD treatment on stochastic differential equations is the core of this work and it is often traded under different names such as *Nonequilibrium Statistical Mechanics*, *Freidlin-Wentzell Theory* or simply *Sample Path Large Deviations*. The centerpiece of the newly suggested control method is the so-called Freidlin-Wentzell theorem which is the culmination of the first part in the development of Freidlin and Wentzell in [51]. In the following, the most important LD results necessary to arrive at the suggested controller design, i.e. the general LD principle and the Freidlin-Wentzell theorem, are summarized. For a more in-depth understanding, [10] provides rigorous introduction of the concepts tailored to an engineering audience.

B. Abstract Large Deviations Principle

LD theory is concerned with the asymptotic probabilities of empirical quantities with respect to their underlying statistics. Thereby, all individual results of LD theory (in the form of theorems) follow a consistent two-stage formal approach:

1. First, it is established that an infinite family of (probability) measures $\{\mathbb{P}_\eta\}_{\eta\in[0,\infty)}$ obeys a LD Principle on a topological space $\mathcal{X}$. As such, the limiting behavior of $\mathbb{P}_\eta$ is characterized as $\eta \to \infty$ by providing asymptotic upper and lower exponential bounds on the values which $\mathbb{P}_\eta$ assigns to measurable subsets of $\mathcal{X}$. These bounds are stated in terms of a rate function. In addition, all technical prerequisites and underlying conditions are stated.
2. The corresponding rate function is determined and an implicit or explicit expression defining its specific appearance is given together with a supplemental characterization.

Although the limiting behavior of a family of measures $\{\mathbb{P}_\eta\}$ is intuitively associated with $\eta \to \infty$, the abstract LD Principle as well as the Freidlin-Wentzell theory are concerned with the limiting behavior of measures $\mathbb{P}_\varepsilon$ for $\varepsilon \to 0$. In terms of LD theory, these two formulations are absolutely equivalent by simply setting $\eta = 1/\varepsilon$. It is now appropriate to introduce abstract and precise definitions for the two entities of interest, the LD Principle and the rate function.

Definition 1 (Large Deviations Principle) *Let $\mathbb{P}_\varepsilon$ be a family of probability measures on $(\mathcal{X},\mathcal{B})$ where $\mathcal{X}$ is a topological space so that open and closed sets are well-defined.*[17] *Let $\overline{S}$ denote the closure of any set $S \in \mathcal{X}$, S^o the interior and S^c the complement of S. Then, $\{\mathbb{P}_\varepsilon\}$ is said to satisfy a* Large Deviations Principle *with a rate function I if, for all $S \in \mathcal{B}$,*

$$-\inf_{x\in S^o} I(x) \leq \liminf_{\varepsilon\to 0} \varepsilon \ln\{\mathbb{P}_\varepsilon(S^o)\} \leq \limsup_{\varepsilon\to 0} \varepsilon \ln\{\mathbb{P}_\varepsilon(\bar{S})\} \leq -\inf_{x\in\bar{S}} I(x)$$

where the infimum over an empty set is interpreted as ∞. The suggestive notation that the family $\mathbb{P}_\varepsilon$ satisfies a Large Deviations Principle is given as

$$\mathbb{P}(S) \asymp e^{-\frac{1}{\varepsilon}I(S)} .$$

[¶¶¶]For details, see book review by Richard Ellis in [43].

The LD limit is the limit needed to retain the dominant exponential term and also called the probability estimate on the logarithmic scale. While specific applications, such as Cramér's theorem or the Freidlin-Wentzell theory, contribute rules to determine specific rate functions, the impression that a rate function can be calculated explicitly for many stochastic processes is false. Closed form expressions are only available for a few simple cases. However, all rate functions share a set of properties by definition.

Definition 2 (Rate Function) *A mapping $I : \mathcal{X} \to [0, \infty]$ is called a* Rate Function *or an* Action Functional *if the function I is lower semicontinuous, i.e. for all real numbers $\alpha \in [0, \infty)$, the level set $\psi_I(\alpha) \triangleq \{x : I(x) \leq \alpha\}$ is a closed subset of $\mathcal{X}$. A rate function I is called a* Good Rate Function *if in addition all the level sets $\psi_I(\alpha)$ are compact subsets of $\mathcal{X}$. The effective domain of I, denoted $\mathcal{D}_I$, is the set of points in $\mathcal{X}$ of finite rate, namely, $\mathcal{D}_I \triangleq \{x : I(x) < \infty\}$.*

Note that the abstract LDP is expressed in terms of probability measures on sets and therefore the mathematical definition applies a priori to any random variable. This rigorous definition of the LDP is due to Varadhan whose contributions immensely advanced LD theory.
This abstract LDP involving limits of lower and upper bounds for sets in topological spaces is rather technical and counter-intuitive; however it encompasses all varieties of situations including pathological ones.[18] Additional or stricter properties than the one in definition 2 might result. Hence, a detailed analysis of the underlying functional spaces based on the appearing natural supremum norm is performed for the control problem in section VI. This results in a few simplifications of the LDP in definition 1 and is discussed in detail in section VI.B.

C. Freidlin-Wentzell Theory

Schilder's theorem[19] furnishes the sample path probability for a Brownian motion realization in the small noise limit. The intricate details of Schilder's theorem (including a detailed proof), its connection to other LD results (such as the Gärtner-Ellis theorem) as well as the irregularities of the Wiener process are discussed in detail in [10].

While Schilder's Theorem established an LDP for the Wiener process, Freidlin and Wentzell extended in [51] the results for Brownian motion to deviations in the small-noise limit for diffusion processes generated by SDEs.[20] Although this is mostly referred to as the *Freidlin-Wentzell theory*, it represents only a part of the full theory by Freidlin and Wentzell which extends much further than SDEs and whose importance and impact cannot be overemphasized. First, the simpler case of an Itō diffusion process generated by a scalar SDE with state-independent noise is considered. Here, the LDP follows from Schilder's theorem (see [10]) via the contraction principle in a rather unambiguous application. The contraction principle allows to immediately substantiate an LDP for a pushforward measure arising from a transformation via a continuous map from a (measurable) space already obeying an LDP. The full version of the Freidlin-Wentzell theorem for a multidimensional Itō diffusion process with state-dependent noise can be found in the appendix. The Freidlin-Wentzell theorem describes the low-noise limit of an SDE by employing LD techniques. The random path generated by an SDE converges to the nominal trajectory in probability where nominal trajectory refers to the solution of the ordinary differential equation arising from Eq. (21) for $\varepsilon \to 0$. In particular, the Freidlin-Wentzell theorem quantifies the likelihood that a realization of the the process ventures away from the deterministic path in the limit $\varepsilon \to 0$.

Theorem 1 (Simple Freidlin-Wentzell Theorem) *Given a scaled nonlinear Itō diffusion process $\{x_\varepsilon(t)\}$ as follows: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a proper defined complete filtered probability space with sample space Ω, σ-field $\mathcal{F}$ and probability measure $\mathbb{P}$, and let the process be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then, $\{x_\varepsilon(t)\}$ is the diffusion process that is the unique solution of the Itō stochastic differential in Eq.* (21)*, i.e.*

$$dx_\varepsilon(t) = b\big(x_\varepsilon(t), t\big)\ dt\ +\ \sqrt{\varepsilon}\ d\mathcal{B}_t, \qquad x_\varepsilon(0) = x_0$$

for all $t \in [0, T]$ with $x_\epsilon : \Omega \times [0, T] \to \mathbb{R}$ and where $b : \mathbb{R} \times [0, T] \to \mathbb{R}$ is uniformly Lipschitz continuous in the first variable (namely, $|b(x) - b(y)| \leq B|x - y|$) and continuous in the second. The existence and uniqueness of the strong solution $\{x_\varepsilon(t)\}$ is standard.

[17] The simplest situation occurs when elements of $\mathcal{B}_{\mathcal{X}}$, the Borel σ-field on $\mathcal{X}$, are of interest. Without loss of generality, probability spaces are assumed to be completed.

[18] For instance, $\mathcal{B}$ not necessarily being the Borel-σ-field or some non-measurable open sets.

[19] See [40], [52], [42], [44] or [51] for details.

[20] The original work was published in Russian in the 1970s while the English translation in [51] as well as the second extended edition in [53] are the standard references to the full body of work by Freidlin and Wentzell on random dynamic systems.

Let $\mathbb{P}_\varepsilon$ denote the probability measure induced by $\{x_\varepsilon(t)\}$ on $C_{x_0}[0,T]$. Then, $\mathbb{P}_\varepsilon = \mathbb{P}_{\mathcal{B}\varepsilon} \circ W^{-1}$, where $\mathbb{P}_{\mathcal{B}\varepsilon}$ is the measure induced by $\{\sqrt{\varepsilon}\, \mathcal{B}_t\}$, and the deterministic map $W : C_0[0,T] \to C_{x_0}[0,T]$ is defined by $w = W(g)$ with w being the unique continuous solution of

$$w(t) = x_0 + \int_0^t b\big(w(s)\big)\, ds + g(t), \qquad t \in [0,T]\,. \tag{6}$$

Then, $\{x_\varepsilon(t)\}$ satisfies an LDP according to definition 1 in $C_{x_0}[0,T]$ with the good rate function

$$I(w) = \begin{cases} \frac{1}{2}\int_0^T |\dot{w}(t) - b\big(w(t)\big)|^2\, dt, & w \in \mathcal{H}^1_{x_0}[0,T] \\ \infty, & w \notin \mathcal{H}^1_{x_0}[0,T] \end{cases}. \tag{7}$$

Note that despite its name, the *Large* Deviations principle in theorem 1 addresses the *small* noise limit of SDEs. The rate function in Eq. (7) is also called the action functional, effective action, Lagrangian, or entropy depending on the context in which the Freidlin-Wentzell theory is presented. The terms 'action' and 'Lagrangian' stem from an analogy with the action of quantum trajectories in the path integral approach to quantum mechanics. The minimum and zero of the rate function appear at the trajectory of the deterministic system obtained in the zero-noise limit. Functional LD principles as in theorem 1 are the most refined LD results available for SDEs. Note that other, more specific or 'coarser' LD principles can again be derived from theorem 1 by contraction. The path minimizing the rate function over a desired set A (i.e. the path with the largest probability) is called the optimal path or maximum likelihood path or instanton of A.

In most literature on probability and stochastic calculus in general - and LD theory in particular - only the state-independent noise version of the Freidlin-Wentzell theorem is stated, if at all. The full version of the Freidlin-Wentzell theorem, i.e. including state-dependent noise, is stated in theorem 2 the appendix. Dembo and Zeitouni in [40] as well as Bucklew in [42], Varadhan in [44] and the original work in [51] are among the few offering a proof for the state-dependent noise case which is significantly more involved than the proof via the contraction principle for theorem 1. A version of the proof adapted to an engineering audience can also be found in [10].

VI. The New Control Framework

After theorem 1 has provided the necessary mathematical tool to consider large deviations of Itō diffusion processes, a proper control objective and the resulting new paradigm following the previous motivating example can be established. We now construct the necessary framework for the newly suggested control paradigm called Minimum Large Deviations (MLD) control. Several formal statements are set up representing initial ideas deemed most appropriate; but those do not comprise an exhaustive treatment of possibilities enabled by the Freidlin-Wentzell theory. The such constructed control can be conveniently extended to higher order systems without suffering from the curse of dimensionality. As a specific application governed by the new paradigm, the stochastic cost problem for nonlinear systems can be addressed in an unprecedented general fashion.

A. Control Objective

The underlying concept in the design of the new controller is to minimize the asymptotic probability that a sample path of a given Itō process strays far away[21] from its mean path. Hence, this concept implies the notion of a distance metric between the sample path and the mean path in the corresponding function space. As realizations of the Itō diffusion process generated by Eq. (21) are continuous almost everywhere[22] as well as continuous wp 1, the proper function space containing all sample paths becomes C_{x_0}, the space of all (vector-valued) continuous functions starting at x_0.

As Brownian motion sample paths are of finite quadratic variation, i.e. of finite energy, the energy norm $\mathcal{L}^2[0,T]$ might appeal as an appropriate distance metric. However, its employment in this context would be highly disadvantageous: Integral norms average singular deviations, effectively counteracting the basic concept of LD and probabilistically dominating sample paths. In addition, the resulting implementation of an optimization scheme on an open $\mathcal{L}^2$ space would prove intractable. A pure technical reason for not utilizing the $\mathcal{L}^2[0,T]$ norm lies in the fact that $C[0,T]$ is not complete with respect to any $\mathcal{L}^p$ norm with $1 \leq p < \infty$. Thus, the design goal will be formulated employing the natural

[21] As a matter of fact, it can also be shown that the controller minimizes small deviations.

[22] This includes Brownian motion itself as the limiting process of Eq. (21) for a vanishing system with $\boldsymbol{b}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \to 0$.

norm on $C[0,T]$, i.e. the maximum or ∞-norm.[23] Yet, the notion of energy is already incorporated, as theorem 2 relates the asymptotic probability of trajectories in a set S to the one particular path in S requiring the least effective action for its deviation. By employing the ∞-norm, the definition of the MLD control objective can now be suggested.

Definition 3 (MLD Control Objective) *The control objective of* Minimum Large Deviations *(MLD) control is to minimize the asymptotic probability that a sample path $x_\varepsilon(t)$ of a scaled Itō diffusion process as specified in definition 4 and theorem 2 deviates more than an amount K from the nominal path $x(t)$ generated by the underlying unperturbed differential equation emerging in the limit $\varepsilon \to 0$. That is, minimizing*

$$\begin{aligned}\mathbb{P}^{as}_{x_\varepsilon}\left(S[0,T]\right) &= \mathbb{P}^{as}_{x_\varepsilon}\left(\|x_\varepsilon(t) - x(t)\|_\infty \geq K\right)\\ &= \mathbb{P}^{as}_{x_\varepsilon}\left(\left\{C_{x_0}[0,T]\setminus\mathbb{B}_K(x(t);\|.\|_\infty)\right\}\right)\\ &= \mathbb{P}^{as}_{x_\varepsilon}\left(\left\{w \in C_{x_0}[0,T]\,\middle|\,|w(t)-x(t)| \geq K \text{ for some } t \in [0,T]\right.\right)\end{aligned} \tag{8}$$

where $C_{x_0}[0,T]$ denotes the space of continuous functions on the interval $[0,T]$ with initial value x_0. The asymptotic probability is interpreted in the LD sense of definition 1 as

$$\ln\left\{\mathbb{P}^{as}_{x_\varepsilon}(S)\right\} = \lim_{\varepsilon\to 0}\varepsilon\ln\left\{\mathbb{P}(x_\varepsilon \in S[0,T])\right\}.$$

Here, a scalar process $x_\varepsilon(t)$ was assumed for ease of notation. The open ball $\mathbb{B}_K(x(t);\|.\|_\infty)$ describes a 'tube' or 'channel' around the nominal path $x(t)$,[24] i.e. the solution of the unperturbed differential equation ($\varepsilon \to 0$) in the time interval $[0,T]$. It is the goal of the design to determine a control law $u(t)$ such that a nominal path results for which the exit probability from the channel defined by $\mathbb{B}_K(x(t);\|.\|_\infty)$, as depicted in Figure 2, is minimized. The definition

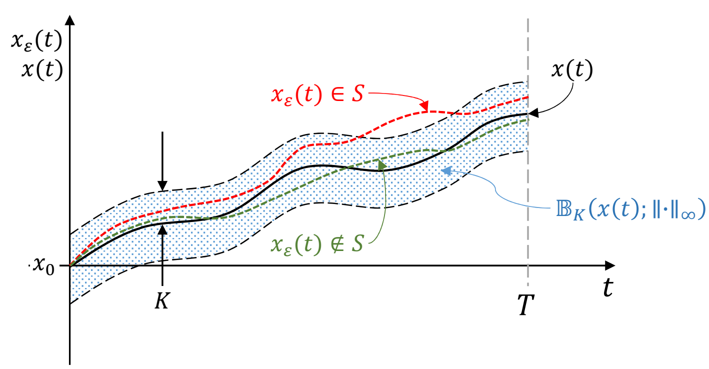


Fig. 2 Illustration of the set $S[0,T]$ in Eq. (8)

of the control objective in Eq. (8) easily extends to the vector-valued case: The supremum norm for a vector-valued process results as the highest value occurring in any dimensions. A design based on this interpretation would correspond to a worst case scenario. In addition, a desired individual bound on the supremum norm of each dimension can be considered, specifying maximum deviations in each coordinate direction. This more general case is reflected in the control statement in section VI.E.

B. Adaptation of the Freidlin-Wentzell Theorem

The formal proposal of the control objective in definition 3 can now be connected in technical terms to the Freidlin-Wentzell theory. Definition 3 already states that the asymptotic probability in Eq. (8) is to be interpreted in

[23] The space of continuous functions on a closed and bounded set $[0,T]$ is a complete metric space with respect to the supremum norm. Occasionally, the LDP in Schilder's theorem is defined on a Sobolev space, but the underlying norm as well as the $\mathcal{H}^1$-norm lack any meaningful usability for the desired control task.

[24] Given that the use of 'tube' implies the notion of something 'round' in higher dimensions, it might intuitively relate to a squared norm. Therefore, the term 'channel' is preferred in the context of this work.

the LDP sense of the Freidlin-Wentzell theorems 1 and 2. This allows to precisely state the underlying meaning of definition 3. The system state is an Itō diffusion generated by Eq. (21) with scaling parameter ε:

$$d\boldsymbol{x}(\omega,t) = \boldsymbol{b}\big(\boldsymbol{x}(\omega,t),\boldsymbol{u}(t),t\big)\ dt\ +\ \sqrt{\varepsilon}\ G\big(\boldsymbol{x}(\omega,t),t\big)\ d\mathcal{B}_t, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{9}$$

where all provisions of definition 4 apply. As before, unit variance Brownian motion $\mathcal{B}_t$ is considered as varying noise intensity is thought to be absorbed in $G(\boldsymbol{x}(t),t)$. The scaling parameter ε is necessary to express the family of probability measures $\mathbb{P}_\varepsilon$ for the application of the Freidlin-Wentzell theorem which establishes the LDP for the process in Eq. (9) as

$$-\inf_{x\in S^o} I(x) \le \liminf_{\varepsilon\to 0} \varepsilon \ln\{\mathbb{P}_\varepsilon(S^o)\} \le \limsup_{\varepsilon\to 0} \varepsilon \ln\{\mathbb{P}_\varepsilon(\bar{S})\} \le -\inf_{x\in\bar{S}} I(x) \tag{10}$$

where $S \subset C_{x_0}$ with S^o and $\bar{S}$ denoting its interior and closure, respectively. The expression $I(x)$ denotes the rate function of Eqs. (7) and (25). The existence of the Itō process requires $b, G \in C[0,T]$ with respect to t as well as b and G uniformly Lipschitz continuous in $\boldsymbol{x}$. This certainly includes most nonlinearities occurring in engineering applications, such as saturation, trigonometric relations, absolute values, state products, convection and advection terms. Note that the Lipschitz condition for the existence of the solution can be relaxed from a global condition to local Lipschitz continuity for every set A_i with $\bigcup_i A_i$ being a covering of $\mathbb{R}$ or $\mathbb{R}^d$, respectively.
The initial condition $\boldsymbol{x}_0$ for the process in Eq. (9) is deterministic. The domain $S[0,T]$ in Eq. (8) is closed with respect to the ∞-norm.[25] However, the domain $S[0,T]$ is only semi-bounded and cannot easily be converted into a domain $D \in \mathbb{R}^d$ of admissible deviations for all $t \in [0,T]$ as required for the impending optimization task (see section VI.C). The LDP for Eq. (9) only provides upper and lower bounds, so the meaning of *the* asymptotic probability as postulated in definition 3 could be called in question. A natural approach for minimization should focus on the upper bound, which applies as $S[0,T]$ in Eq. (8) is closed.

In addition, an argument for the coinciding upper and lower bounds in Eq. (10) can be made: The probability mass of a deviation set is concentrated in one dominating point or one path, respectively. In a simple case, such as Cramér's theorem, the infimum of the rate function is attained at the point closest to the sample mean. The same plausibility argument can be extended to the case of sample paths: It seems reasonable to assume that the infimum of the rate function $I(w)$ is attained in the neighborhood of a path closest to the nominal trajectory, that is, in the neighborhood of a path w^* with $\|w^*(t) - x(t)\|_\infty = K$. This is confirmed by theorem 2.3 and associated corollary in chapter 3 of [51].

This reasoning is substantiated in rigorous terms and discussed in detail in [10]. For the remainder of this section, it shall be assumed that the upper and lower bounds in Eq. (10) coincide and that the rate function attains its minimum on the set $S[0,T]$. This simplifies the adaptation of the Freidlin-Wentzell theorem to the suggested control objective in definition 3 significantly. The log-asymptotic probability in Eq. (8) can now be explicitly stated as

$$\begin{aligned}\ln\Big\{\mathbb{P}^{\text{as}}_{x_\varepsilon}\Big(\big\{C_{x_0}\backslash\mathbb{B}_K\big(x(t);\|.\|_\infty\big)\big\}\Big)\Big\} &= \lim_{\varepsilon\to 0}\ \varepsilon\ln\Big\{\mathbb{P}_\varepsilon\Big(\big\{C_{x_0}\backslash\mathbb{B}_K(x(t);\|.\|_\infty)\big\}\Big)\Big\}\\ &= -\min_{w\in S[0,T]\cap\mathcal{H}^1_{x_0}} I(w)\ .\end{aligned} \tag{11}$$

Equation (11) just repeats the LD principle from definition 1 for sample path LD problems, but it is now fully adapted to the control objective in definition 3.

Remark 1 *The question arises how the limit in Eq.* (11) *can be reasonably converted into a control task and why the asymptotic probability is a meaningful measure. There is always uncertainty in realizations that cannot be accounted for by any control. Yet, the probability of certain deviating sample paths described by the set S decays with an exponential rate which can be subject to manipulation. The sublimit expression*

$$\ln\Big\{\mathbb{P}^{as}_{x_\varepsilon}(S)\Big\} \approx -\frac{1}{\varepsilon}\min_{w\in S[0,T]\cap\mathcal{H}^1_{x_0}} I(w)$$

might not quite be the limiting rate function, but it is the most refined available estimate. The rate function reflects the sole access point affecting the 'asymptotical steepness' of the exit probability. Therefore, maximization of the infimum of

[25] This is indeed the case despite $S[0,T]$ being unbounded. As ∞ is not a number, sequences whose norms do tend to it are therefore non-convergent and have no limit in $C[0,T]$. Therefore, it can be easily inferred that $S[0,T]$ is closed since every converging sequence in $S[0,T] \subset C[0,T]$ with respect to the supremum norm will have a limiting supremum distance to $x(t)$ in $[K,\infty)$. In particular, the complement of $S[0,T]$ is just the open ball $\mathbb{B}_K(x(t);\|\cdot\|_\infty)$, hence $S[0,T]$ must be closed.

the rate function will be the focus of the suggested design. The terminology might be confusing as $I(x)$ is simply called 'rate function' while a more meaningful denomination would be 'asymptotic rate function'. Note that the exact rate is derived from

$$\mathbb{P}(w \in \mathcal{H}^1_{x_0}) = \exp\left\{ -\frac{1}{\varepsilon} I(w) + o\left(\frac{1}{\varepsilon}\right)\right\} \tag{12}$$

with additional sublinear terms in w. However, neglecting these sublinear terms in an equal fashion for all possible trajectories does not affect the successful determination of the optimal path for a large class of system dynamics $\boldsymbol{b}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$. Thus, it is a matter of understanding that this does not constitute a limitation to the control design since the controller still chooses the nominal path relative to the lowest exponential deviation probability.

There are some additional aspects of the Freidlin-Wentzell theorem to be discussed in order to fully provide the framework for the new control paradigm. The LDP in theorem 1 and 2 relates sample path probabilities to deterministic analysis and in particular to calculus of variations techniques. The probability measure of the Brownian motion $\boldsymbol{\mathcal{B}_t}$ induces a probability measure $\mathbb{P}_\varepsilon$ on the space of continuous functions C_{x_0} via the Itō diffusion generated by the stochastic differential in Eq. (9). By nature, the domain on which the LDP in theorems 1 and 2 is satisfied corresponds to C_{x_0}. However, the effective domain $\mathcal{D}_I$, i.e. the domain for which the rate function in Eqs. (7) and (25) obtains a value less than infinity, is only comprised of the Cameron-Martin space $\mathcal{H}^1_{x_0}$.[26] For readers with engineering background quite some confusion might arise when consulting the referenced literature on the actual function space characterizing the effective domain. The problem arises from the fact that the classical Wiener space can be extended to an abstract Wiener space and some literature discusses the Cameron-Martin space in this abstract setting. In addition, diversified definitions exist for the identical space. For the applications in mind, the classical Wiener space suffices, i.e. the space of continuous paths. Here, the Cameron-Martin norm becomes the classical Wiener measure and $\mathcal{H}^1_{x_0}[0,T]$ coincides with $\mathcal{L}^{2,1}_{x_0}[0,T]$, the space of all functions $f(t) \in \mathbb{R}^d$ starting at x_0 admitting a first derivative in the Lebesgue space $\mathcal{L}^2[0,T]$. Yet, to be technically precise, this does not require the functions $f(t)$ to be continuously differentiable as absolute continuity suffices.[27]

In engineering terms and in spirit of the proof of the Freidlin-Wentzell theorem via polygonal approximations in [10], it is without loss of generality to interpret $\mathcal{H}^1[0,T]$ as the space of piecewise differentiable functions on $[0,T]$ with bounded first derivative. In other words $f \in \mathcal{H}^1_{x_0}[0,T]$ if $f \in C^1_{x_0}[0,T]$ for all but countably many points $t \in [0,T]$ and $\sup_{t\in[0,T]}\{|f'(t)|\} < \infty$.[28]

Remark 2 *The space $S[0,T]$ in the MLD control objective of definition 3 has been defined as*

$$\begin{aligned} S[0,T] &= \left\{ C_{x_0}[0,T] \backslash \mathbb{B}_K(x(t); \|.\|_\infty) \right\} \\ &= \left\{ w \in C_{x_0}[0,T] \,\middle|\, |w(t) - x(t)| \geq K \textit{ for some } t \in [0,T] \right\} . \end{aligned}$$

But following the previous discussion, the analysis of the rate function in Eqs. (7) *and* (25) *is limited to its effective domain $\mathcal{D}_I$ and hence, the Cameron-Martin space $\mathcal{H}^1_{x_0}$. The infimum of the rate function is therefore sought on the space $S[0,T] \cap \mathcal{H}^1_{x_0}$. In order to avoid excessive notation by consistently denoting the intersection whenever the infimum of the rate function is concerned in the remainder of this study, the space S of Eq.* (8) *could be redefined as*

$$\begin{aligned} S[0,T] &= \left\{ \mathcal{H}^1_{x_0}[0,T] \backslash \mathbb{B}_K(x(t); \|.\|_\infty) \right\} \\ &= \left\{ w \in \mathcal{H}^1_{x_0}[0,T] \,\middle|\, |w(t) - x(t)| \geq K \textit{ for some } t \in [0,T] \right\} \end{aligned}$$

for ease of notation. Yet, this would create another excessive notation issue for the considerations in sections VI.D. Therefore, it is assumed without loss of generality for the remainder of this study, that the value ∞ is assigned to the rate functional $I(w)$ for all $w \in \{C_{x_0} \backslash \mathcal{H}^1_{x_0}\}[0,T]$ without explicit statement.

Note that in order to arrive at the underlying stochastic differential in Eq. (9) Itō calculus has to be applied. However, differential and integral terms based on the rate function are subject to ordinary differential calculus. This is yet another great advantage of the LD approach to statistical control problems.

[26]The Cameron-Martin space corresponds to the functions in C_{x_0} with finite energy. Therefore, it is rather obvious statement that the asymptotic probability of sample paths with infinite energy tends to zero.

[27]This can be seen from the proof of lower-semicontinuity of the rate function in Schilder's theorem in [10] where absolute continuity was required and established in order to apply the fundamental theorem of Lebesgue integral calculus. There, the fact has been used that $f(t)$ being absolute continuous is equivalent to $f(t)$ having a Lebesgue integrable derivative almost everywhere.

[28]Note that the following inclusion holds on a closed interval of the real line: $\mathcal{L}^\infty \subseteq \mathcal{L}^2 \subseteq \mathcal{L}^1$.

C. Boundedness of the Domain

With an impending implementation of an optimization scheme in mind, the subspace expressed in Eq. (8) reveals itself to be a highly unfavorable choice as it defines an unbounded domain in C_{x_0}. Therefore, it might be tempting to regard the complementary problem by maximizing the probability of a sample path to remain within the defined channel around the mean, i.e. maximizing

$$\mathbb{P}^{\text{as}}_{x_\varepsilon}(S^c) = \mathbb{P}^{\text{as}}_{x_\varepsilon}\Big(\mathbb{B}_K\big(x(t); \|.\|_\infty\big) \subset C_{x_0}\Big) . \tag{13}$$

This would require, in terms of theorem 2, to find a nominal path such that the infimum of the rate function over the subspace $\mathbb{B}_K(x(t); \|.\|_\infty)$ is minimized. However, this standard technique from probability calculus would constitute a major mistake in the context of asymptotic probabilities: LD theory simply yields the *law of the large number* as soon as the mean path is an element of the space over which the infimum is sought. The rate function attains its minimum of zero exactly at the nominal path. This property of the rate function is proven for sequences of i.i.d. random variables as well as illustrated in detail for one-dimensional Brownian motion using Schilder's theorem in [10].

D. Transformation of the Unbounded Domain

The space $S \in C_{x_0}[0,T]$ of Eq. (8) over which the infimum is sought, constitutes an ill-conditioned, unbounded functional domain not suitable for an optimization task. It can neither be converted into a simple set $D \in \mathbb{R}^d$ with $w(t) \in D$ for all $t \in [0,T]$ nor into a time-varying set $D(t) \in \mathbb{R}^d$ with $w(t) \in D(t)$. Therefore, any application of calculus of variations is prevented. The question arises if the problem can be reformulated in order to make it suitable for an optimization technique, be it numerically or analytically. The considerations in this section are an essential contribution of the presented work as they specify the minimization over the semi-bounded and ill-conditioned function space $S \subset C_{x_0}[0,T]$[29] in terms of a sequence of two-point boundary value problems. These are expressed via a parametrized, bounded and closed function space $S_\tau[0,\tau] \in \mathcal{H}^1[0,\tau]$ with parameter $\tau \in [0,T]$. The set S in definition 3 has been given as

$$S[0,T] = \Big\{w \in C_{x_0}[0,T] \,\Big|\, \sup_{0\le t\le T} |w(t) - x(t)| \ge K\Big\}$$

where $x(t)$ is the nominal path resulting from the unperturbed version of the stochastic differential equation (9), i.e. $\lim_{\varepsilon\to\infty} x_\varepsilon(t) = x(t)$. Now, it is assumed that the infimum of the rate function is actually attained by a path $w^*(t) \in \{S[0,T] \cap \mathcal{H}^1_{x_0}[0,T]\}$, yielding

$$\inf_{w\in S[0,T]} I(w) = I(w^*) .$$

As $w \in S \subset C_{x_0}[0,T]$, every $w(t)$ is continuous and therefore, $w(\tau) = K$ has to hold for at least one $\tau \in [0,T]$. This allows to define the set of function $\mathcal{W}'$ with its members $w'(t) \in \mathcal{W}'$ depicted in Figure 3 and given by

$$w'(t) = \begin{cases} w^*(t) & \text{for } 0 \le t \le t_1 \\ w(t) \in C_{w^*(t_1)}(t_1,T) & \text{for } t_1 < t \le T \end{cases} . \tag{14}$$

The functions $w'(t)$ coincide with the optimal trajectory $w^*(t)$ on $[0,t_1]$ and are arbitrary continuous functions with $w'(t_1) = w^*(t_1)$ for $[t_1,T]$. Note that $w^*(t_1) = x(t_1) \pm K$. Furthermore, note carefully that $w'(t) \in S[0,T]$ as $w^*(t) \in S[0,t_1]$ and

$$S'[0,T] \equiv \Big\{S[0,t_1] \cup C_{w^*(t_1)}[t_1,T]\Big\} \subseteq S[0,T] .$$

Then, the rate function in Eq. (7) evaluated at any $w'(t)$ can be expressed as

$$I(w') = \frac{1}{2}\int_0^{t_1} \big\|\dot{w}'(t) - b\big(w'(t),u(t)\big)\big\|^2\,dt + \frac{1}{2}\int_{t_1}^{T} \big\|\dot{w}'(t) - b\big(w'(t),u(t)\big)\big\|^2\,dt .$$

Now, with $w^*(t) \in S'[0,T]$, the infimum can be expanded in

$$\begin{aligned}\inf_{w\in S[0,T]} I(w) = \inf_{w\in S'[0,T]} I(w) = &\inf_{w\in S[0,t_1]} \frac{1}{2}\int_0^{t_i} \big\|\dot{w}(t) - b\big(w(t),u(t)\big)\big\|^2\,dt \\ &+ \inf_{w\in C_{w^*(t_1)}[t_1,T]} \frac{1}{2}\int_{t_1}^{T} \big\|\dot{w}(t) - b\big(w(t),u(t)\big)\big\|^2\,dt .\end{aligned} \tag{15}$$

[29]See remark 2 on considering the subspace of $S[0,T]$ which is in $\mathcal{H}^1_{x_0}$.

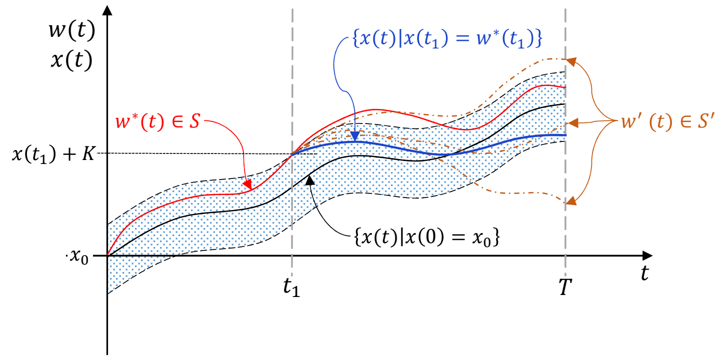


Fig. 3 Examples for Deviation Paths $w'(t)$ **in Eq.** (14)

But in the interval $[t_1, T]$, the infimum is taken over any function $w \in \{\mathcal{C}_{w^*(t_1)} \cap \mathcal{H}^1\}$ which includes the solution $x(t)$ to the ordinary differential equation emerging from the unperturbed limit of the Itō diffusion in Eq. (9) with initial condition $x_{t_1}^K = x(t_1) \pm K$, i.e.

$$w(t) = x_{t_1}^K \pm K + \int_{t_1}^{t} b(w(t), u(\tau), \tau) d\tau$$

where $t \in [t_1, T]$. Hence, the contribution of the second integral in Eq. (15) yields

$$\inf_{w \in \mathcal{H}'_{x_{t_1}^K}[t_1,T]} \frac{1}{2} \int_{t_1}^{T} \left\| \dot{w}(t) - b\big(w(t), u(t)\big) \right\|^2 dt \ = \ 0$$

without loss of generality. Now, this allows for the following statement being equivalent to the infimum of the rate function in Eq. (7) over the set S of definition 3:

$$\inf_{w \in S[0,T]} \frac{1}{2} \int_0^T \|\dot{w}(t) - b\big(w(t), u(t)\big)\|^2 \, dt = \min_{\tau \in [0,T]} \inf_{w \in S_\tau[0,\tau]} \frac{1}{2} \int_0^\tau \left\|\dot{w}(t) - b\big(w(t), u(t)\big)\right\|^2 dt \tag{16}$$

where the set $S_\tau[0, \tau]$ is defined as

$$S_\tau[0, \tau] := \Big\{ w \in C_{x_0}[0, \tau] \,\Big|\, w(0) = x_0, \ |w(\tau) - x(\tau)| = K \Big\} \, . \tag{17}$$

Hence, the ill-conditioned, unbounded original minimization problem can be expressed as the minimization over a set of well-conditioned two-point boundary value problems.

E. The Large Deviation Control Problem Statement

The preliminary analysis and considerations in the preceding sections allows for a precise MLD control statement. It shall be noted that merely finding the one nominal path with the lowest asymptotic probability of deviation is hardly ever a suitable control task. On the contrary, the overall control objective will in almost all situations be related to a deterministic control task and/or contain additional constraints. These underlying tasks will often include but are not limited to standard optimal control statements. For the general MLD control statement, let $x_\varepsilon(t)$ be a scaled Itō diffusion process as in theorem 2 with all provisions of definition 4. Then, the newly suggested problem statement reads as follows:[30]

[30] This is the multidimensional version of the control statement with K_i indicating individual bounds for each state $x_i(t), \ i = 1 \ldots n$.

Control Problem 1 (General MLD Control)

$$
\begin{aligned}
&\textit{Minimize} && J(\boldsymbol{u}) = \lim_{\varepsilon\to 0} \varepsilon \ln \left\{ \mathbb{P}\left(S^u\right) \right\} = -\inf \left\{ I^u(\boldsymbol{w}) \,\middle|\, \boldsymbol{w} \in S^u \cap \mathcal{H}^1_{x_0} \right\} \\
&\quad \textit{subject to} && d\boldsymbol{x}_\varepsilon(t) = \boldsymbol{b}\big(\boldsymbol{x}_\varepsilon(t), \boldsymbol{u}(t), t\big)dt + \sqrt{\varepsilon}\, G\big(\boldsymbol{x}(t), t\big)\, d\mathcal{B}_t \\
& && \boldsymbol{x} \in \mathcal{X}[0,T] \subseteq C^1_{x_0}[0,T] \\
&\quad \textit{over all} && \boldsymbol{u} \in \mathcal{U}[0,T] \\
&\quad \textit{with} && \boldsymbol{x}_\varepsilon(0) = \boldsymbol{x}(0) = x_0 \\
& && \varepsilon > 0 \\
&\quad \textit{where} && S^u := \Big\{ \boldsymbol{w} \in C_{x_0}[0,T] \,\Big|\, \|w_i(t) - x_i(t)\|_\infty \geq K_i \;\; \textit{for } t \in [0,T] \Big\} \\
& && I^u(\boldsymbol{w}) = \frac{1}{2} \int_0^T \left\| \dot{\boldsymbol{w}}(t) - \boldsymbol{b}\big(\boldsymbol{w}(t), \boldsymbol{u}(t), t\big) \right\|^2_{a^{-1}(\boldsymbol{w}(t),t)} \, dt \\
& && \boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{b}\big(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau\big) d\tau \\
& && a^{-1}\big(w(t), t\big) = \big[GG^T\big]^{-1}\big(w(t), t\big) \\
& && \left\| \boldsymbol{e}(t) \right\|^2_{A(t)} = \big\langle A(t)\boldsymbol{e}(t), \boldsymbol{e}(t) \big\rangle \;.
\end{aligned}
$$

Note that the validity of statement 1 is, in general, limited to the existence of $[GG^T]^{-1}$. In control statement 1, the spaces $\mathcal{X}[0,T]$ and $\mathcal{U}[0,T]$ allow to incorporate any additional constraints, arising from design restrictions or superordinate other control requirements. Now, it shall be assumed that another control task is already given for the underlying deterministic differential equation system in the form of an optimal control principle, i.e. minimizing a certain objective function. For the states being generated by the diffusion process $\boldsymbol{x}_\varepsilon(t)$, this deterministic performance index becomes a stochastic process itself. There are different possibilities to incorporate the stochastic dependence into the cost function. Several exemplary ways of absorbing the underlying deterministic control task into the Freidlin-Wentzell based control paradigm can be stated via augmented objective functions. A simple idea for the inclusion challenge would be the additive combination of the superordinate deterministic performance index and of the asymptotic exit probability in a modified MLD cost function. Therefore, a weighted sum could be formed as shown in [10]. In this formulation, additional constraints could again be included via the choices of $\mathcal{X}$ and $\mathcal{U}$. However, a different incorporation of a deterministic cost function is possible by taking inspiration from Sain's method in MCV control (see section III): A certain amount of slack in the performance index is allowed to define admissible state and control spaces, $\mathcal{X}[0,T]$ and $\mathcal{U}[0,T]$, over which the asymptotic probability of deviation from the nominal path is then minimized. The mean performance index is constrained by an upper limit $D > D_{\min}$, where $D_{\min}$ can be chosen to either correspond to the deterministic optimal control cost or to the minimum average performance. This suggested approach is by no means exhaustive, but represents a method which is deemed most appropriate. The following statement embodies the idea:

Control Problem 2 (Cost Constrained MLD Control)

$$\begin{aligned}
&\textit{Minimize} && J(\boldsymbol{u}) = \lim_{\varepsilon\to 0} \varepsilon \ln \left\{ \mathbb{P}\left(S^u\right) \right\} = -\inf \left\{ I^u(\boldsymbol{w}) \,\middle|\, \boldsymbol{w} \in S^u \cap \mathcal{H}^1_{x_0} \right\} \\
&\textit{subject to} && d\boldsymbol{x}_\varepsilon(t) = \boldsymbol{b}\big(\boldsymbol{x}_\varepsilon(t), \boldsymbol{u}(t), t\big)dt + \sqrt{\varepsilon}\, G\big(\boldsymbol{x}(t), t\big)\ d\mathcal{B}_t \\
&\textit{over all} && \boldsymbol{u} \in \mathcal{U} = \left\{ \boldsymbol{u} : \left[\theta\big(\boldsymbol{x}(T), \boldsymbol{u}(T)\big) + \int_0^T \phi\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big) dt \right] \leq D \right\} \\
&\textit{with} && \boldsymbol{x}_\varepsilon(0) = \boldsymbol{x}(0) = x_0 \\
& && \varepsilon > 0 \\
&\textit{where} && S^u := \left\{ \boldsymbol{w} \in C_{x_0}[0,T] \,\middle|\, \|w_i(t) - x_i(t)\|_\infty \geq\ K_i \textit{ for } t \in [0,T] \right\} \\
& && I^u(\boldsymbol{w}) = \frac{1}{2} \int_0^T \left\| \dot{\boldsymbol{w}}(t) - \boldsymbol{b}\left(\boldsymbol{w}(t), \boldsymbol{u}(t), t\right) \right\|^2_{a^{-1}(\boldsymbol{w}(t),t)}\ dt \\
& && \boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{b}\left(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau\right)\ d\tau \\
& && D > \min_{\boldsymbol{u}'(t)} \left\{ \mathcal{E}\left\{ \theta\big(\boldsymbol{x}_\varepsilon(T), \boldsymbol{u}'(T)\big) + \int_0^T \phi\big(\boldsymbol{x}_\varepsilon(t), \boldsymbol{u}'(t)\big) dt \right\} \right\} = D_{min} \\
& && a^{-1}\big(w(t), t\big) = \left[GG^T\right]^{-1}\big(w(t), t\big) \\
& && \left\|\boldsymbol{e}(t)\right\|^2_{A(t)} = \big\langle A(t)\boldsymbol{e}(t), \boldsymbol{e}(t) \big\rangle\ .
\end{aligned}$$

Yet, another variation of cost constrained MLD control can be achieved if the dynamics of the accumulated total cost are expressed by a stochastic differential depending on the perturbed state trajectory. Then, an augmented SDE system can be formed. This is a quite appealing way of handling the cost constrained MLD problem, however its derivation is rather involved and left for future publication. Note that in all these variations, the original objective function does not necessarily need to be formulated as a mean minimization. On the contrary, the Freidlin-Wentzell theory in theorems 1 and 2 characterizes deviations from the strong solution of the unperturbed ODE and not from the mean process.

VII. Numerical Evaluation of MLD Control

This section provides an initial numerical evaluation of MLD control as suggested in statements 1 and 2. Before two nonlinear examples are considered, applied evaluation metrics and difficulties to be expected from numerical simulation are discussed.

A. Evaluation Criteria

Two integration schemes are most often employed for the simulation of SDEs: the Euler-Maruyama method and the Milstein scheme. Both produce linear polygonal approximations of the process, $x_{\varepsilon,\delta}(t)$. If it is of importance that the numerical solution $x_{\varepsilon,\delta}(t)$ reflects path properties, i.e. if the whole path plays a role in further analysis, numerical schemes with strong convergence should be considered. If, on the other hand, the focus is on path independent quantities, then weakly convergent schemes reflect the point dependent properties well. The Euler-Maruyama method converges strongly and weakly if $b(\cdot)$ and $G(\cdot)$ are four times continuous differentiable with bounded first derivatives and if both functions do not grow too fast.[31] The Milstein scheme exhibits strong convergence of order 1. However, the presented numerical examples in this section all exhibit state-independent noise terms such that the Euler-Maruyana and Milstein methods coincide.

As this study of the newly suggested control paradigm considers asymptotic probabilities, it has to be discussed if the numerical integration schemes properly reflect the desired quantities. Strong convergence might guarantee that certain path properties are reflected by the approximation scheme, but only in the expectation sense. Most existing integration schemes for SDEs do not converge in probability. Therefore, difficulties are to be expected if empirical probabilities on SDE sample paths generated by the above numerical methods are employed as statistical performance measures. Caution is advised when immediate conclusions on probability distributions are drawn from numerically generated SDE realizations. For a more detailed account on numerical integration of SDEs, it is referred to the subject

[31] This requirement should be satisfied by the Lipschitz continuity demanded earlier.

primers by Higham in [54] and by Malham et al. in [55]. A standard reference for the topic [56] by Kloeden et al. while Iacus treats the subject with equal detail in [57]. A more thorough discussion on the issues to be expected in context with the new control paradigm can be found in [10] (including the slow speed of convergence for empirical probabilities based on Brownian motion and the striking efficiency of Schilder's theorem).

We will now briefly discuss the metrics applied to the numerical examples. In the motivating example of section IV.B, the sample probability for exceeding a certain bound K has been employed. But the choice of the particular channel width was an arbitrary design parameter. The choice of this bound is not so trivial as it might appear at first: Too low of a bound creates difficulties with a standard Monte Carlo simulation[32] as almost all paths will exceed the bound. Any bound in the magnitude of the noise variance becomes meaningless as the exit probability will simply converge to the probability distribution of a Gaussian random variable. Too large of a channel width might prohibit a meaningful interpretation as only few paths will exceed its bounds for any control, unless some advanced simulation technique employing, for instance, boosted SDEs is utilized. Thus, the choice of K for evaluation purposes has to be adapted to the specific problem noise level at hand and needs to be regarded with caution.

Additional measures can also be utilized. In section VI.D, the most probable exit time, i.e. the time at which the infimal deviation – with respect to the effective action – exceeds K for the first time, has been discussed repeatedly. For simplicity, this time shall be denominated simply as *first exit* for the remainder of this section. As $J_\tau[0,\tau]$ will reach its minimum at the first exit, it can be predicted by the new control paradigm and therefore might be a suitable parameter for evaluation. However, it suffers from the same lack of convergence in terms of numerical integration of SDEs as the exit probability. The lower the bound K is chosen the sooner the first exit will appear.[33] In addition, the numerical determination of empirical probabilities, and thus of the exit probability, creates further challenges. The time domain has to be clustered into intervals of size larger than the integration step size in order to compute an empirical distribution. Then, the number of exit occurrences per interval is simply counted, and the result can be depicted using a histogram. The first exit is then assigned to the interval of highest count. Yet, due to the properties of numerical SDE integration, this empirical first exit distribution will always be tilted to the left.

As the direct empirical determination of the exit time by Monte Carlo methods emerges to be unreliable, different additional measures - based on expectations or averages - for the prediction accuracy of the new control paradigm have to be selected. The average error at each time t over all Monte Carlo iterations has been deemed to be a highly effective and very useful metric. Since it results from an expectation operation, convergence with respect to the employed numerical technique is guaranteed. The average error directly reflects the deviation probability at each point in time t. In addition, the accumulated average error, i.e. the integral over the average error curve, serves as a measure of the total deviation probability in the time interval $[0,T]$. The first exit, i.e. the most probable exit time, corresponds to the occurrence of the maximum average error. This is a stable criterion which is independent from the arbitrary choice of the exceedance level K.

The average error allows for two additional metrics which are employed for evaluation in this chapter: The maximum average error and the area under the average error curve which is designated as the cumulative or accumulated average error. These two metrics effectively allow comparison of trajectories in terms of worst case behavior (maximum average error) and overall deviation potential (cumulative average error).

Finally, five additional quantities are calculated where applicable: the mean cost for the quadratic cost criterion, the cost variance and the deterministic cost associated with the nominal path. The average maximum error and the most probable occurrence of the maximum error are likewise provided. Yet, the latter measures have to be regarded with great care. Again due to the convergence properties of numerical schemes for SDEs, the maximum error will appear at the very last time interval for the majority of realizations, not reflecting the true result.[34] The deviation from the nominal path is denoted as $e_\varepsilon(t) = |x^u_\varepsilon(t) - x^u(t)|$.

[32] This refers to Monte Carlo methods which are not based on importance sampling or boosted SDEs.

[33] To give an extreme example, if the bound K is selected to be zero, the first exit in a polygonal approximation will always occur after the first sampling interval when a Gaussian random disturbance is applied.

[34] This can be substantiated by considering the probability mass associated with it. Empirical distributions are determined based on the probability mass in finite intervals and not at singular points. A single point of a continuous distribution has zero probability mass which is clearly not reflected by the numerical integration scheme. It is tempting to use a fine sampling grid when constructing histograms based on large scale Monte Carlo evaluations. A such constructed histogram is often thought to resemble a probability density function. Yet, this is a misleading approach as it is not the number of Monte Carlo evaluations that determines a meaningful grid but the step size of the numerical integration technique. As there is no probability mass associated with the final time peak, it quickly vanishes for an increasingly coarse grid.

B. Scalar Nonlinear MLD Example

In order to develop a first idea for LPD in general, the scalar nonlinear example in Eq. (4) of section IV.B is employed again, i.e.

$$dx_\varepsilon(t) = \left[x^2(t) - 1 + u(t)\right] dt + \sqrt{\varepsilon}\, d\mathcal{B}_t \ . \tag{18}$$

Four possible trajectories from initial condition −1 to final condition 1 have been explored in section IV.B by comparing the associated deviation probabilities arising from Monte Carlo simulation as given in Table 1. Now, the problem is revisited when the optimal control $u^*(t)$ and corresponding trajectory $x^*(t)$ are computed according to the new control paradigm. This system has been selected in particular as it exhibits a nontrivial nonlinear structure, but yet provides intuitive access. As discussed previously, lower deviation probability is related to the time which the particular nominal trajectory spends in a region with high energy dissipation. Thus, more 'effort' is required by the disturbance to create a deviation. With this explanation in mind, an optimal nominal trajectory minimizing the deviation potential should stay as long as possible in the most dissipative area. Certainly, this would result in an unbounded control effort for the system in question. For the minimization task at hand, the control is therefore confined to an area between upper and lower bounds of ±5. Now, the optimal law subject to this bound and subject to the initial condition of $x_0 = -1$ as well as the final condition of $x_T = 1$ is determined according to the MLD criterion and depicted in Figure 4.

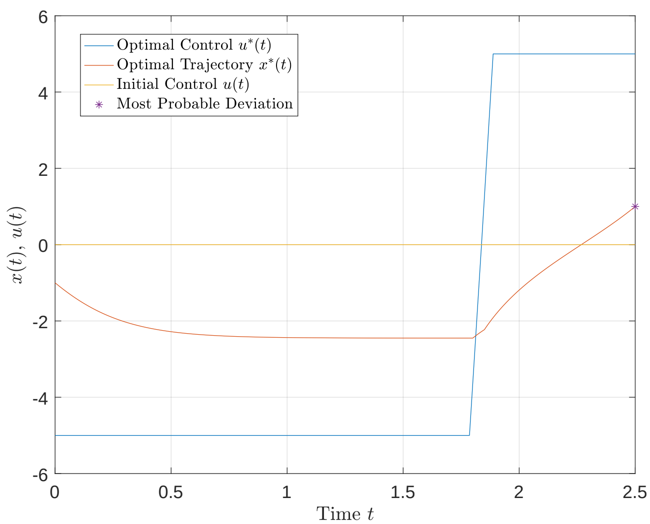


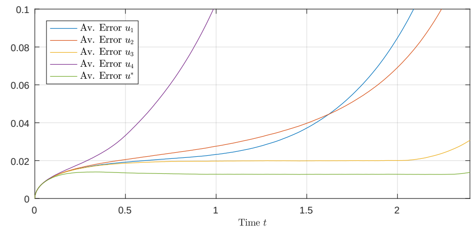
Fig. 4 Optimal MDP Control for $\dot{x}(t) = x(t)^2 - 1 + u(t)$**,** $x_0 = -1$**,** $x_T = 1$

The control exhibits precisely the anticipated behavior as the maximum control effort is provided to move the trajectory in the most dissipative area. The lowest possible value of the state $x(t)$ is maintained until exactly the point in time is arrived at, from which the final constraint is lastly reachable given the bounds on control. In that sense, the numerical determination of the MLD control can be deemed as successful.

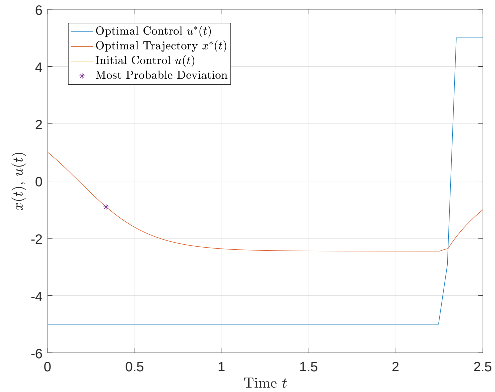
For comparison, the average error for the four control laws from −1 to 1 in Eq. (5) is shown in Figure 5(a). In addition, the optimal MLD trajectory from 1 to −1 for the problem in Eq. (18) is calculated and depicted in Figure 5(b). The maximum value of the associated average error resulting from $M = 10^6$ Monte Carlo evaluations appears at $t = 0.3325$ corresponding almost precisely to the first exit at $t = 0.3350$ as predicted by the new controller. Table 2 supplies the introduced statistical measures for the nonlinear scalar problem in Eq. (18) and for each of the control trajectories in Eq. (5) in comparison to the optimal LPD trajectory.

C. Cost Constrained MLD control

In this section, the Cost Constrained MLD controller, as detailed in statement 2, is computed for two example problems: The scalar quadratic system in Eq. (18) and the two-dimensional nonlinear Duffing oscillator. Both systems exhibit strong nonlinear behavior. The starting points for the cost constraints are the minimum deterministic quadratic costs achieved by the corresponding optimal controller for the non-perturbed system arising from $\varepsilon \to 0$. The deterministic optimal control emerges as the solution to a nonlinear two-point boundary value problem. Every simulation



(a) Average Error for the Different Control Laws with Noise Level $\sqrt{\varepsilon} = 0.05$



(b) Optimal MDP Control for $x_0 = -1$, $x_T = 1$

Fig. 5 Control Strategies for Eq. (5)**,** $\dot{x}(t) = x(t)^2 - 1 + u(t)$**, with** $x(0) = -1$ **and** $x(5) = 1$

Table 2 Statistical Evaluation of Optimal LPD Control and Control Laws in Eq. (5) **for** $\dot{x}(t) = x^2(t) - 1 + u(t)$

	Simulation Run				
	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u_4(t)$	$u^*(t)$
Simulation Parameter					
Initial Value, x_0	-1				
Final Value, x_T	1				
Final Time, T	2.5				
No. of Control Inputs, N_C	50				
No. of Integration Steps, N_I	$5 \cdot 10^3$				
Noise Level, $\sqrt{\varepsilon}$	0.05				
No. of Monte Carlo Evaluations, M	10^5				
Channel Width, K	0.06				
Statistical Analysis					
Cum. Avg. Err.: $\int \mathcal{E}\{e_\varepsilon\}$	0.1320	0.1151	0.0497	∞	0.0321
Occurr. of Max. Avg. Err.: $t(\max \mathcal{E}\{e_\varepsilon\})$	2.500	2.500	2.500	1.6905	2.500
Max. Avg. Err.: $\max \mathcal{E}\{e_\varepsilon\}$	0.2280	0.1626	0.0386	∞	0.0155
Occurr. of Max. Err.: $t(\max\{e_\varepsilon\})$	2.4875	2.4875	2.4875	2.4875	2.4875
Avg. of Max. Err.: $\mathcal{E}\{\max\{e_\varepsilon\}\}$	0.2355	0.1732	0.0633	∞	0.0435
Occurr. of Max. Err. for $e \geq K$: $t(\max\{e_\varepsilon \geq K\})$	2.4875	2.4875	2.4875	2.4875	2.4875
First Exit Time for $e \geq K$: $\tau_{e \geq K}$	1.6375	1.6875	2.4875	0.5375	2.4875
% of Paths Exceeding Limit: $\mathbb{P}_S\{e_\varepsilon \geq K\}$	90.42	88.63	49.07	98.747	3.49

follows a continuous-discrete interpretation of the underlying system, i.e. the control is applied in a sample-and-hold manner at $N_C = 50$ different points in the time interval. The statistical analysis is based on $N_I = 5 \cdot 10^3$ integration steps in order to generate the sample paths for the M Monte Carlo simulations. The noise level $\sqrt{\varepsilon}$, the channel width K as well as the initial and final conditions are specified individually for each example.

1. Scalar Example: Quadratic System

Again, the nonlinear quadratic system is considered generated by the associated scaled Itō diffusion process

$$dx_\varepsilon(t) = [x_\varepsilon^2(t) - 1 + u(t)]\, dt \ + \ \sqrt{\varepsilon}\, d\mathcal{B}_t\,, \quad x_\varepsilon(0) = 2 \tag{19}$$

where $\mathcal{B}_t$ denotes Brownian motion with unit variance. The scaling parameter $\sqrt{\varepsilon}$ is utilized to adapt for different noise levels. The deterministic quadratic cost criterion is given as

$$D(u) = \int_0^T \phi\big(x(t), u(t), t\big)\, dt = \int_0^T \big[q x^2(t) + r u^2(t)\big]\, dt$$

where $q = 10$ and $r = 1$ are selected. The emerging optimal control law, the associated optimal deterministic trajectory as well as the costate minimizing the quadratic cost function are shown in Figure 6. In addition, this figure illustrates

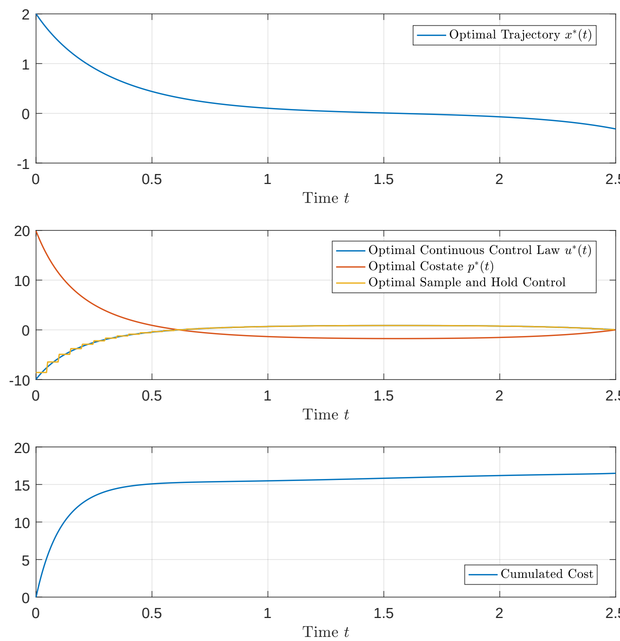


Fig. 6 Optimal Deterministic Control for Eq. (19)

the applied sample-and-hold control and the cumulated cost. The minimal deterministic performance index for the sample-and-hold control is calculated as $D_{\min} = 16.46$. Now, three control strategies are computed according to statement 2 to determine the Cost Constrained MLD control. Thereby, the maximum allowed deterministic cost for the $\varepsilon \to 0$ limit is given by $[1.3 \cdot D_{\min}]$, $[1.5 \cdot D_{\min}]$ and $[2.0 \cdot D_{\min}]$, respectively. The corresponding optimal MLD control laws shall be denoted by CCMLD13, CCMLD15 and CCMLD20 for the remainder of text while OPTDET is assigned to the optimal deterministic control. Figure 7 depicts the resulting optimal CCMLD13 control law $u^*(t)$, the associated optimal trajectory $x^*(t)$ and the predicted most probable exit time $\tau_{\min}$. The statistical metrics as discussed in section VII.B are determined for each controller and are summarized in Table 3 together with the simulation parameters.

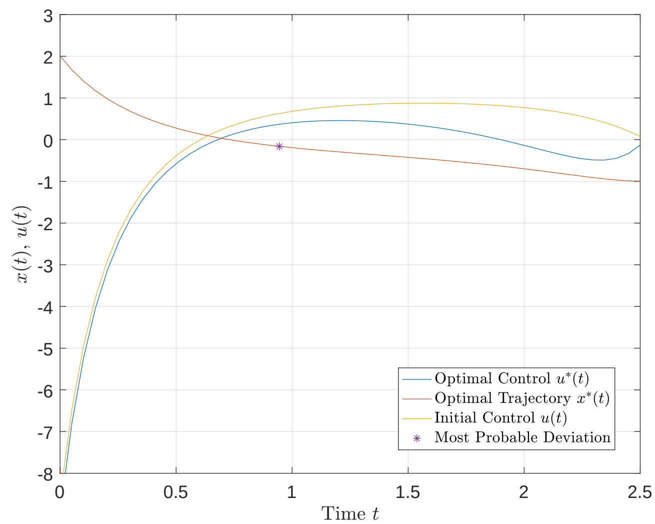


Fig. 7 Optimal LPD Control and Trajectory for Eq. (19) **with** $D \leq [1.3 \cdot D_{\min}]$

Table 3 Statistical Evaluation Constrained LPD for $\dot{x}(t) = x^2(t) - 1 + u(t)$

	Simulation Run			
	OPTDET	CCLPD13	CCLPD15	CCLPD20
Simulation Parameter				
Initial Value, x_0	2			
Final Time, T	2.5			
No. of Control Inputs, N_C	50			
No. of Integration Steps, N_I	$5 \cdot 10^3$			
Noise Level, $\sqrt{\varepsilon}$	0.05			
No. of Monte Carlo Evaluations, M	10^6			
Channel Width, K	0.1			
Control Parameter				
State Weight for Cost Funct., q	10			
Control Weight for Cost Funct., r	1			
Pred. First Exit, $\tau_{\min}$		0.945	0.910	0.880
Determ. Cost, $D_{\min}$	16.2373	21.3763	24.6792	32.9208
Statistical Analysis				
Mean Cost: $\mathcal{E}\{D\}$	16.4247	21.3915	24.6894	32.9267
Cost Variance: Var$\{D\}$	0.3163	0.3150	0.3732	0.4400
Cum. Avg. Err.: $\int \mathcal{E}\{e_\varepsilon\}$	0.1641	0.0932	0.088	0.0845
Occurr. of Max. Avg. Err.: $t(\max \mathcal{E}\{e_\varepsilon\})$	2.1805	0.9315	0.8945	0.8540
Max. Avg. Err.: $\max \mathcal{E}\{e_\varepsilon\}$	0.0874	0.0496	0.0486	0.0478
Occur. of Max. Err.: $t(\max\{e_\varepsilon\})$	2.4875	0.7875	0.7875	0.7875
Avg. of Max. Err.: $\mathcal{E}\{\max\{e_\varepsilon\}\}$	0.1287	0.0869	0.0849	0.0830
Occurr. of Max Err. for $e \geq K$: $t(\max\{e_\varepsilon \geq K\})$	2.4875	08625	0.8125	0.8625
First Exit Time for $e \geq K$: $\tau_{e \geq K}$	0.5375	0.4875	0.4875	0.4875
% of Paths Exceeding Limit: $\mathbb{P}_S\{e_\varepsilon \geq K\}$	59.1362	28.9182	26.8575	25.1233

D. Multivariate Example: Duffing Oscillator

Now, a multivariate example is considered via the nonlinear Duffing oscillator. For parameters $\alpha = 1$, $\beta = -0.1$ and $\gamma = 0.25$, the scaled Itō diffusion process is generated by

$$\begin{bmatrix} dx_{\varepsilon,1}(t) \\ dx_{\varepsilon,2}(t) \end{bmatrix} = \left[\begin{bmatrix} x_{\varepsilon,2}(t) \\ -\alpha x_{\varepsilon,2}(t) - \beta x_{\varepsilon,1}(t) - \gamma x_{\varepsilon,1}^3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \right] dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\mathcal{B}_t \tag{20}$$

where $\mathcal{B}_t$ denotes Brownian motion with unit variance. The scaling parameter $\sqrt{\varepsilon}$ is utilized to adapt for different noise levels. The deterministic quadratic cost criterion is given as

$$D(u) = \int_0^T \phi\big(x(t), u(t), t\big)\ dt = \int_0^T \left[q_1 x_1^2(t) + q_2 x_2^2(t) + r u^2(t)\right]\ dt$$

where $q_1 = q_2 = 10$ and $r = 1$ is selected. The resulting optimal control law (see [10] for further details), the associated optimal deterministic trajectory as well as the costate minimizing the quadratic cost function are shown in Figure 8(a). In addition, Figure 8(b) illustrates the applied sample-and-hold control and the cumulated cost. The minimal

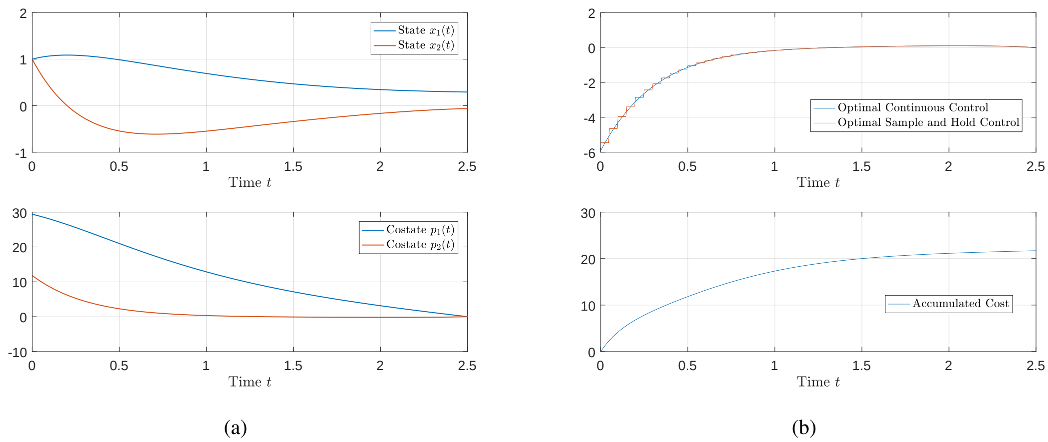


Fig. 8 Optimal Deterministic Control for Eq. (20)

deterministic cost for the sample-and-hold control is calculated as $D_{\min} = 21.69$. Again, three control strategies are computed according to statement 2 to determine the Cost Constrained MLD control. As before, the maximum allowed deterministic cost for the $\varepsilon \to 0$ limit is given by $1.3 \cdot D_{\min}$, $1.5 \cdot D_{\min}$ and $2.0 \cdot D_{\min}$, respectively. Figure 9 depicts the resulting optimal control law $u^*(t)$ for the CCMLD20 case as well as the associated optimal trajectory $x^*(t)$ and the predicted most probable exit time $\tau_{\min}$. Once more, the statistical measures as discussed are determined for each controller and are summarized in Table 4 together with the simulation parameters. As before, amplified average deviation bounds around the nominal paths are shown as well as the average error with a marked maximum value.

VIII. Conclusion

A new comprehensive approach to statistical control based on a functional interpretation of stochastic processes over the space of sample paths has been presented. In contrast to earlier development, the probability of realizations to deviate from the deterministic solution is addressed directly. This is enabled by the creation of a formal control framework embedding LD techniques and, in particular, the Freidlin-Wentzell theory. The account does not only include the precise definition of a control objective, but also comprises a discussion of the adaptation of the Freidlin-Wentzell theorem to the particular situation. The suggested objective consists of two nested minimization tasks necessary to arrive at the desired control. Thereby, the domain of interest is identified as being of ill-conditioned nature with respect to standard optimization techniques. A necessary transformation of the inner minimization task over an unbounded domain into a sequential minimization of parametrized well-conditioned function spaces – which can be expressed as subsets of $\mathbb{R}^d$ – is presented. Subsequently, formal control statements are established building the basis of Minimum

Table 4 Statistical Evaluation Constrained LPD for Duffing Oscillator in Eq. (20)

	Simulation Run			
	OPTDET	CCLPD13	CCLPD15	CCLPD20
Simulation Parameter				
Initial Value, $x_{1,0}$	1			
Initial Value, $x_{2,0}$	1			
Final Time, T	2.5			
No. of Control Inputs, N_C	50			
No. of Integration Steps, N_I	$5 \cdot 10^3$			
Noise Level, $\sqrt{\varepsilon}$	0.01			
No. of Monte Carlo Evaluations, M	10^5			
Channel Width, K	0.018			
Control Parameter				
State Weight for Cost Funct., q_1	10			
State Weight for Cost Funct., q_2	10			
Control Weight for Cost Funct., r	1			
Predicted First Exit Time, $\tau_{\min}$		2.5000	2.5000	2.5000
Determ. Cost, $D_{\min}$	21.5824	28.1458	32.4990	43.3657
Statistical Analysis				
Mean Cost: $\mathcal{E}\{D\}$	21.5839	28.1464	32.4995	43.3659
Cost Variance: Var$\{D\}$	0.0022	0.0326	0.0449	0.0651
Cum. Avg. Err.: $\int \mathcal{E}\{e_{\varepsilon,1}\}$	0.0099	0.0092	0.0089	0.0085
Cum. Avg. Err.: $\int \mathcal{E}\{e_{\varepsilon,2}\}$	0.0121	0.0115	0.0114	0.0113
Occurr. of Max. Avg. Err.: $t(\max \mathcal{E}\{e_{\varepsilon,1}\})$	2.5000	2.5000	2.5000	2.5000
Occurr. of Max. Avg. Err.: $t(\max \mathcal{E}\{e_{\varepsilon,2}\})$	2.4730	1.9480	2.4990	2.5000
Max. Avg. Err.: $\max \mathcal{E}\{e_{\varepsilon,1}\}$	0.0081	0.0067	0.0063	0.0055
Max. Avg. Err.: $\max \mathcal{E}\{e_{\varepsilon,2}\}$	0.0056	0.0051	0.0051	0.0051
Occurr. of Max. Err.: $t(\max\{e_\varepsilon\})$	2.4875	2.4875	24875	2.4875
Avg. of Max. Err.: $\mathcal{E}\{\max\{e_\varepsilon\}\}$	0.0137	0.0132	0.0131	0.0130
Occurr. of Max. Err. for $e \geq K$: $t(\max\{e_\varepsilon \geq K\})$	2.4875	2.4875	2.4875	2.4875
First Exit Time for $e \geq K$: $\tau_{e \geq K}$	2.4125	2.4625	2.2875	2.4875
% of Paths Exceeding Limit: $\mathbb{P}_S\{e_\varepsilon \geq K\}$	13.92	9.85	8.98	8.49

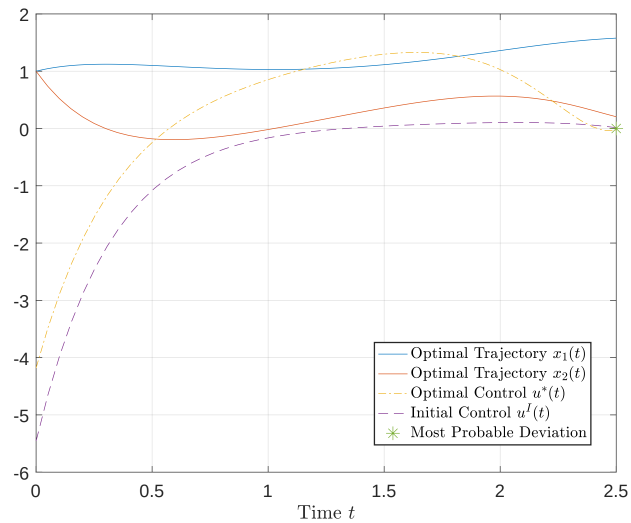


Fig. 9 Optimal LPD Control and Trajectory for Eq. (20) **with** $D \leq [2.0 \cdot D_{\min}]$

Large Deviations (MLD) control. This study represents the first work of its kind in a new control design which is, by nature, still in its infancy. A stochastic control paradigm for general nonlinear systems is enabled. The suggested structure has revealed a significant potential for expansion beyond the ideas in this work and allows for adaptation to a great variety of related problems. In addition, the significant paradigm shift (with respect to traditional control engineering) by moving from point-in-time statistics to sample path statistics replaces the previously necessary spatial integrals with time integrals, thus relieving the control design from the curse of dimensionality. An initial numerical evaluation of MLD control and Cost Constrained MLD indicates the successful control design while expected numerical issues with the simulation of SDEs are briefly discussed.

This work does not address the efficient numerical solution of the control problems in statements 1 and 2. The nested character of the required optimizations renders this a formidable, computationally expensive challenge. The minimization of the inner loop, i.e. the determination of the minimum of the rate function for a nominal path emerging from a particular control input, in statements 1 and 2 can be identified as the key factor in the success of the new control paradigm and its applicability. Therefore, the need for a closed-form performance index reflecting the infimum of the rate function is the necessary next step in future development. This closed form perfomance index should allow for extension to systems subject to state-dependent noise and to systems of high order.

Appendix

Definition 4 (Itō Diffusion Process) *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$ *denote the proper defined complete filtered probability space, i.e. containing the sample space* Ω*, a* σ*-field* $\mathcal{F}$ *consisting of subsets of* Ω *such that* $(\Omega, \mathcal{F})$ *is a measurable space with associated measure* $\mathbb{P}$ *called probability.* $\{\mathcal{F}_t\}_{t \geq 0}$ *is a collection of nested sub-*σ*-fields, called a filtration, providing the notion of time-dependent information. The filtration* $\{\mathcal{F}_t\}_{t \geq 0}$ *is adapted to the Brownian motion* $\boldsymbol{\mathcal{B}}_t$ *by construction. Then, the non-anticipating process* $\{\boldsymbol{x}(t)\}$ *is called an* Itō diffusion process *generated by the stochastic differential equation*

$$d\boldsymbol{x}(\omega, t) = \boldsymbol{b}\big(\boldsymbol{x}(\omega, t), \boldsymbol{u}(t), t\big)\; dt \;+\; G\big(\boldsymbol{x}(\omega, t), t\big)\; d\boldsymbol{\mathcal{B}}_t, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{21}$$

where $\boldsymbol{x} : \Omega \times [0, T] \to \mathbb{R}^n$ *is the stochastic system state,* $\boldsymbol{b} : \mathbb{R}^n \times \boldsymbol{R}^m \times [0, T] \to \mathbb{R}^n$ *a deterministic vector function,* $\boldsymbol{u} : [0, T] \to \mathbb{R}^m$ *the control input,* $G : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times q}$ *the noise gain matrix, and* $\boldsymbol{\mathcal{B}}_t : \Omega \times [0, T] \to \mathbb{R}^q$ *a unit variance Brownian motion. Equation* (21) *is the symbolic notation of*

$$\boldsymbol{x}(\omega, t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{b}\big(\boldsymbol{x}(\omega, t), \boldsymbol{u}(t), t\big)\; d\tau + \int_0^t G\big(\boldsymbol{x}(\omega, t), t\big)\; d\boldsymbol{\mathcal{B}}_t\ . \tag{22}$$

Theorem 2 (Full Freidlin-Wentzell Theorem) *Given the provisions of definition 4, then, the Itō diffusion process* $\{\boldsymbol{x}_\varepsilon(t)\}$ *is the unique solution of the stochastic differential equation*

$$d\boldsymbol{x}_\varepsilon(t) = \boldsymbol{b}\big(\boldsymbol{x}_\varepsilon(t),t\big)\ dt + \sqrt{\varepsilon}\ \boldsymbol{G}\big(\boldsymbol{x}_\varepsilon(t),t\big)\ d\boldsymbol{\mathcal{B}}_t\ , \qquad \boldsymbol{x}_\varepsilon(0) = \boldsymbol{x}_0 \tag{23}$$

for all $t \in [0,T]$*, where* $\boldsymbol{x}_\varepsilon : \Omega \times [0,T] \to \mathbb{R}^n$ *is deterministic,* $\boldsymbol{b} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ *is uniformly Lipschitz continuous in* $\boldsymbol{x}_\varepsilon$ *and continuous in t, all the elements of the diffusion matrix G are bounded, uniformly Lipschitz continuous in* $\boldsymbol{x}_\varepsilon$ *and continuous in t, and* $\boldsymbol{\mathcal{B}}_t$ *is a standard Brownian motion in* $\mathbb{R}^q$*. The existence and uniqueness of the strong solution* $\{\boldsymbol{x}_\varepsilon(t)\}$ *is standard. Then,* $\{\boldsymbol{x}_\varepsilon(t)\}$*, the solution of the above Itō process, satisfies an LDP according to definition 1 in* $C_{x_0}[0,T])$ *with the good rate function*

$$I_{x_0}(\boldsymbol{w}) = \inf_{\{\boldsymbol{g}\in\mathcal{H}^1:\ \boldsymbol{w}(t)=\boldsymbol{x}_0+\int_0^t \boldsymbol{b}(\boldsymbol{w}(s),s)ds+\int_0^t G(\boldsymbol{w}(s),s)\dot{\boldsymbol{g}}(s)ds} \ \frac{1}{2}\int_0^t |\dot{\boldsymbol{g}}(t)|^2 dt, \tag{24}$$

where the infimum over an empty set is taken as $+\infty$*, and* $|\cdot|$ *denotes both the usual Euclidean norm on* $\mathbb{R}^d$ *and the corresponding operator norm of matrices. The spaces* $\mathcal{H}^1$ *and* $\mathcal{L}^2$ *for* $\mathbb{R}^n$*-valued functions are defined using this norm. For* $G(\cdot,\cdot)$ *being a square matrix and for nonsingular diffusions, the term* $a(\boldsymbol{x}_\varepsilon,t) = G(\boldsymbol{x}_\varepsilon,t)G^T(\boldsymbol{x}_\varepsilon,t)$ *is uniformly positive definite and bounded. Thus,* $a^{-1}(\cdot,\cdot)$ *exists and is bounded, such that the rate function in Eq.*(24) *can be stated in compact form as*

$$\begin{aligned} I_{x_0}(\boldsymbol{w}) =& \frac{1}{2}\int_0^T \Big[\dot{\boldsymbol{w}}(t) - \boldsymbol{b}\big(\boldsymbol{w}(t),t\big)\Big]^T a^{-1}\big(\boldsymbol{w}(t),t\big)\Big[\dot{\boldsymbol{w}}(t) - \boldsymbol{b}\big(\boldsymbol{w}(t),t\big)\Big]\ dt \\ & \text{for } \boldsymbol{w} \in \mathcal{H}^1_{x_0}[0,T],\ \textit{and} \\ I_{x_0}(\boldsymbol{w}) =& \infty \\ & \text{for } \boldsymbol{w} \notin \mathcal{H}^1_{x_0}[0,T]\ . \end{aligned} \tag{25}$$

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